STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 1, March 1998

ON THE REMAINDER TERM IN MULTIVARIATE APPROXIMATION

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Dedicated to Professor Ioan Purdea at his 60th anniversary

1. Introduction An efficient procedure to construct multivariate approximation operators is to extend the known results from the univariate case. An algebraic approach of this technique has been developed in [4]. It was shown that any collection of commuting projectors generates a distributive lattice, each of whose elements provide an approximation for a given function. The maximal element of the lattice, which is the "Boolean sum" of the lattice generator operators was identified as "algebrically maximal" approximation operator and the minimal element, which is the "product" of the lattice generators as the "algebrically minimal" approximation operator of the lattice.

Next, in [1], the algebrically maximal and the algebrically minimal operators were characterized by there approximation order: the Boolean sum operator has the maximum approximation order while the product operator has the minimum approximation order among the all elements of the lattice. The proof of these extremally properties is based on the representation of the corresponding remainder operators: the remainder operator corresponding to the Boolean sum of the lattice generators is the product of the remainder operators corresponding to the generator operators and the remainder corresponding to the product of the generators is the Boolean sum of the corresponding remainder operators.

The problem which appears here is to find the remainder operator corresponding to an arbitrary element of the lattice, which is the purpose of this paper.

Also, for the characterization of an interpolation operators will be used the degree of exactness (dex).

1991 *Mathematics Subject Classification.* 41A35,41A65.

Key words and phrases. operator approximation, multivariate approximation, remainder.

2. Let X be a real linear space and P_1, P_2 projectors defined on X. One denotes by P_1P_2 the product and by $P_1 \oplus P_2$ $(P_1 \oplus P_2 = P_1 + P_2 - P_1P_2)$ the Boolean sum of the projectors P_1 and P_2. If $P_1P_2 = P_2P_1$ then P_1 and P_2 are commuting projectors.

Let $P_1, \ldots, P_n$ be commuting projectors defined on X. The algebraic operations of product and Boolean sum yield now projectors.

Let us remind some useful properties of the commuting projectors: if P_1, P_2, P_3 are commuting projectors then:

$$P_1P_2 \text{ and } P_1 \oplus P_2 \text{ are projectors} \tag{1}$$

$$P_1 \oplus P_2 = P_2 \oplus P_1 \tag{2}$$

$$P_1 \oplus (P_2 \oplus P_3) = (P_1 \oplus P_2) \oplus P_3 \tag{3}$$

$$P_1(P_2P_3) = (P_1P_2)P_3 \tag{4}$$

$$P_1(P_2 \oplus P_3) = (P_1P_2) \oplus (P_1P_3) \tag{5}$$

$$P_1 \oplus (P_2P_3) = (P_1 \oplus P_2)(P_1 \oplus P_3) \tag{6}$$

One denoted by $\mathcal{P}$ the set of all projectors generated from the projectors $P_1, \ldots, P_n$ by the operations of product and Boolean sum. $P_1, \ldots, P_n \in \mathcal{P}$, are said to be generator (or primaly) projectors of $\mathcal{P}$. With respect to the order relation " $\leq$ ": $P \leq Q$ iff $PQ = P$, for $P, Q \in \mathcal{P}$, $\mathcal{P}$ is a lattice, i.e. $\inf\{P, Q\} = PQ$ and $\sup\{P, Q\} = P \oplus Q$ for all $P, Q \in \mathcal{P}$. More than that, $\mathcal{P}$ is a distributive lattice (properties (5) and (6)).

3. Mutivariate approximation

Let $D \subset \mathbf{R}^n$ be a rectangular domain, say $D = X_{i=1}^{n}[a_i, b_i]$, and $\mathcal{F}_n$ a set of real functions defined on D.

One considers as generator projectors, the interpolation operators P_i, $P_i : \mathcal{F}_n \to \mathcal{G}_i$, that interpolate a function $f \in \mathcal{F}_n$ with respect to the variable x_i, for $i = 1, \ldots, n$. So, $\mathcal{G}_i$, $i = 1, \ldots, n$, are sets of functions of $n-1$ variables. One reminds that $P_1, \ldots, P_n$ are commuting projectors. Let $\mathcal{P}_n$ be the lattice generated by $P_1, \ldots, P_n$. $S = P_1 \oplus \cdots \oplus P_n$ and $P = P_1 \ldots P_n$ are the maximal respectively the minimal element of $\mathcal{P}_n$. As, P_if can be considered an approximation of f, one denotes by R_if the remainder term, where R_i is the remainder operator: $R_i = I - P_i$, with I the identity operator, for all $i = 1, \ldots, n$.

The arising problem is: *for a given $Q \in \mathcal{P}_n$ which is the corresponding remainder operator, say R_Q.*

It was already mentioned that $R_S = R_1 \dots R_n$ and $R_P = R_1 \oplus \dots \oplus R_n$. Hence, we have the following decompositions of the identity operator:

$$I = S + R_S \tag{7}$$

and

$$I = P + R_P \tag{8}$$

The proof of these identities are based on the mathematical induction principle. So, for $n = 2$, (7) becomes

$$I = P_1 \oplus P_2 + R_1 R_2 \tag{9}$$

Taking into account that $R_i = I - P_i,\ i = 1, 2$, (9) is easy to verify.

Using the associativity property of the Boolean sum and product operations, the relation (7) follows for any $n > 2$. In the same way can be justified the identity (8).

Each decomposition of the identity operator

$$I = P + R$$

generates an approximation formula

$$f = Pf + Rf.$$

Now, let Q be an arbitrary element of $\mathcal{P}_n$. The problem is to determine the remainder operator R_Q, i.e.

$$Q + R_Q = I.$$

Theorem. *If $Q \in \mathcal{P}_n$ is of the form*

$$Q = (P_1 \dots P_{i_1}) \oplus (P_{i_1+1} \dots P_{i_2}) \oplus \dots \oplus (P_{i_{n-1}+1} \dots P_{i_n})$$

then

$$R_Q = (R_1 \oplus \dots \oplus R_{i_1})(R_{i_1+1} \oplus \dots \oplus R_{i_2}) \dots (R_{i_{n-1}+1} \oplus \dots \oplus R_{i_n}). \tag{10}$$

Proof. From (7) it follows that

$$R_Q = R_{P_1 \dots P_{i_1}} R_{P_{i_1+1} \dots P_{i_2}} \dots R_{P_{i_{n-1}+1} \dots P_{i_n}}.$$

But, from (8), we have

$$R_{P_1 \dots P_{i_1}} = R_1 \oplus \cdots \oplus R_{i_1}$$

$$R_{P_{i_{n-1}+1} \dots P_{i_n}} = R_{i_{n-1}+1} \oplus \cdots \oplus R_{i_n}$$

and (10) is proved.

It follows the rule: *the remainder operator R_Q corresponding to the interpolating operator Q is obtained by changing in Q each generator operator P_i by the corresponding remainder operator R_i and the product operation by Boolean sum and the Boolean sum by product.*

Some simple examples are:

$$Q_1 = P_1(P_2 \oplus P_3), \quad R_{Q_1} = R_1 \oplus (R_2 R_3) \tag{11}$$

$$Q_2 = P_1 \oplus (P_2 P_3), \quad R_{Q_2} = R_1(R_2 \oplus R_3) \tag{12}$$

$$Q_3 = (P_1 P_2) \oplus (P_3 P_4), \quad R_{Q_3} = (R_1 \oplus R_2)(R_3 \oplus R_4) \tag{13}$$

$$Q_4 = (P_1 \oplus P_2)(P_3 \oplus P_4), \quad R_{Q_4} = (R_1 R_2) \oplus (R_3 R_4) \cdot \tag{14}$$

Homogeneous approximation formulas

Let $f \in \mathcal{F}_n$ be given and $Q \in \mathcal{P}_n$. The decomposition of the identity operator $I = Q + R_Q$ generates the approximation formula for the function f:

$$f = Qf + R_Q f,$$

with $R_Q f$, the remainder term. For example, the two extremal elements of $\mathcal{P}_n$, S and P generate the so called algebrical maximal respectively algebrical minimal formulas, i.e.

$$f = Sf + R_S f$$

and

$$f = Pf + R_P f.$$

Definition 1. Let $Q \in \mathcal{P}_n$ be given. The number $r \in \mathbb{N}$, with the property that $Qf = f$ for all $f \in \mathbf{P}_r^n$ (the set of all polynomials in n variables of the total degree at most r) and there exists a polynomial $g \in \mathbf{P}_{r+1}^n$ such that $Qg \neq g$, is called the degree of exactness of the operator Q, i.e. $\mathrm{dex}(Q) = r$.

Remark 1. The conditions $Qf = f$ for all $f \in \mathbf{P}_r^n$ and there exists $g \in \mathbf{P}_{r+1}^n$ such that $Qg \neq g$ are equivalente with $Qe_{ij} = e_{ij}$ for all $i, j \in \mathbb{N},\ i + j \leq r$ and there exists $p, q \in \mathbb{N}$ with $p + q = r + 1$ such that $Qe_{pq} \neq e_{pq}$, where $e_{ij}(x, y) = x^i y^j$.

It is known that the approximation order of the operators S and P are given by:

$$\operatorname{ord}(S) = \operatorname{ord}(P_1) + \cdots + \operatorname{ord}(P_n)$$

respectively

$$\operatorname{ord}(P) = \min\{\operatorname{ord}(P_1), \ldots, \operatorname{ord}(P_n)\}.$$

We also have:

Theorem 2. $\operatorname{dex}(S) = \operatorname{dex}(P_1)+\cdots+\operatorname{dex}(P_n)$, $\operatorname{dex}(P) = \min\{\operatorname{dex}(P_1), \ldots, \operatorname{dex}(P_n)\}$ *and* $\operatorname{dex}(P) \leq \operatorname{dex}(Q) \leq \operatorname{dex}(S)$ *for all* $Q \in \mathcal{P}_n$.

Following the Remark 1, the proof is reduced to a direct verification.

So, the Boolean sum operator S has the maximum degree of exactness while the product P has the minimum degree of exactness, among all elements of $\mathcal{P}_n$.

But, Sf approximates the function f in terms of functions of $n - 1, \ldots, 1$ variables, while Pf is a scalar approximation of f.

Remark 2. $P \in \mathcal{P}_n$ is the only element of $\mathcal{P}_n$ with the property that Pf is a scalar approximation of f. For any $Q \in \mathcal{P}_n$, $Q \neq P$, Qf has at least one free variable of f. For example,

$$Q_1 f := (P_1P_2 + P_1P_3 - P_1P_2P_3)f$$

with Q_1 from (11), contains two free variables: x_3 in the term P_1P_2f and x_2 in P_1P_3f.

Starting with an approximation formula

$$f = Qf + R_Q f, \quad Q \in \mathcal{P}_n,\ Q \neq P,$$

in order to obtain a scalar approximation formula, we can use next approximation levels. If $Q_1 = P_1^1(P_2^1 \oplus P_3^1)$ (example from (11)), where the upper index marks the approximation level number, then the corresponding approximation formula is generated by the identity

$$I = (P_1^1P_2^1 + P_1^1P_3^1 - P_1^1P_2^1P_3^1) + (R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1).$$

If, in a second level of approximation, it is used the operators P_3^2 and P_2^2 with $R_3^2 = I - P_3^2$ and $R_2^2 = I - P_2^2$, one obtains

$$I = (P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_3^1) + (R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2),$$

which generates a scalar approximation formula: $(P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_3^1)f$ is a scalar approximation of f and $(R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2)f$ is the corresponding remainder term.

It is obviously to see that the remainder operator R_Q for $Q \neq S$ and $Q \neq P_i$, $i = 1,\ldots,n$, is the sum of many terms. The approximation order of the interpolation operator Q must be taken with respect to each term of R_Q. In the above example the number of the terms is five: R_1^1, $R_2^1R_3^1$, $R_1^1R_2^1R_3^1$, $P_1^1P_2^1P_3^2$ and $P_1^1P_3^1R_2^2$. The degree of exactness corresponding to these terms are $\mathrm{dex}(P_1^1)$, $\mathrm{dex}(P_2^1) + \mathrm{dex}(P_3^1)$, $\mathrm{dex}(P_1^1) + \mathrm{dex}(P_2^1) + \mathrm{dex}(P_3^1)$, $\mathrm{dex}(P_3^2)$ respectively $\mathrm{dex}(P_2^2)$.

Let $Q \in \mathcal{P}_n$, $Q \neq S$ and $Q \neq P_i$, $i = 1,\ldots,n$ and R_Q the corresponding remainder operator.

Definition 2. If the degree of exactness corresponding to each term of the remainder operator R_Q is the same then Q is called a homogeneous approximation operator and

$$f = Qf + R_Qf$$

a homogeneous approximation formula.

In the considered example

$$Q = P_1^1P_2^1P_3^2 + P_1^1P_2^2P_3^1 - P_1^1P_2^1P_2^1$$

and

$$R_Q = R_1^1 + R_2^1R_3^1 - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1 + R_2^2,$$

Q is a homogeneous approximation operator if

$$\mathrm{dex}(P_1^1) = \mathrm{dex}(P_2^1) + \mathrm{dex}(P_3^1) = \mathrm{dex}(P_1^1) + \mathrm{dex}(P_2^1) + \mathrm{dex}(P_3^1) = \mathrm{dex}(P_2^2) = \mathrm{dex}(P_3^2)$$

that has the only solution $\mathrm{dex}(P_1^1) = \mathrm{dex}(P_2^1) = \mathrm{dex}(P_3^1) = \mathrm{dex}(P_2^2) = \mathrm{dex}(P_3^2) = 0$. But the remainder operator R_Q can be changed in a convenable way. One of them is

$$\tilde{R}_Q = R_1^1 + R_2^1R_3^1(P_1^1 + R_1^1) - R_1^1R_2^1R_3^1 + P_1^1P_2^1R_3^2 + P_1^1P_3^1R_2^2 \quad (P_1^1 + R_1^1 = I)$$

or

$$\tilde{R}_Q = R_1^1 + P_1^1R_2^1R_3^1 + P_1^1P_2^1P_3^2 + P_1^1P_3^1R_2^2.$$

It follows that the formula

$$f = Qf + \tilde{R}_Qf \tag{15}$$

with

$$Q = P_1^1 P_2^1 P_3^2 + P_1^1 P_2^2 P_3^1 - P_1^1 P_2^1 P_3^1$$

is a scalar homogeneous interpolation formula if

$$\mathrm{dex}(P_1^1) = \mathrm{dex}(P_2^1) + \mathrm{dex}(P_3^1) = \mathrm{dex}(P_3^2) = \mathrm{dex}(P_2^2). \tag{16}$$

For example, such a formula is obtained for: $P_1^1 := H^x_{2m+1}$, $P_2^1 := H^y_{m+1}$, $P_3^1 := L^z_m$, $P_2^2 := H^y_{2m+1}$ and $P_3^2 := H^z_{2m+1}$, where L^v_n and H^v_n are Lagrange respectively Hermite interpolation operator of the degree n, which interpolate a function f with respect to the variable v. The degree of exactness of the obtained operator is $2m+1$.

So, starting with the formula given by the operators Q_1 and R_{Q_1} from (11), we can obtain scalar interpolation operators (scalar interpolation formulas) of any degree of exactness.

A second example, we are looking for, is given by the operator of example (12). Initial formula is

$$f = (P_1 + P_2P_3 - P_1P_2P_3)f + (R_1R_2 + R_1R_3 - R_1R_2R_3)f,$$

that is not a homogeneous one. In order to get a homogeneous formula the remainder operator can be change as follows:

$$R_1R_2 + R_1R_3 - R_1R_2R_3 = R_1R_2(P_3+R_3) + R_1R_3 - R_1R_2R_3 = P_3R_1R_2 + R_1R_3 \quad (P_3 + R_3 = I).$$

This way, one obtains

$$f = (P_1 + P_2P_3 - P_1P_2P_3)f + (P_3R_1R_2 + R_1R_3)f. \tag{17}$$

Obviously, if $\mathrm{dex}(R_2) = \mathrm{dex}(R_3)$ then (17) is a homogeneous approximation formula. But, it is not a scalar formula, as can be seen, $P_1 + P_2P_3 - P_1P_2P_3$ is not a scalar approximation operator.

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