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LACUNARY INTERPOLATION BY CUBIC SPLINE

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Dedicated to Professor V. Ureche on his 60^{th} anniversary

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REZUMAT. - Interpolare lacunară prin funcţii spline cubice. În această notă se studiază o problemă de interpolare prin funcţii spline cubice. Se dă o metodă de construire a soluţiei şi se evaluează eroarea aproximării.

As a generalization of polynomial Birkhoff interpolation, I.J.Schoenberg [22] had initiated the studying of lacunary interpolation by spline functions. Next, involving the values of a given function and of certain of its derivatives it was studied different particular cases of lacunary spline interpolation problems.

The goal of this note is to study such a lacunary spline problem and to give a method to construct corresponding solution. Also the approximation error is evaluated.

Let $S_p(3.\Delta_n)$ be the set of the cubic splines for the partition

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$$\Delta_n: a = x_0 < x_1 < \ldots < x_n = b,\ n \in \mathbf{N},\ n > 1, \tag{1}$$

i.e.:

$$\begin{cases} 1)\ s|_{[x_{i-1}, x_i]} \in \mathrm{P}_3, \\ & i = \overline{1,n} \\ 2)\ s \in C^2[a,b] \end{cases} \tag{2}$$

For a spline $s \in S_p(3,\Delta_n)$ we can write

$$s''(x) = M_{i-1} + \frac{M_i - M_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}) \text{ for all } x \in [x_{i-1}, x_i],\ i = \overline{1,n} \tag{3}$$

where $M_k = s''(x_k)$, $k = \overline{0,n}$.

For $f \in C[a,b]$ one considers the conditions

$$\begin{cases} s(x_i) = f(x_i) \\ & i = \overline{0,n}. \\ s'(x_i) = m_i \end{cases} \tag{4}$$

As a solution of the differential problem (3)-(4), one obtains [24]

$$s(x) = \frac{M_i - M_{i-1}}{6(x_i - x_{i-1})}(x - x_{i-1})^3 + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + m_{i-1}(x - x_{i-1}) + f(x_{i-1}) \tag{5}$$

for $x \in [x_{i-1}, x_i]$, $i = \overline{1,n}$.

In some supplementary conditions [24] the solution s of the above differential problem can be unique determined.

Now, one considers the following lacunary spline interpolation problem: find the cubic spline $s \in S_p(3,\Delta_n)$ that interpolates the data

$$\begin{cases} f(x_0), \ldots, f(x_n) \\ f''(t_1), \ldots, f''(t_n),\quad t_i \in (x_{i-1}, x_i),\ i = \overline{1,n} \\ f''(x_0) \end{cases} \tag{6}$$

Without lost of generality, it can be considered the interval [0,1] and its uniform partition Δ_n given by the nodes

$$x_i = \frac{i}{n}, \quad i = \overline{0,n}$$

with the norm $h = 1/n$.

The points t_i can be written as

$$t_i = x_{i-1} + \alpha h, \text{ with } \alpha \in (0,1), \quad i = \overline{1,n} \tag{7}$$

THEOREM 1. *Let f: [0,1] → **R** be a given function for which there exist* $f''(0)$ *and* $f''(t_i)$, $t_i \in (x_{i-1}, x_i)$ *for all* $i = \overline{1,n}$. *Then, there exists a unique cubic spline* $s \in S_p(3,\Delta_n)$ *such that*

$$\begin{aligned} & s\left(\frac{i}{n}\right) = f\left(\frac{i}{n}\right), \quad i = \overline{0,n} \\ & s''(0) = f''(0) \\ & s''(t_i) = f''(t_i), \quad i = \overline{1,n} \end{aligned} \tag{8}$$

and

$$s(x) = \frac{n}{6}(\lambda_i - \lambda_{i-1})\left(x - \frac{i-1}{n}\right)^3 + \frac{\lambda_{i-1}}{2}\left(x - \frac{i-1}{n}\right)^2 + \left[n\left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right) - \frac{\lambda_i + 2\lambda_{i-1}}{6n}\right]\left(x - \frac{i-1}{n}\right) + f\left(\frac{i-1}{n}\right), \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right), \quad i = \overline{1,n} \tag{9}$$

with λ_i *some parameters.*

Proof. We look for the function s as a solution of the differential problem

$$s''(x) = \frac{\lambda_i - \lambda_{i-1}}{h}\left(x - \frac{i-1}{n}\right) + \lambda_{i-1}, \quad x \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \tag{10}$$

$$s\left(\frac{i}{n}\right)=f\left(\frac{i}{n}\right),\quad i=\overline{1,n}\,. \tag{11}$$

We have

$$s(x)=\frac{\lambda_i-\lambda_{i-1}}{6h}\left(x-\frac{i-1}{n}\right)^3+\frac{\lambda_{i-1}}{2}\left(x-\frac{i-1}{n}\right)^2+C_1\left(x-\frac{i-1}{n}\right)+C_2,$$

$x\in\left(\frac{i-1}{n},\frac{i}{n}\right)$, $i=\overline{1,n}$, where the constants C_1 and C_2 are obtained from the conditions (11). One obtains

$$C_2=f\left(\frac{i-1}{n}\right)$$

$$C_1=n\left[f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)\right]-\frac{\lambda_i+2\lambda_{i-1}}{6h},\ i=\overline{1,n}$$

and (9) follows.

To find the parameters λ_i, $i=\overline{0,n}$, there are used the interpolation conditions (8).

One obtains the follow ng system

$$\begin{cases}\lambda_0=f''(0)\\ \lambda_{i-1}+n(\lambda_i-\lambda_{i-1})\left(t_i-\frac{i-1}{n}\right)=f''(t_i),\ i=\overline{1,n}\end{cases} \tag{12}$$

Using the relation

$$t_i=\frac{i-1}{n}+\alpha\frac{1}{n},\ \alpha\in(0,1)$$

the system (12) becomes

$$\begin{cases}\lambda_0=f''(0)\\ \alpha\lambda_i+(1-\alpha)\lambda_{i-1}=f''(t_i),\ i=\overline{1,n},\ \alpha\in(0,1)\end{cases} \tag{13}$$

This system has a unique solution, so the theorem is completely proved.

THEOREM 2. *If* $f \in C^2[0,1]$ *and* $f \in \mathrm{Lip}_L[0,1]$, *with* L *the Lipschitz constant, then for the corresponding cubic spline interpolation* s, *we have:*

$$\text{i)}\ |s(x)-f(x)| \le \frac{2L}{n} + \frac{1}{6n^2}\left[|\lambda_i-\lambda_{i-1}| + |\lambda_i+2\lambda_{i-1}| + 3|\lambda_{i-1}|\right]$$

$$\text{ii)}\ |s'(x)-f'(x)| \le L + \frac{1}{6n}\left[3|\lambda_i-\lambda_{i-1}| + |\lambda_i+2\lambda_{i-1}| + 6|\lambda_{i-1}|\right] + + \omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right| \tag{14}$$

$$\text{iii)}\ |s''(x)-f''(x)| \le |\lambda_i-\lambda_{i-1}| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|,$$

where $\omega_j\left(\frac{1}{n}\right)$ *is the modulo of continuity of* $f^{(j)}$, $j = 1,2$ *on* $\left[\frac{i-1}{n},\frac{i}{n}\right]$, $i=\overline{1,n}$.

Proof. Taking into account that

$$\left|x-\frac{i-1}{n}\right| \le \left|\frac{i}{n}-\frac{i-1}{n}\right| = \frac{1}{n},\quad x\in\left[\frac{i-1}{n},\frac{i}{n}\right],\quad i=\overline{1,n},$$

$$|f(x)-f(y)| \le L|x-y|,$$

and (9), one obtains:

$$\text{i)}\ |s(x)-f(x)| \le \frac{n}{6}|\lambda_i-\lambda_{i-1}|\cdot\frac{1}{n^3} + \frac{|\lambda_{i-1}|}{2}\cdot\frac{1}{n^2} + nL|x_i-x_{i-1}|\cdot\frac{1}{n} + + \frac{|\lambda_i+2\lambda_{i-1}|}{6n}\cdot\frac{1}{n} + L|x_{i-1}-x| \le$$

$$\le \frac{|\lambda_i-\lambda_{i-1}|}{6n^2} + \frac{|\lambda_{i-1}|}{2n^2} + \frac{|\lambda_i+2\lambda_{i-1}|}{6n^2} + \frac{2L}{n}$$

$$\text{ii)}\ |s'(x)-f'(x)| \le \frac{n}{2}|\lambda_i-\lambda_{i-1}|\cdot\frac{1}{n^2} + |\lambda_{i-1}|\cdot\frac{1}{n} + L + \frac{|\lambda_i+2\lambda_{i-1}|}{6n} + + |f'(x)| \le \frac{1}{2n}|\lambda_i-\lambda_{i-1}| + \frac{1}{6n}|\lambda_i+2\lambda_{i-1}| + + \frac{1}{n}|\lambda_{i-1}| + L + \left|f'(x)-f'\left(\frac{i-1}{n}\right)\right| + \left|f'\left(\frac{i-1}{n}\right)\right| \le$$

$$\leq L + \omega_1\left(\frac{1}{n}\right) + \frac{1}{6n}\left[3\,|\lambda_i - \lambda_{i-1}| + |\lambda_i + 2\lambda_{i-1}| + 6\,|\lambda_{i-1}|\right] + $$
$$+\left|f'\left(\frac{i-1}{n}\right)\right|$$

and

$$\text{iii)}\ \ |s''(x) - f''(x)| \leq n\,|\lambda_i - \lambda_{i-1}|\left|x - \frac{i-1}{n}\right| + |\lambda_{i-1}| + \left|f''(x) - f''\left(\frac{i-1}{n}\right)\right| +$$
$$+\left|f''\left(\frac{i-1}{n}\right)\right| \leq |\lambda_i - \lambda_{i-1}| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

and the theorem is completely proved.

Remark 1. If $\alpha = 1/2\left(t_i = \dfrac{x_{i-1} + x_i}{2}\right)$ from (13) it follows that $\lambda_i = 2f''(t_i) - \lambda_{i-1}$ and the inequalities (14) becomes:

$$\text{i)}\ \ |s(x) - f(x)| \leq \frac{2L}{n} + \frac{1}{6n^2}\left[\left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + \left|2f''\left(\frac{2i-1}{2n}\right) + \lambda_{i-1}\right| + \right.$$
$$\left. + 3\,|\lambda_{i-1}|\right]$$

$$\text{ii)}\ \ |s'(x) - f'(x)| \leq L + \frac{1}{6n^2}\left[3\left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + \right.$$
$$\left. + \left|2f''\left(\frac{2i-1}{2n}\right) + \lambda_{i-1}\right| + 6\,|\lambda_{i-1}|\right] + \omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right| \qquad (14')$$

$$\text{iii)}\ \ |s''(x) - f''(x)| \leq \left|2f''\left(\frac{2i-1}{2n}\right) - 2\lambda_{i-1}\right| + |\lambda_{i-1}| + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

Remark 2. If $\lambda_i \geq \lambda_{i-1} \geq 0,\ i = \overline{1,n}$ then

$$\text{i)}\ \ |s(x) - f(x)| \leq \frac{2L}{n} + \frac{1}{3n^2}(\lambda_i + 2\lambda_{i-1})$$

$$\text{ii) } |s'(x)-f'(x)| \le L + \frac{1}{6n}(4\lambda_i+5\lambda_{i-1}) + \omega_1\left(\frac{1}{n}\right) + \left|f'\left(\frac{i-1}{n}\right)\right| \tag{14"}$$

$$\text{iii) } |s''(x)-f''(x)| \le \lambda_i + \omega_2\left(\frac{1}{n}\right) + \left|f''\left(\frac{i-1}{n}\right)\right|$$

for all $x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $i=\overline{1,n}$.

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