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DEFICIENT SPLINE APPROXIMATIONS FOR SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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REZUMAT. - Aproximări prin funcţii spline cu deficienţă pentru ecuaţii diferenţiale de tip neutral cu întârziere. În această lucrare se consideră un procedeu de rezolvare numerică a ecuaţiei diferenţiale de ordinul al doilea cu argument modificat utilizând funcţii spline polinomiale de gradul $m \geq 3$ şi clasă de continuitate C^{m-2}. Se studiază estimarea erorii procedeului de colocaţie dat, împreună cu convergenţa metodei. Un exemplu numeric ilustrează eficienţa metodei.

Abstract. A collocation procedure with polynomial spline functions of degree $m \geq 3$ and continuity class C^{m-2} is considered for numerical solution of a second order initial value problem for neutral delay differential equations. The estimation of the error as well as the convergence of the deficient cubic spline approximations is investigated.

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1. Introduction. In recent years functional differential equations have been applied in various fields of science and consequently, a large number of papers on the theory of functional differential equations has been published. The divergence of higher degree spline function methods arises from the enforcement of the continuity requirement on the spline functions. It is possible to generate convergent higher order methods by relaxing the continuity. This type of method will be considered in this paper. Here a spline approximation for the numerical solution of neutral delay differential equations has been introduced with degree $m \geq 3$ and continuity class C^{m-2}.

2. Description of the spline collocation method. Consider the following second order initial value problem for neutral delay differential equations:

$$y''(t) = f(t, y(t), y(g(t)), y'(g(t))), \quad t \in [a,b]$$

$$y(t) = \varphi(t), \ y'(t) = \varphi'(t), \ t \in [a,b], \ \alpha \leq a < b \tag{2.1}$$

The function g, called the delay function, is assumed to be continuous on the interval $[\alpha, b]$, and to satisfy the inequality $\alpha \leq g(t) < t$, $t \in [a,b]$, and $\varphi \in C^{m-1}[\alpha, a]$, where $m > 2$. Assume that the functional

$$f\colon [a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b] \to \mathbf{R}$$

satisfies the following conditions H_1 and H_2:

H_1. For any $x \in C^1[\alpha,b]$, the mapping $t \rightarrow f(t,x(t),x(\cdot),x'(\cdot))$ is continuous on $[a,b]$.

H_2. The following Lipschitz condition holds:

$$\|f(t,x_1(t),y_1(\cdot),z_1(\cdot)) - f(t,x_2(t),y_2(\cdot),z_2(\cdot))\|$$

$$\leq L_1\left(\|x_1 - x_2\|_{[\alpha,t]} + \|y_1 - y_2\|_{[\alpha,t-\delta]} + \|z_1 - z_2\|_{[\alpha,t-\delta]}\right)$$

$$+ L_2\|z_1 - z_2\|_{[\alpha,t]}$$

with $L_1 \geq 0$, $0 \leq L_2 < 1$, $\delta^* > 0$, for any $t \in [a,b]$, $x_1,x_2 \in C^1[a,b]$, $y_1,y_2,z_1,z_2 \in C[\alpha,a]$. Under conditions H_1 and H_2, the problem (2.1) has a unique solution $y \in C^2[a,b] \cap C[\alpha,b]$; see [1,2].

As it is well known, jump discontinuities can occur in various higher order derivatives of the solution even if f,g,φ are analytic in their arguments. Such jump discontinuities are caused by the delay function g and propagate from the point a as the order of derivative increases. We denote the jump discontinuities by $\{\xi_i\}$, which are the roots of the equations $g(\xi_i) = \xi_{i-1}$; $\xi_0 = a$ is the jump discontinuity of φ. Since in this paper g does not depend on y, we can consider the jump discontinuities to be known for sufficiently high order derivatives and to be such that

$$\xi_0 < \xi_1 < \ldots < \xi_{k-1} < \xi_k < \ldots, \xi_M$$

We shall construct a deficient polynomial spline approximating function of degree $m \geq 3$ and deficiency 2, denoted by s: $[a,b] \rightarrow \mathbf{R}$.

Consider the interval $[\xi_j,\xi_{j+1}]$, $j = 0, \ldots, M-1$, subdivided by a uniform partition by the knots

$$\xi_j = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_N = \xi_{j+1},$$

where $t_k = t_0 + kh$ and $h = (\xi_{j+1} - \xi_j)/N$. The spline function s approximating the solution of (2.1) is defined on each subinterval $[t_k,t_{k+1}]$ by

$$s(t) = \sum_{i=0}^{m-2} \frac{s^{(i)}(t_k)}{i!}(t-t_k)^i + \frac{a_k}{(m-1)!}(t-t_k)^{m-1} + \frac{b_k}{m!}(t-t_k)^m, \qquad (2.2)$$

where $s^{(i)}(t_k)$, $0 \leq i \leq m-1$, are left-hand limits of the derivatives as $t \rightarrow t_k$ of the segment of s defined on $[t_{k-1},t_k]$, and the parameters a_k and b_k are determined from the following collocation conditions:

$$s''(t_k + h/2) =$$

$$f\Big(t_k + h/2,\ s(t_k + h/2),\ s(g(t_k + h/2)),\ s'(g(t_k + h/2))\Big) \qquad (2.3)$$

$$s''(t_{k+1}) = f\Big(t_{k+1}, s(t_{k+1}), s(g(t_{k+1})), s'(g(t_{k+1}))\Big) \qquad (2.4)$$

In this way, we obtain a spline function of degree $m \geq 3$ and class C^{m-2} over the entire interval $[\xi_j,\xi_{j+1}]$, with the knots $\{t_k\}_{k=0}^{N}$. It remains to show that, for h sufficiently small, the parameters a_k and b_k can be uniquely determined from

(2.3) and (2.4).

THEOREM 2.1. *If f satisfies conditions* H_1 *and* H_2, $\varphi \in C^{m-1}[\alpha,a]$, $\alpha \leq g(t) \leq t$, $t \in [\alpha,b]$, *and if h is sufficiently small, then there exists a unique spline approximating solution of problem* (2.1) *given by* (2.3)-(2.4).

Proof. If is sufficient to proof that a_k and b_k can be uniquely determined from (2.3) and (2.4). Substituting (2.2) in (2.3) and (2.4), we have

$$a_k = \frac{(m-3)!}{h^{m-3}}\Bigg\{2^{m-2} f\Bigg(t_k + \frac{h}{2}, A_k(t_k + \frac{h}{2}) + \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1} + \frac{b_k}{m!}(\frac{h}{2})^m,$$

$$A_k(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1} + \frac{b_k}{m!}\left(g(t_k + \frac{h}{2}) - t_k\right)^m,$$

$$A_k'(g(t_k + \frac{h}{2}) + \frac{a_k}{(m-2)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-2} + \frac{b_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1}\Bigg)$$

$$- 2^{m-2} A_k''(t_k + \frac{h}{2})$$

$$-f\Bigg(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m,$$

$$A_k(g(t_{k+1})) + \frac{a_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} + \frac{b_k}{m!}(g(t_{k+1}) - t_k)^m,$$

$$A_k'(g(t_{k+1})) + \frac{a_k}{(m-2)!}(g(t_{k+1}) - t_k)^{m-2} + \frac{b_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1}\Bigg) + A_k''(t_{k+1})\Bigg\}$$

$$b_k = \frac{2(m-2)!}{h^{m-2}}\Bigg\{f\Bigg(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m,$$

$$A_k(g(t_{k+1})) + \frac{a_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1} + \frac{b_k}{m!}(g(t_{k+1}) - t_k)^m,$$

$$A_k'(g(t_{k+1})) + \frac{a_k}{(m-2)!}(g(t_{k+1}) - t_k)^{m-2} + \frac{b_k}{(m-1)!}(g(t_{k+1}) - t_k)^{m-1}\Bigg) - A_k''(t_{k+1})$$

$$-2^{m-3} f\left(t_k + \frac{h}{2}, A_k(t_k + \frac{h}{2}) + \frac{a_k}{(m-1)!}(h)^{m-1} + \frac{b_k}{m!}(h)^m,\right.$$

$$A_k(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-1)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-1} + \frac{b_k}{m!}\left(g(t_k + \frac{h}{2}) - t_k\right)^m,$$

$$\left.A_k'(g(t_k + \frac{h}{2})) + \frac{a_k}{(m-2)!}\left(g(t_k + \frac{h}{2}) - t_k\right)^{m-2} + \frac{b_k}{(m-1)!}(g(t_k + \frac{h}{2}) - t_k)^{m-1}\right) + 2^{m-3} A_k''\, t_k + \frac{h}{2})\Bigg\}$$

where

$$A_k(t) = \sum_{j=0}^{m-2} \frac{S^{(j)}(t_k)}{j!}(t - t_k).$$

Thus we have

$$a_k = G_1(a_k, b_k)$$

$$b_k = G_2(a_k, b_k). \tag{2.5}$$

Using assumption H_2 for $\frac{5Lh(m+5h)}{m(m-1)} < 1$ the application G: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(a_k, b_k) \to \left(G_1(a_k, b_k), G_2(a_k, b_k)\right) \tag{2.6}$$

is a contraction mapping and has a unique solution (a_k, b_k), which can be found by iteration. This completes the proof of the theorem.

In order to make a connection between the spline method and discrete multistep methods, we present the following theorem, which gives the relationship between the value of any spline function $s \in S_m$ and its second derivative at the knots.

THEOREM 2.2. [3] *For any spline function* $s \in S_m$, $s \in C^{m-2}[a,b]$, $m \geq 3$, *there exists a unique linear consistency relation between the quantities* $s(t_k)$ *and* $s''(t_k)$, $k = 0, 1, \ldots, m-2$, *namely*

$$\sum_{j=0}^{m-2} a_j^{(m)} s(t_{j+\nu}) = h^2 \sum_{j=0}^{m-3} b_j^{(m)} s''(t_{j+\nu}),\ 0 \leq \nu \leq N-m+1, \tag{2.7}$$

where

$$a_j^{(m)} := (m-1)!\left[Q_{m-1}(j+1) - 2Q_{m-1}(j) + Q_{m-1}(j-1)\right],$$

$$b_j^{(m)} := (m-1)!\, Q_{m+1}(j+1), \tag{2.8}$$

and

$$Q_k(t) := \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (t-i)_+^{k-1}.$$

THEOREM 2.3. *The spline approximating values* $s(t_k)$, $k = \overline{0,N}$ *of the above procedure are exactly the values furnished by the following multidiscrete method*

$$\sum_{j=0}^{m-1} a_j^{(m)} Y_{j+\nu} = h^2 \sum_{j=0}^{m-1} b_j^{(m)} Y''_{j+\nu},\ 0 \leq \nu \leq N-m+1, \tag{2.9}$$

where the coefficients $a_j^{(m)}$ *and* $b_j^{(m)}$ *are given by* (2.8), *if the starting values*

$$y_0 = s(t_0), y_1(t_1), \ldots, y_{m-3} = s(t_{m-2}) \tag{2.10}$$

are used.

Proof. For h small enough, only one set of values $\{y_k\}_k$ is satisfying the relation (2.9) with the starting values (2.10). But obviously the spline values $\{s_k\}_k$ are satisfying (2.9) on the basis of the consistency relation (2.7) and also the starting values (2.10). That means the spline values must coincide with the values given by the discrete multistep method (2.9).

In the sequel, we shall be concerned with estimating the error in the approximation of the solution of (2.1) by splines as well as with the convergence of the approximation s to the exact solution y as $h \to 0$.

We now define the step function $s^{(m)}$ at the knots $\{t_k\}_{k=1}^{N-1}$ by the usual arithmetic mean:

$$s^{(m)}(t_k) = \frac{1}{2}\left[s^{(m)}(t_k - \frac{1}{2}h) + s^{(m)}(t_k + \frac{1}{2}h)\right] \tag{2.11}$$

We need the following lemmas:

LEMMA 2.1. *Let* $s:[a,b] \to \mathbf{R}$ *be the spline approximating function and* y *be the unique solution of problem* (2.1). *If*

$$|s(t_k) - y(t_k)| < Kh^p, \quad |s(g(t_k)) - y(g(t_k))| < Kh^p,$$

$$|s'(g(t_k)) - y'(g(t_k))| < Kh^p,$$

where K *is a constant, then there exists a constant* K_1 *such that*

$$|s(t_k) - y(t_k)| < K_1 h^p, \ |s''(t_k) - y''(t_k)| < K_1 h^p.$$

Proof. Using the Lipschitz condition H_2, we have

$$|s''(t_k) - y''(t_k)|$$

$$= |f(t_k, s(t_k), s(g(t_k)), s'(g(t_k))) - f(t_k, y(t_k), y(g(t_k), y(g(t_k)), y'(g(t_k)))|$$

$$\leq L\{|s(t_k) - y(t_k)| + |s(g(t_k)) - y(g(t_k))| + |s'(g(t_k)) - y'(g(t_k))|\}$$

$$L\{Kh^p + Kh^p + Kh^p\} = 3KLh^p.$$

If $K_1 := \max\{K, 3KL\}$, then we have

$$|s(t_k) - y(t_k)| < K_1 h^p, \ |s''(g(t_k)) - y''(g(t_k))| < K_1 h^p.$$

LEMMA 2.2. [4] *Let $y \in C^{m+1}[a,b]$, and s be the spline function of degree m and class $C^{(m-2)}$ with the knots $\{t_k\}_k$.* Suppose that the following relations hold:

$$|s^{(r)}(t_k) - y^{(r)}(t_k)| = O(h^{p_r}), \ |s^{(r)}(g(t_k)) - y^{(r)}(g(t_k))| = O(h^{p_r}),$$

$$0 \leq r \leq m-2, \ 0 \leq k \leq N,$$

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \ t_k < t < t_{k+1}, \ 0 \leq k \leq N.$$

Then it follows:

$$|s(t) - y(t)| = O(h^p), \ p := \min\{p_1, 1 + p_1, \ldots, (m-2) + p_{m-2}\},$$

$$p_m = 1, \ \forall t \in [a,b]$$

and

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \ t \in [a,b].$$

3. Cubic spline function approximating the solution. By Theorem 2.3 for $m = 3$, the cubic approximating spline function of degree 3 and defficiency 2 yields the same values at the knots as the discrete multistep method based on the following recurrence formula:

$$y_{k+1} - 2y_k + y_{k-1} = \frac{h^2}{6}\left[y_{k+1} + 4y_k + y_{k-1}\right] = \frac{h^2}{6}\left[f_{k+1} + 4f_k + f_{k-1}\right] \tag{3.1}$$

where

$$f_j = f\big(t_j, y(t_j), y(g(t_j)), y'(g(t_j))\big),$$

if the starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ are used. The discrete method (3.1) has degree of exactness three provided that the starting values y_0 and y_1 have third order accuracy.

As in [3], it is easy to prove that the starting values $y_0 = s(t_0)$ and $y_1 = s(t_0+h)$ have the same order of exactness as the recurrence formula (3.1); therefore we can conclude that

$$|s(t_k) - y(t_k)| = O(h^3), \ |s''(t_k) - y''(t_k)| = O(h^3).$$

The second relation follows from Lemma 2.1 for $p = 3$.

THEOREM 3.1. *If* $f \in C^2([a,b] \times C^1[a,b] \times C^1[\alpha,b] \times C[\alpha,b])$ *and* s *is* *the cubic spline function of degree* 3 *and defficiency* 2 *approximating the solution of* (2.1), *then there exists a constant K, independent of h, such that, for h sufficiently small and* $t \in [a,b]$,

$$|y(t) - s(t)| < Kh^3, \ |y'(t) - s'(t)| < Kh^2, \ |y'''(t) - s'''(t)| < Kh.$$

Proof. The proof is similar to the proof of Theorem 3.1 in [5].

4. Numerical Example. Consider the following neutral delay differential equation.

$$y''(t) = \cos t - \frac{1}{2}y(t) + \frac{1}{2}y(t-\pi) - y'(t-\pi), \ t \geq 0$$

$$y(t) = 1, \ -\pi \leq t \leq 0$$

The exact solution for this problem with the given initial function is:

$$y(t) = 1 - 2\cos t + 2\cos\left(\frac{\sqrt{3}}{3}t\right), \text{ for } t \in [0,\pi].$$

Table I shows cubic approximations and Table II shows deficient spline approximations of order 3 and continuity class 1, for $h = \frac{22}{1400}$.

Table I

k	a_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
0	0.369943	1.000000	1.000000	2.0161678549E-07
1	-0.208136	1.000002	1.000000	1.3896369637E-06
2	-0.015305	1.000004	1.000000	3.2549742173E-06

k	a_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
3	0.231232	1.000006	1.000001	5.7729739638E-06
4	-0.385548	1.000010	1.000001	9.2825812317E-06
5	0.429349	1.000015	1.000001	1.3213615603E-05

Table II

k	a_k	b_k	$s(t_k)$	$y(t_k)$	$e(t_k)$
0	3.206419	0.013867	1.000396	1.000000	3.9586596358E-04
1	2.790963	0.012070	1.001532	1.000000	1.5321755618E-03
2	2.429338	0.010506	1.003313	1.000000	3.3129796484E-03
3	2.114568	0.009145	1.005655	1.000001	5.6547611457E-03
4	1.840582	0.007960	1.008486	1.000001	8.4848242204E-03
5	1.602096	0.006928	1.011741	1.000001	1.1739892165E-02

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