STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLIII, Number 1, March 1998

TORSION IN Γ-LATTICES

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. After general properties of Γ-lattices, a new notion of torsion is given and some of its connections with purity are established.

1. Introduction

Let $(\Gamma, \cdot, 1)$ be a monoid. A lattice L is called **Γ-latice** ([3]) if it is provided with a multiplication $\varphi : \Gamma \times L \to L$ (we shall denote by $\gamma a = \varphi(\gamma, a)$) which satisfies the following axioms

$\Gamma 1 : \gamma a \leq a$

$\Gamma 2 : \gamma(a \vee b) = \gamma a \vee \gamma b$

$\Gamma 3 : (\gamma\gamma')a = \gamma(\gamma' a)$

$\Gamma 4 : 1.a = a$

The source of this notion is the lattice of all the submodules of a given module M over a commutative ring R with identity on which the monoid of the principal ideals of R operates in a natural way: $\varphi(rR, A) = rA$ $(r \in R, A \leq M)$.

Remark 1.1. *This monoid naturally acts also on quotient R-modules.*

Moreover, this monoid has a special element : the zero ideal. In order to get suitable definitions for purity, divisibility and torsion and to recover some of the standard results one must consider a zero element in the monoid Γ. This is called a Γ_0-lattice if it satisfies the axiom

$\Gamma 0$: for each $a \in L$, $0.a = 0$ holds.

We say that Γ **has no zero-divisors** if $\gamma \neq 0, \delta \neq 0$ imply $\gamma \cdot \delta \neq 0$.

A subset $C \subsetneq L$ is called a **system of generators** for L if each element of L is a union of elements from C. A system of generators is called **closed** if $\Gamma \cdot C \subseteq C$.

This research was completed in the Universita degli Studi of Padova under a Nato-CNR fellowship.

As in [2] we use the quotient sublattice notation $b/a = \{c \in L | a \leq c \leq b\}$. An element $c \in L$ is called **cycle** if $c/0$ is a noetherian and distributive sublattice. Clearly, using $\Gamma 1$, for any cycle c and any $\gamma \in \Gamma$, γc is a cycle too.

In a Γ_0-lattice an element $d \in L$ is called **divisible** if $\forall 0 \neq \gamma \in \Gamma : \gamma d = d$.

In a Γ-lattice L an element p is called **pure** (see [1]) if $\gamma p = p \wedge \gamma 1, \forall \gamma \in \Gamma$.

2. Elementary results

In what follows Γ will denote a (non-necessary commutative) monoid. For the proofs of the following simple results see [1].

Lemma 2.1. *In any* Γ*-lattice,* $\gamma.0 = 0, \forall \gamma \in \Gamma$.

Consequence 2.1. 0 *is divisible in each* Γ_0*-lattice.*

- One can consider, for a fixed $\gamma \in \Gamma$, the upper semi-morphism (according to $\Gamma 2$) $\varphi_\gamma : L \to L, \varphi_\gamma(a) = \gamma a, \forall a \in L$. Hence

Lemma 2.2. φ_γ *is an order-preserving morphism.*

Hence

Lemma 2.3. *(i)* $a \leq b \Rightarrow \gamma a \leq \gamma b$. *Moreover,*

(ii) $\gamma(a \wedge b) \leq \gamma a \wedge \gamma b$.

A subset B of a Γ-lattice L is called a **Γ-stable** if $\forall \gamma \in \Gamma, \gamma B \subseteq B$.

Clearly (using $\Gamma 1$) the sublattices $a/0$ are Γ-stable and in general not every sublattice $1/a$ (or b/a) is Γ-stable.

Proposition 2.1. *A sublattice* b/a *is* Γ*-stable iff* a *is divisible.*

Lemma 2.4. *Each divisible element is also pure.*

Reconsidering 1.1 we consider on quotient sublattices b/a the following Γ-lattice structure:

$\forall \gamma \in \Gamma, \forall c \in b/a : \gamma * c = (\gamma c) \vee a$

enlarging in this way the notion of **Γ-sublattice** (Γ-stable sublattices).

Obviously, if a is divisible this is the natural Γ-lattice structure on b/a obtained by restriction.

3. Torsion and purity in Γ_0-lattices

In this section we give some properties of a new notion of torsion in a Γ_0-lattice L connected with purity (continuing [1]). In this context special new conditions on Γ_0-lattices seem to be necessary.

Observe that the inequality $\bigvee\{\gamma a | \gamma a \leq b\} \leq b \wedge \gamma 1$ holds for each $b \in L$ and each $\gamma \in \Gamma$.

Clearly, if $b = \gamma x$ for a suitable $x \in L$ this is an equality: indeed, both members are equal to b. Generally, if $b \notin \Gamma \cdot L$ this could be no equality.

In the sequel we shall call a Γ-lattice **dense** if for each $\gamma \in \Gamma$ and each $b \in L$ the equality $\bigvee\{\gamma a | \gamma a \leq b\} = b \wedge \gamma 1$ holds.

We use **bounded** elements in a Γ_0-lattice, i.e. elements $b \in L$ such that there is an $0 \neq \gamma \in \Gamma$ with $\gamma b = 0$. We shall denote by B the set of all the bounded elements of L.

For a Γ_0-lattice L the **torsion part** $t(L)$ is defined as the union of all the bounded (compact) elements. Then L is called a **torsion lattice** if $t(L) = 1$ resp. $t \in L$ is called a **torsion element** if $t = t(t/0)$. The lattice L is called **torsion-free** if $t(L) = 0$ resp. $u \in L$ is called a **torsion-free element** if $t(u/0) = 0$.

A closed system of generators $C \subseteq L$ is called **good** if $C \cap (t(L)/0) \subseteq B$, i.e., the generators $c \in C$ such that $c \leq t(L)$ are bounded (as concrete examples one could consider the compact elements in algebraic H-noetherian lattices or the cycles in cyclic generated lattices).

We first record in a Γ_0-lattice L the following simple properties:

(a) If $a \leq b$ and b is bounded then a is also bounded. In particular, by $\Gamma 1$, if b is bounded, γb is bounded too, for each $\gamma \in \Gamma$.

(b) Each atom is bounded or divisible.

(c) If C is a system of generators for L then any bounded element b is an union of bounded generators $\{c_i\}_{i \in I} \subseteq C$. Moreover, if for $0 \neq \gamma$ we have $\gamma b = 0$ then $\forall i \in I : \gamma c_i = 0$.

Consequently, *if the Γ_0-lattice L has no divisible atoms*

(d) The socle $s(L) \leq t(L)$.

(e) If u is a torsion-free element then $u/0$ has no atoms.

If Γ has no zero-divisors

(f) For each $0 \neq \gamma \in \Gamma$, b is bounded iff γb is bounded.

Indeed, if γb is bounded there is $0 \neq \delta \in \Gamma : \delta(\gamma b) = (\delta\gamma) b = 0$. Γ having no zero-divisors, $\delta\gamma \neq 0$ and so b is bounded. The rest is (a).

Proposition 3.1. *If Γ has no zero-divisors the "radical" property:*

$t(1/t(L)) = t(L)$,

holds in a Γ_0-lattice L with a good system of generators C.

Proof. By definitions: $1/t(L)$ is a torsion-free sublattice $\Leftrightarrow$

$\forall b \in 1/t(L), \exists 0 \neq \gamma \in \Gamma : \gamma * b = t(L) \Rightarrow b = t(L) \Leftrightarrow$

$\exists 0 \neq \gamma \in \Gamma : \gamma b \leq t(L) \leq b \Rightarrow b = t(L)$. The lattice L having a (good) system of generators C, the inequality $b \leq t(L)$ can be verified as follows: $\forall c \in C, c \leq b \Rightarrow c \leq t(L)$.

Indeed, if $c \leq b$, by 2.3 $\gamma c \leq \gamma b \leq t(L)$. C being a good system of generators γc is also a generator and it is bounded. Hence by (f) c is bounded too and $c \leq t(L)$.□

Consequence 3.1. *If Γ has no zero-divisors, L is cycle generated Γ_0-lattice and the cycles in $t(L)/0$ are bounded then L has the "radical" property.*□

Consequence 3.2. *If Γ has no zero-divisors and L is an algebraic and H-noetherian Γ_0-lattice then L has the "radical" property.*□

Another condition we need in the propositions that follows is:

for each $0 \neq \gamma \in \Gamma, \gamma a \leq \gamma b \Rightarrow a \leq b$ for elements in torsion-free Γ_0-lattices $(*)$.

Proposition 3.2. *If for an element p in a dense Γ_0-lattice L the sublattice $1/p$ is torsion-free then p is pure. Conversely, if p is pure in a torsion-free Γ_0-lattice L with $(*)$ then $1/p$ is also torsion-free.*

Proof. Indeed, $1/p$ is torsion-free iff $\exists 0 \neq \gamma \in \Gamma : \gamma u \leq p \leq u \Rightarrow u = p$. Using the density of L we prove the inequality $p \wedge \gamma 1 \leq \gamma p$ as follows: $\gamma a \leq p \wedge \gamma 1 \Rightarrow \gamma(a \vee p) = \gamma a \vee \gamma p \leq p \wedge \gamma 1 \leq p \Rightarrow a \vee p = p \Rightarrow a \leq p \Rightarrow \gamma a \leq \gamma p$.

Conversely, for $0 \neq \gamma \in \Gamma : \gamma u \leq p \leq u$ and $\gamma p = p \wedge \gamma 1$ we have to prove that $u = p$.

First, observe that $\gamma u \leq p, \gamma u \leq \gamma 1 \Rightarrow \gamma u \leq p \wedge \gamma 1 = \gamma p$ and $p \leq u \Rightarrow \gamma p \leq \gamma u$ so that $\gamma u = \gamma p$. One has finally to use $(*)$.□

Proposition 3.3. *If Γ has no zero-divisors, in a dense Γ_0-lattice L with a good system of generators, $t(L)$ is pure.*

Proof. This is an immediate consequence of 3.1 and 3.2.□

Proposition 3.4. *In a dense torsion-free* Γ_0*-lattice* L *with* $(*)$, *an intersection of pure elements is pure.*

Proof. Let $\{p_i\}_{i\in I}$ be a family of pure elements of L and let $\overline{p} = \bigwedge_{i\in I} p_i$. The lattice being dense it suffices to verify that for each $0 \neq \gamma \in \Gamma$: $\gamma a \leq \overline{p} \wedge \gamma 1$ implies $\gamma a \leq \gamma\overline{p}$.

Indeed, $\gamma a \leq \overline{p} \wedge \gamma 1 = (\bigwedge_{i\in I} p_i) \wedge \gamma 1 = \bigwedge_{i\in I}(p_i \wedge \gamma 1) = \bigwedge_{i\in I}(\gamma p_i)$ implies $\gamma a \leq \gamma p_i$ for each $i \in I$. Now the condition $(*)$ implies $a \leq p_i$ for each $i \in I$ and so $a \leq \overline{p}$. Hence $\gamma a \leq \gamma\overline{p}$ by 2.3.□

Final remark. Although with a promising start, Γ-lattices require too much special conditions in order to obtain important results.

References

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