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PARABOLIC SYSTEMS WITH DISCONTINUOUS NONLINEARITY

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Dedicated to Professor V. Ureche on his 60th anniversary

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REZUMAT. - Sisteme parabolice cu nonlinearitate discontinuă. Se studiază rezolvabilitatea problemei Cauchy-Dirichlet pentru sisteme parabolice cu neliniaritate discontinuă.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipshitz boundary $\partial\Omega$. We consider the Cauchy-Dirichlet problem

$$\frac{\partial u_i}{\partial t} - L_i u = f_i(\cdot, u) \quad \text{in} \quad D_T = \Omega \times (0,T), \quad i = 1,\dots,N \tag{1}$$

$$u_i(x;t)|_{x\in\partial\Omega} = 0, \quad u_i(x,0) = \varphi_i(x) \quad x \in \Omega, \quad i = 1,\dots,N \tag{2}$$

where L_i are second order linear differential operators with real coefficients of the form

$$L_i(u) = \sum_{j=1}^{N} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} \left[a_{kl}^{ij} \frac{\partial u_j}{\partial x_l} \right] - \sum_{j=1}^{N} a_0^{ij} u_j \quad i = 1,\dots,N \tag{3}$$

and $f_i : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}$ are given functions.

Let

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$$L^2(\Omega, \mathbb{R}^N) = \{u = (u_1,...,u_N) \mid u_i \in L^2(\Omega) \quad i = 1,...,N\}$$

with the scalar product resp. norm

$$(u,v)_{L^2(\Omega,\mathbb{R}^N)} = \int_\Omega \sum_{i=1}^n u_i v_i \, dx, \quad \|u\|^2_{L^2(\Omega,\mathbb{R}^N)} = \int_\Omega \sum_{i=1}^n |u_i|^2 \, dx; \tag{4}$$

$$H_0^1(\Omega, \mathbb{R}^N) = \left\{ u \in L^2(\Omega, \mathbb{R}^N) \,\middle|\, \frac{\partial u_i}{\partial x_k} \in L^2(\Omega), u_i|_{\partial\Omega} = 0 \quad i = 1,...,N, \quad k = 1,...,n \right\}$$

with the scalar product resp. norm

$$(u,v)_{H_0^1(\Omega,\mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \frac{\partial u_i}{\partial x_k} \frac{\partial v_i}{\partial x_k} \, dx, \quad \|u\|^2_{H_0^1(\Omega,\mathbb{R}^N)} = \int_\Omega \sum_{i=1}^N \sum_{k=1}^n \left|\frac{\partial u_i}{\partial x_k}\right|^2 dx \tag{5}$$

and $H^{-1}(\Omega, \mathbb{R}^N)$ the dual space of $H_0^1(\Omega, \mathbb{R}^N)$.

Besides these we need some other spaces too.

Let X a Banach or Hilbert space. We denote by $C([0,T], X)$ the linear space of the continuous functions $u : [0,T] \to X$ with the norm

$$\|u\|_{C([0,T],X)} = \sup_{t \in [0,T]} \|u(t)\|_X$$

Analogously if X is a Hilbert space let $L^2(0,T; X)$ the set of the measurable functions $u : (0,T) \to X$ for which $\int_0^T \|u(t)\|_X^2 dt < +\infty$. In $L^2(0,T;X)$ we use the scalar product

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X \, dt. \tag{6}$$

If X' is the dual space of X we can similarly define the spaces $C[[0,T], X']$, $L^2(0,T; X')$.

We shall use the following notations:

$$\begin{aligned} V &= L^2(0,T;L^2(\Omega,\mathbf{R}^N)) = L^2(D_T,\mathbf{R}^N), \quad & W &= L^2(0,T;H_0^1(\Omega,\mathbf{R}^N)) \\ W' &= L^2(0,T;H^{-1}(\Omega,\mathbf{R}^N)), & Z &= C([0,T), L^2(\Omega,\mathbf{R}^N)) \end{aligned} \tag{7}$$

If the system $Lu = (L_1u,\dots,L_Nu)$ is elliptic and weakly closed [2] for all $t \in (0,T)$ or is strongly elliptic [13], the coefficients a_0^{ij} satisfy some "sign" conditions, then for all $f_i \in L^2(D_T)$ (here f_i does not depend on u) and for all $\varphi_i \in L^2(\Omega)$ there exists a unique weak solution $u \in W \cap Z$ of the problem (1) - (2) and an estimate of the form

$$\|u\|_W \le C\left(\|f\|_V + \|\varphi\|_{L^2(\Omega,\mathbf{R}^N)}\right) \tag{8}$$

is true.

For Cauchy-Dirichlet problem see [1, 6, 8, 14].

If the functions f_i depend on u and satisfy Caratheodory type conditions, then many existence results were obtained using various methods for nonlinear operatos [6, 8, 10].

In the tehnical applications appear various problems for parabolic systems with initial and boundary condition which contain discontinuous nonlinearities. In the study of these problems usually the inclusions differentials are applied. We use here a simple method proposed by S. Carl [4].

In this paper we study the solvability of the Cauchy-Dirichlet problem (1)-

(2) in the case when f_i does not depend explicitly on x and t and f_i has discontinuities in $u_1,\ldots,u_N$. We assume in the sequel that a_{kl}^{ij}, $a_0^{ij} \in L^\infty(D_T)$ and we build the bilinear forms $a_i : W \times W \to \mathbf{R}$

$$a_i(u,v_i) = \int_{D_T} \sum_{j=1}^{N} \left[\sum_{k,l=1}^{n} a_{kl}^{ij} \frac{\partial u_j}{\partial x_k} \frac{\partial v_i}{\partial x_l} + a_0^{ij} u_j v_i \right] dx\,dt \tag{9}$$

DEFINITION 1. We say that $u \in W$ is a weak solution of (1) - (2) if $u \in Z$, $\frac{\partial u}{\partial t} \in W'$, $\left\langle \frac{\partial u}{\partial t}, v \right\rangle \in L^1(0,T)$, $f_i(u) \in L^2(D_T)$ and

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(u,v_i) = (f(u),v)_{L^2(D_T,\mathbf{R}^N)} \quad \forall v \in W \tag{10}$$

$$u(x,0) = \varphi(x) \qquad \text{a.e. on } \Omega \tag{11}$$

Here $\left\langle \frac{\partial u}{\partial t}, v \right\rangle$ stays for the pairing of the functional $\frac{\partial u}{\partial t}(t) \in H^{-1}(\Omega,\mathbf{R}^N)$ with $v(\cdot,t) \in H_0^1(\Omega,\mathbf{R}^N)$.

We introduce in $L^2(D_T, \mathbf{R}^N)$ a partially ordering relation. One says that $u \le v$ if and only if $v - u \in L_+^2(D_T,\mathbf{R}^N) = \{w \in L^2(0,T;\mathbf{R}^N) \mid w_i(x) \ge 0 \text{ a.e. on } D_T\}$. Let $W_+ = W \cap L_+^2(D_T,\mathbf{R}^N)$. If $\underline{u}, \overline{u} \in L^2(D_T,\mathbf{R}^N)$ and $\underline{u} \le \overline{u}$, we denote

$$[\underline{u},\overline{u}] = \{u \in L^2(D_T,\mathbf{R}^N) \mid \underline{u} \le u \le \overline{u}\}.$$

DEFINITION 2. We call $u \in W$ a weak upper solution of (1) - (2) if in definition 1 condition (10) is changed into

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(u,v_i) \ge (f,v)_{L^2(D_T,\mathbf{R}^N)} \quad \forall v \in W_+ \tag{12}$$

Similarly we define the weak lower solutions changing the sign "$\ge$" in

(12) into "$\leq$".

We assume that

α1) The system $(L_1 u, \dots, L_N u)$ is strongly elliptic or weakly closed

α2) There exists a positive constant M_1 such that for all $M \geq M_1$ the Cauchy-Dirichlet problem

$$\frac{\partial u}{\partial t} - Lu + Mu = g \quad \text{in } D_T, \quad u(x,t)\,|_{x \in \partial\Omega} = 0, \quad u(x,0) = \varphi(x) \tag{13}$$

has a unique weak solution u for all $g \in L^2(D_T, \mathbf{R}^N)$ and $\varphi \in L^2(\Omega, \mathbf{R}^N)$. For the parabolic operator $\frac{\partial}{\partial t} - L + MI$ the weak maximum and minimum principle are true in the sense that: $u \in W$, $u(x,0) = 0$ on Ω and

$$A_M(u,v) := \int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(u, v_i) + M \int_{D_T} \sum_{i=1}^N u_i v_i \, dx\, dt \geq 0 \quad \forall v \in W_+, \tag{14}$$

implies $u(x,t) \geq 0$ a.e. on D_T; and from $u \in W$, $u(x,0) = 0$ on Ω, and from $A_M(u,v) \leq 0$ $\forall v \in W_+$ results that $u(x,t) \leq 0$ a.e. on D_T.

Conditions α1 and α2 are obviously fulfilled if $L_i u$ contains only the function u_i

$$L_i u = \sum_{k,l=1}^n \frac{\partial}{\partial x_k} \left[a_{kl}^i \frac{\partial u_i}{\partial x_l} \right] - a_0^i u_i \qquad i = 1, \dots, N \tag{15}$$

and there exists $\mu > 0$ such that

$$\sum_{k,l=1}^n a_{kl}^i(x,t) \xi_k \xi_l \geq \mu \sum_{k=1}^n \xi_k^2 \ \text{for a.e. } (x,t) \in D_T, \ \forall \xi \in \mathbf{R}^n, \ i = 1, \dots, N. \tag{16}$$

For the maximum and minimum principles see [3, 5, 7, 12].

β1) There exists a positive constant M_2 such that the functions

$$F_i(\tau) = f_i(\tau) + M\tau_i \quad \tau \in \mathbb{R}^N \quad i = 1,\dots,N \tag{17}$$

are monotone increasing for every $M \geq M_2$, e.g.

$$F_i(\tau^1) \leq F_i(\tau^2) \text{ if } \tau_j^1 \leq \tau_j^2 \; j = 1,\dots,N$$

β2) There exist a finite or countable number of surfaces $S_k \subset \mathbb{R}^N$ for which we have a representation

$$S_k = \{\tau = (\tau_1,\dots,\tau_N) \in \mathbb{R}^N \mid \tau_N = \psi_{Nk}(\tau'), \quad \tau' = (\tau_1,\dots,\tau_{N-1}) \in \mathbb{R}^{N-1}\}, \tag{18}$$

where $\psi_{Nk} \in C^1(\mathbb{R}^{N-1})$ and

$$\psi_{N,k}(\tau') > \psi_{N,k-1}(\tau') \quad \forall \tau' \in \mathbb{R}^{N-1}, \quad \forall k$$

The functions $f_i : \mathbb{R}^N \to \mathbb{R}$ are continuous on $\mathbb{R}^N \setminus \bigcup_k S_k$, f_i has one-side limits on S_k e.g.

$$f^-(\tau) = \lim_{\substack{\xi \to \tau \\ \xi_N < \tau_N}} f(\xi_1,\dots,\xi_N), \quad f^+(\tau) = \lim_{\substack{\xi \to \tau \\ \xi_N > \tau_N}} f(\xi_1,\dots,\xi_N)$$

exist and are finite.

γ) The Cauchy-Dirichlet problem (1) - (2) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ such that $\underline{u} \leq \overline{u}$.

LEMMA 1. *We assume that the conditions* β1), β2) *and* γ) *are fulfilled and* $M \geq \max\{M_1, M_2\}$ *is a constant. Then*

1° *For every* $u \in [\underline{u}, \overline{u}]$ *the function* $F(u) = f(u) + Mu$ *belongs to* $L^2(D_T, \mathbb{R}^N)$.

2° If $u, v \in [\underline{u}, \overline{u}]$ and $u \leq v$ then $F(u) \leq F(v)$.

3° *The set* $\{F(u) \mid u \in [\underline{u}, \overline{u}]\}$ *is bounded in* $L^2(D_T, \mathbb{R}^N)$.

For the proof see [11].

Let $M_0 = \max\{M_1, M_2\}$, $M \geq M_0$ a constant, $\varphi \in L^2(\Omega, \mathbb{R}^N)$ a fixed element and $w \in [\underline{u}, \overline{u}]$ an arbitrary function.

THEOREM 1. *If the hypotheses* α1, α2, β1, β2 *and* γ *are satisfied, then the Cauchy-Dirichlet problem*

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu + Mu = f(w) + Mw \quad \text{on } D_T \\ &u(x,t)\big|_{x\in\partial\Omega} = 0, \quad u(x,0) = \varphi(x) \end{aligned} \tag{19}$$

has a unique weak solution $u \in [\underline{u}, \overline{u}]$. If $\varphi \in H_0^1(\Omega, \mathbb{R}^N)$ *then* $u \in W \cap L^2(0, T, H^2(\Omega, \mathbb{R}^N))$ *and* $\frac{\partial u}{\partial t} \in V$.

Proof. By Lemma 1 $F(w) = f(w) + Mw \in L^2(D_T, \mathbb{R}^N)$. In this case the unique solvability of (19) results from α1) and α2). Let u the weak solution of (19). The function $\overline{u}$ is a weak upper solution of (1) - (2) with the same $\varphi \in L^2(\Omega, \mathbb{R}^N)$, thus we have

$$A_M(u, v) = \int_{D_T} \sum_{i=1}^{N} F_i(w)\, v_i\, dx\, dt \qquad \forall v \in W$$

$$A_M(\overline{u}, v) \geq \int_{D_T} \sum_{i=1}^{N} F_i(\overline{u})\, v_i\, dx\, dt \qquad \forall v \in W_+$$

$$u(x, o) = \overline{u}(x, 0) = \varphi(x) \qquad \text{a.e. on } \Omega$$

For the function $\overline{u} - u$ we obtain then

$$A_M(\overline{u} - u, v) \geq \int_{D_T} [F_i(\overline{u}) - F_i(w)]\, v_i\, dx\, dt \geq 0 \qquad \forall v \in W_+$$

and $(\overline{u} - u)(x,0) = 0$.

Applying the maximum principles the last two formulae give $\overline{u} - u \geq 0$ a.e. on D_T. Simillarly we obtain $\underline{u} - u \leq 0$, and then $\underline{u} \leq u \leq \overline{u}$.

If $\varphi \in H_0^1(\Omega, \mathbf{R}^N)$ then $u \in L^2(0, T, H^2(\Omega, \mathbf{R}^N))$ and $\dfrac{\partial u}{\partial t} \in V$ (see [1]).

Let $M_3 > M_0$. We consider the family of the cauchy-Dirichlet problem (19) when w describes the interval $[\underline{u}, \overline{u}]$, $M \in [M_0, M_3]$ and φ is the same function for all problems. We denote by u_{wM} the weak solution of (19).

THEOREM 2. *There exist positive constants* C_2 and C_3 *such that*

$$\|u_{wM}\|_W \leq C_2 \qquad \forall w \in [\underline{u}, \overline{u}], \quad \forall M \in [M_0, M_3] \tag{20}$$

$$\left\| \frac{\partial u_{wM}}{\partial t} \right\|_{W'} \leq C_3 \qquad \forall w \in [\underline{u}, \overline{u}], \quad \forall M \in [M_0, M_3] \tag{21}$$

Proof. $F(w) = f(w) + Mw \in V$ so from conditions $\alpha 1$) and $\alpha 2$) results that there exists a constant $C > 0$ such that for the solutions u_{wM} of the problem (19) we have

$$\|u_{wM}\|_W \leq C(\|F(w)\|_V + \|\varphi\|_{L^2(\Omega, \mathbf{R}^N)}) \qquad \forall w \in [\underline{u}, \overline{u}] \tag{22}$$

According to Lemma 1 $\{ \|F(w)\|_V \mid w \in [\underline{u}, \overline{u}] \}$ is bounded, φ is the same for all M, therefore there exists $C_2 > 0$ such that (20) is true. The estimate (21) is a consequence of (20) and

$$\int_0^T \left\langle \frac{\partial u_{wM}}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(u_{wM}, v_i) + M(u_{wM}, v)_V = \int_{D_T} \sum_{i=1}^N F_i(w)\, v_i \, dx\, dt \quad \forall v \in W$$

LEMMA 2. *Let* $u^1, u^2, \dots, u^k, \dots$ *a bounded monotone sequence (increasing or decreasing) in* W *for which* $\{ \| \frac{\partial u_k}{\partial t} \| \mid k = 1,2,\dots\}$ *is also bounded. Then* $(u^k)_{k\geq 1}$ *is weakly convergent in* W, *strongly convergent in* $L^2(D_T, \mathbf{R}^N)$ *and* $\int_0^T \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle dt \to \int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt \quad \forall v \in W \ (u = \lim u^k)$.

Proof. The monotonicity of $(u^k)_{k\geq 1}$ means here the monotonicity of the components of $u^k = (u_1^k, \dots, u_N^k)$. Then Lemma 2 results from [4].

THEOREM 3. *Let* $\underline{u}, \overline{u} \in W$ *be one lower resp. upper solution of the Cauchy-Dirichlet problem* (1) - (2). *Assume that the conditions* α1), α2), β1), β2) *and* γ) *are fulfilled and* $f^+(\tau) = f(\tau)$ (or $f^-(\tau) = f(\tau)$) *for every* $t \in \bigcup_k S_k$. *Then there exists at least one weak solution* $u \in [\underline{u}, \overline{u}]$ *of problem* (1) - (2).

Proof. We use a constructive iterative method proposed by S. Carl [4] solving an infinite sequence of problems. Let $\varphi \in L^2(\Omega, \mathbf{R}^N)$ be the given function in (2). We chose an $M \in [M_0, M_3]$ and start with the problem

$$\begin{gathered} \frac{\partial U^1}{\partial t} - L U^1 + M U^1 = f(U^0) + M U^0 \quad \text{in } \Omega \times (0, T) \\ U^1(x,t)|_{x \in \partial\Omega} = 0, \quad U^1(x, 0) = \varphi(x) \end{gathered} \tag{23}$$

where $U^0 = \overline{u}$.

(23) has a unique weak solution U^1. Thus we have

$$A_M(U^0, v) \geq \int_{D_T} \sum_{i=1}^{N} f_i(U^0)\, v_i\, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i^0\, v_i\, dx\, dt \qquad \forall v \in W_+$$

$$A_M(U^1, v) = \int_{D_T} \sum_{i=1}^{N} f_i(U^0)\, v_i\, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i^0\, v_i\, dx\, dt \qquad \forall v \in W$$

The last two formulae give

$$A_M(U^0 - U^1, v_i) \geq 0 \quad \forall v \in W_+$$

In the same way we get

$$A_M(\underline{u} - U^1, v) \leq 0 \quad \forall v \in W_+$$

Using the maximum resp. minimum principle we obtain

$$\underline{u} \leq U^1 \leq U^0 = \overline{u}\ .$$

In the same manner the sequence $U^1, U^2, \ldots, U^k, \ldots$ is built solving the Cauchy-Dirichlet problems

$$\begin{cases} \dfrac{\partial U^{k+1}}{\partial t} - L\, U^{k+1} + M U^{k+1} = f(U^k) + M U^k \\ U^{k+1}(x, t)\,\big|_{x \in \partial\Omega} = 0, \quad U^{k+1}(x, 0) = \varphi(x) \quad x \in \Omega \end{cases} \tag{24}$$

It is obvious that

$$\underline{u} \leq U^{k+1} \leq U^k \leq \ldots \leq U^1 \leq U^0 = \overline{u}\ .$$

By Theorem 2 the sequence $(u^k)_{k\geq 1}$ is bounded in W, and $\left\| \dfrac{\partial U^k}{\partial t} \right\|_{W'} \leq C_3$. Then from Lemma 2 results that $(u^k)_{k\geq 1}$ is strongly convergent in V and weakly convergent in W. U^k is the weak solution of (24), thus

$$\int_0^T \left\langle \frac{\partial U^{k+1}}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(U^{k+1}, v_i) + M \int_{D_T} \sum_{i=1}^{N} U_i^{k+1} v_i \, dx\, dt$$
$$= \int_{D_T} \sum_{i=1}^{N} f_i(U^k) v_i \, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i^k v_i \, dx\, dt \tag{25}$$

But according to Lemma 2 U_k converges strongly to an $U \in L^2(D_T, \mathbf{R}^N)$, $U^k \to U$ weakly in W, $\frac{\partial u}{\partial t} \in W'$ and $\int_0^T \left\langle \frac{\partial U^k}{\partial t}, v \right\rangle dt \to \int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt$.

Consequently after passing to limit the left side of (25) is

$$\int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt + \sum_{i=1}^{N} a_i(U, v_i) + M \int_{D_T} \sum_{i=1}^{n} U_i v_i \, dx\, dt \tag{26}$$

We shall show that the limit of the right side of (25) exists and is equal to

$$\int_{D_T} \sum_{i=1}^{N} f_i(U) v_i \, dx\, dt + M \int_{D_T} \sum_{i=1}^{N} U_i v_i \, dt$$

$U^k(x,t)$ converges decreasing to $U(x,t)$ a.e. on D_T, f_i is continuous on $\mathbf{R}^N \setminus \bigcup_j S_j$, $f_i(\tau) = f_i^+(\tau)$ on S_j, thus $f_i(U^k(x,t)) \to f_i(U(x,t))$ a.e. on D_T and from Theorem 2 results that

$$\int_{D_T} f_i(U^k(x,t)) \cdot v_i(x,t) \, dx\, dt \mid \leq \| f_i(U^k) \|_{L^2(D_T)} \cdot \| v_i \|_{L^2(D_T)} \leq C$$

where C is a conveniently chosen constant. Thus we can pass to limit in the right side of (25), too, and we obtain

$$\int_0^T \left\langle \frac{\partial U}{\partial t}, v \right\rangle dt + \sum_{i=1}^N a_i(U, v_i) = \int_{D_T} \sum_{i=1}^N f_i(U)\, v_i\, dx\, dt \qquad \forall v \in W$$

which means that U is a weak solution of problem (1) - (2).

If $f_j(\tau) = f_j^-(\tau)$ on S_k then starting with $u^0 = \underline{u}$ (lower solution of (1) - (2)) we can build a convergent sequence $u^1, u^2, \dots, u^k, \dots$, u_k is the solution of

$$\begin{cases} \dfrac{\partial u^{k+1}}{\partial t} - L u^{k+1} + M u^{k+1} = f(u^k) + M u^k \quad \text{in } D_T \\ u^{k+1}(x,t)|_{x \in \partial\Omega} = 0, \quad u^{k+1}(x,0) = \varphi(x) \end{cases} \tag{27}$$

For the solution $u = \lim u^{k+1}$ we have then

$$\underline{u} \le u \le \overline{u}$$

REMARK 1. For both cases $f_i(\tau) = f_i^+(\tau)$ and $f_i(\tau) = f_i^-(\tau)$ we may start the iteration method with any $U^0, u^0 \in [\underline{u}, \overline{u}]$. The sequences built by the method (24) resp. (27) may converge to an element different from U resp. u obtained in Theorem 3. We have the following

THEOREM 4. a) *If in Theorem* 3 $f_i(\tau) = f_i^+(\tau)$ $\tau \in \bigcup_k S_k$, *then the solution* U *of the Cauchy-Dirichlet problem* (1) - (2) *obtained in the proof of Theorem* 3 *is maximal in the sense that for all solutions* $w \in [\underline{u}, \overline{u}]$ *of problem* (1) - (2) *we have* $w \le U$.

b) *If* $f_i(\tau) = f_i^-(\tau) \quad \tau \in \bigcup_k S_k$, *then the solution* u *obtained by algorithm* (27) *is minimal, that is* $u \leq w$ *for any solution* $w \in [\underline{u}, \overline{u}]$.

For the proof see [11].

REMARK 2. using differential inclusions we may weaken the assumptions about the operator L and functions f_i [9], but in this case we can not apply the simple constructive method offered by the monotone iterative technique.

REFERENCES

1. V. Barbu, *Probleme la limită pentru ecuații cu derivate parțiale.* Ed. Acad. Române 1993.
2. A. V. Bitsadse, *Kraevâe zadaci dlja elliptičeskix uravnenij vtorogo porjadka.* Nanka, Moskva, 1966
3. L. Byszewski, *Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with arbitrary functionals.* Journal of Mathematical analysis and applications. Vol. 156. Nr. 2, 1991,, 456-470.
4. S. Carl, *The monotone iterative technique for a parabolic boundary value problem with discontinuous nonlinearity.* Nonlinear Analysis, Theory, Methods & Applications Vol. 13. Nr. 12. 1399-1407.
5. R. K. Inberg, *Several observations concerning a maximum principle for parabolic systems.* University of California Irvine 1967
6. D. A. Ladâjenskaia, V. A. Solomnikov, N. N. Uraltzeva, *Lineinie i cvasilinienie uravnenia paraboliceskava tipa.* Nauka, Moskva, 1967
7. I. Kačur, *A generalized maximum principle and estimates of max vrai u for nonlinear parabolic boundary value problems.* Czechoslovak Mathematical Journal v. 26 (101), 1976, No. 4 507-526

8. N. V. Krylov, *Nonlinear elliptic and parabolic equations of the second order.* D. Reidel Publishing Company, Dordrecht, Holland, 1987

9. S. A. Marano, *Implicit elliptic differential equations.* Set-Valued Analysis 2 (1994), 545-558

10. T. Stays, *On the unique solvability of the firs Fourier problem for a parabolic system of linear differential equations of second order.* Prace Mat. 9. (1965) 283-289

11. P. Szilágyi, *Elliptic systems with discontinous nonlinearity.* Studia Univ. Babeş-Bolyai, Mathematica, XXXIX, 4, 1994, 11-20.

12. A. W. Tarski, *A modern proof of the maximum principle.* Annales Polonici Mathematici, LIII, 2 (1991) 147-152

13. M. I. Visik, *Strongly elliptic systems of differential equations.* Math. Sbornik, 1951, 29

14. W. von Wahl, *On parabolic systems with discontinuous nonlinearities.* Appl. Analysis 20 (1985) 82-102