STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XL, 4, 1995

PARTIAL DIFFERENTIAL SUBORDINATIONS FOR HOLOMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES

Gabriela KOHR* and Mirela KOHR-ILE*

Dedicated to Professor V. Ureche on his 60[th] anniversary

Received: January 20, 1995
AMS subject classification: 32H02, 30C45

REZUMAT. - Subordonări diferenţiale pentru funcţii olomorfe de mai multe variabile complexe. În această lucrare autorii consideră clase speciale de subordonări diferenţiale precum şi inegalităţi cuprinzând derivate parţiale de ordinul întâi pentru transformări olomorfe definite pe polidiscul unitate din $\mathbb{C}^n$.

Abstract. In this paper the authors consider special classes of subordinations and inequalities involving first partial derivatives of holomorphic mappings in the unit polidisc of $\mathbb{C}^n$.

1. Introduction. In several papers [6], [7], [8] S.S. Miller and P.T. Mocanu considered the analytic functions defined on the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$, which satisfy some differential inequalities or subordinations. Using the technique of subordination were obtained severeal

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

results including inclusion relations, inequalities and some sufficient conditions for univalence. K. Dobrowolska, P. Liczberski in [3] and also, S. Gong, S.S. Miller in [4] proved that if an analytic function of several complex variables defined on a complete circular domain satisfy certain partial differential inequalities or subordinations, then the function itself must satisfies an associated inequality or subordination. P. Liczberski in [5] obtained some results concerning partial differential inequalities for holomorphic mappings on the open unit Euclidian ball.

In this paper we obtain a new generalization of Jack-Miller-Mocanu Lemma and then, using this result we will obtain some properties of holomorphic mappings defined on the unit polydisc of $\mathbf{C}^n$.

2. Preliminaries. We let $\mathbf{C}^n$ denote the space of n complex variables $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$ with the norm $\|z\| = \max\limits_{1 \le j \le n} |z_j|$. By U_r^n and $H(U_r^n)$ we shall denote the open polydisc in $\mathbf{C}^n$ i.e. the set $\{z \in \mathbf{C}^n : \|z\| < r\}$, and the family of all holomorphic mappings f: $U_r^n \to \mathbf{C}^n$, respectively. If Ω is a region in $\mathbf{C}^n$ and f is a holomorphic mapping in Ω, then we denote by $Df(z)$ the Frechet derivative of f at $z \in \Omega$. Also, if F is a holomorphic function in Ω, then by

$DF(z)$ we denote the complex vector $\begin{pmatrix} \frac{\partial F(z)}{\partial z_1} \\ \dots \\ \frac{\partial F(z)}{\partial z_n} \end{pmatrix}$ for all $z \in \Omega$ and $D^2F(z)$ the complex matrix $\left[\frac{D^2F(z)}{\partial z_i \partial z_j}\right]_{1 \le i,j \le n}$. Let $L(\mathbf{C}^n)$ the space of all linear continuous operators from $\mathbf{C}^n$ into $\mathbf{C}^n$ with the standard operator norm $\|\cdot\|$ and let I be the identity in $L(\mathbf{C}^n)$, then the restriction $D^2f(z)(z,\cdot)$ of the continuous symmetric bilinear operator $D^2f(z)$ to $z \times \mathbf{C}^n$ belongs to $L(\mathbf{C}^n)$. If $z \in \mathbf{C}^n$, then z' will represent its transpose.

If Ω is a region in $\mathbf{C}^n$ and f is holomorphic in Ω, then we say that f is biholomorphic mapping in Ω if the inverse mapping f^{-1} does exist, is holomorphic on an open set $V \subseteq \mathbf{C}^n$ and $f^{-1}(V) = \Omega$.

The main results are based on the following lemmas.

LEMMA 2.1 [6,7]. *Let g be a holomorphic function in the unit disc U with $g(0) = 0$ and suppose that at $\zeta_0 \in U$ with $|\zeta_0| = r_0$, where $0 < r_0 < 1$, g satisfies the following condition*

$$|g(\zeta_0)| = \max\{|g(\zeta)|: |\zeta| \le |\zeta_0|\}$$

then there exists a real number $m \ge 1$ such that

$$\zeta_0 g'(\zeta_0) = mg(\zeta_0)$$

and

$$\operatorname{Re}\left[1 + \frac{\zeta_0 g''(\zeta_0)}{g'(\zeta_0)}\right] \geq m.$$

LEMMA 2.2 [3,4]. *Let g: $\bar{U} \to \mathbf{C}$ be a function which is holomorphic and univalent in $\bar{U}$ without at most one point $\zeta \in \partial U$, which is a simple pole. Let f be a holomorphic function in U_1^n with $f(0) = g(0)$. Suppose that $f(U_1^n) \not\subset g(U)$, then there exists $\zeta_0 \in \partial U$, $r_0 \in (0,1)$, $z_0 \in \bar{U}_{r_0}^n$ and a real number $m \geq 1$ such that $f(z_0) = g(\zeta_0)$ and $[Df(z_0)]'(z_0) = m\zeta_0 g'(\zeta_0)$.*

3. Main results.

Now we give the main result.

THEOREM 3.1. *Let $f \in H(U_1^n)$ with $f(0) = 0$ and $f(z) \neq 0$. Let r be a real number from the open interval $(0,1)$. If for $z_0 \in \bar{U}_r^n$ we have*

$$\|f(z_0)\| = \max\{\|f(z)\| : z \in \bar{U}_r^n\}, \tag{3.1}$$

then there exist the real numbers m, s such that $s \geq m \geq 1$ and the following relations are satisfied

(i) $$\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k \frac{[Df_k(z_0)]'(z_0)}{f_k(z_0)} = m;$$

(ii) $$\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k \operatorname{Re}\left\{\frac{(z_0')D^2 f_k(z_0)(z_0)}{f_k(z_0)}\right\} \geq m(m-1),$$

where $t_k \geq 0$ *for each* k *and* $\sum_{|f_k(z_0)| = \|f(z_0)\|} t_k = 1$;

(iii) $\|Df(z_0)(z_0)\| = s\|f(z_0)\|$.

Proof. Let us denote $b = f(z_0)$, then according the condition (3.1) we can assume that $b \neq 0$. Because $(\mathbf{C}^n, \|\cdot\|)$ is normed space, Hahn Banach Theorem guarantees that there exists Λ a continuous linear function from $\mathbf{C}^n$ into $\mathbf{C}$ such that $\Lambda(b) = \|b\|$ and $|\Lambda(u)| \leq \|u\|$, for all $u \in \mathbf{C}^n$. But, it is well known that Λ can be written under the form $\Lambda(z) = \sum_{|b_k| = \|b\|} t_k \frac{\|b\|}{b_k} z_k$, for all $z \in \mathbf{C}^n$, where $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix}$, where $t_k \geq 0$ for each k and $\sum_{|b_k| = \|b\|} t_k = 1$.

Now, if we consider the complex function g, defined in the unit disc U by the formula

$$g(\zeta) = \Lambda \circ f(\zeta z_0 \|z_0\|^{-1}),\ \zeta \in U,$$

then g satisfy $g(0) = 0$ and $|g(\zeta_0)| = \max\{g(\zeta)|: |\zeta| \leq |\zeta_0|\}$, where $\zeta_0 = \|z_0\|$. So, from Lemma 2.1 we conclude that there exists a real number m, with $m \geq 1$, such that the condition (i) and (ii) are satisfied. On the other hand, the first equality can be rewritten as follows

$$\Lambda(Df(z_0)(z_0)) = m\|f(z_0)\|,$$

and using the inequality $|\Lambda(u)| \leq \|u\|$, for each $u \in \mathbf{C}^n$, then we deduce that there exists a real number s, with $s \geq m \geq 1$ and such that the last equality is satisfied.

Let us consider M be a positive number and let D be a domain in $\mathbb{C}^{2n}$ such that $(0,0)\in D$, where $0 = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}$.

Let $M_n = \bigcup_{s\geq 1} M_n^s(M)$, where $M_n^s(M) = \{(u,v)\in\mathbb{C}^{2n}: \|u\| = M, \|v\| = sM\}$ and suppose that $M_n \subseteq D$. Also let $G_n(D,M) = \{h: D \to \mathbb{C}^n: h$ continuous in D, $\|h(0,0)\| < M$, $\|h(u,v)\| \geq M$, for all $(u,v) \in M_n\}$.

Using these classes and from the result of Theorem 3.1, we obtain the following result:

THEOREM 3.2. *Let* $D \subseteq \mathbb{C}^{2n}$ *be a domain, let* $f \in H(U_1^n)$ *with* $f(0) = 0$ *and* $f(z) \neq 0$, $z \in U_1^n$. *Suppose that there exists a mapping* $h \in G_n(D,M)$ *such that*

$$(f(z), Df(z)(z)) \in D$$

and

$$\|h(f(z), Df(z)(z))\| < M,$$

for all $z \in U_1^n$. *Then* $\|f(z)\| < M$, $z \in U_1^n$.

Proof. If we suppose that there exists $z_0 \in \bar{U}_r^n$, $r \in (0,1)$ with $\|f(z_0)\| = M = \max\{\|f(z)\|: z \in \bar{U}_r^n\}$, then using Theorem 3.1 we conclude that there exists a real number s, with $s \geq 1$ and such that

$\|Df(z_0)(z_0)\| = s\|f(z_0)\|$. Hence, if we denote by $u=f(z_0)$ and $v=Df(z_0)(z_0)$, then $(u, v) \in M_n^s(M)$ and because $h \in G_n(D, M)$, we deduce that $\|h(u, v)\| \geq M$, but this is a contradiction with the hypothesis. So, $\|f(z)\| < M$, for all $z \in U_1^n$.

Remark 3.1. It is interesting that this result can be used to show that some first order partial differential equations in $\mathbf{C}^n$ have bounded solution.

COROLLARY 3.1. *Let $F \in H(U_1^n)$ with $F(0) = 0$ and $\|F(z)\| < M$, for all $z \in U_1^n$. Let $h \in G_n(D, M)$ such that the differential equation*

$$h(f(z), Df(z)(z)) = F(z), f(0) = 0,$$

has a holomorphic solution f. Then $\|f(z)\| < M$, for all $z \in U_1^n$.

DEFINITION 3.1. Let f and g be holomorphic mappings on the unit polydisc U_1^n. We say that f is subordinate to g (written $f \prec g$ or $f(z) \prec g(z)$) if there exists $w \in H(U_1^n)$ with $w(0)=0$, $\|w(z)\| < 1$, for all $z \in U_1^n$ and $f = g \circ w$.

Remark 3.2. If f is subordinate to g, then $f(0) = g(0)$ and $f(U_1^n) \subseteq g(U_1^n)$. But, if g is biholomorphic in U_1^n, then easily we show that $f \prec g$ if and only if $f(0) = g(0)$ and $f(U_1^n) \subseteq g(U_1^n)$.

Now applying the result of Theorem 3.1, we obtain the following result.

THEOREM 3.3. *Let $f \in H(U_1^n)$, g be a biholomorphic in U_ρ^n, for some $\rho > 1$, with $f(0) = g(0)$. Suppose that f is not subordinate to g, then there exist* *an* $r_0 \in (0,1)$, *the points* $z_0 \in U_1^n$, $\|z_0\| \leq r_0$, $\zeta_0 \in \partial U_1^n$ *and* m, s *be real numbers such that* $s \geq m \geq 1$ *and at* $z = z_0$ *the following relations are satisfied*

(i) $$f(z_0) = g(\zeta_0),\ f(\bar{U}_{r_0}^n) \subseteq g(\bar{U}_1^n);$$

(ii) $$\sum_{|\zeta_0^k|=1} t_k \frac{[D\tilde{g}_k(w_0)]'\, Df(z_0)(z_0)}{\zeta_0^k} = m,$$

where $w_0 = g(\zeta_0)$, $g^{-1}(w_0) = \begin{pmatrix} \tilde{g}_1(w_0) \\ \dots \\ \tilde{g}_n(w_0) \end{pmatrix}$, $\zeta_0 = \begin{pmatrix} \zeta_0^1 \\ \dots \\ \zeta_0^n \end{pmatrix}$ *and* $t_k \geq 0$ *for each* k, $\sum_{|\zeta_0^k|=1} t_k = 1$;

(iii) $$s\|[Dg(\zeta_0)]^{-1}\|^{-1} \leq \|Df(z_0)(z_0)\| \leq s\|Dg(\zeta_0)\|.$$

Proof. Since f is not subordinate to g and $f(0)=g(0)$ then $f(U_1^n) \not\subset g(U_1^n)$, hence we can get $r_0 \in (0,1)$ and the points $z_0 \in U_1^n$, $z_0 \in \bar{U}_{r_0}^n$, $\zeta_0 \in \partial U_1^n$, with $f(z_0) = g(\zeta_0)$ and $f(\bar{U}_{r_0}^n) \subseteq g(\bar{U}_1^n)$. On the other hand, if we consider the mapping $h(z) = (g^{-1} \circ f)(z)$, $z \in U_{r_0}^n$, then $h \in H(\bar{U}_{r_0}^n)$, $h(0) = 0$ and $1 = \|h(z_0)\| = \max\{\|h(z)\| : z \in \bar{U}_{r_0}^n\}$. Using the result of Theorem 3.1 and the properties of the norm of a linear operator from $\mathbf{C}^n$, a straightforward calculation shows our result.

Now, we are able to define the concept of "admissible class" in the case of several variables. This concept is given in the following definition.

DEFINITION 3.2. Let D and Ω be domains from $\mathbf{C}^n$ and $\mathbf{C}^{2n}$, respectively.

Let g be a biholomorphic in U_ρ^n for some $\rho > 1$, $\zeta_0 \in \partial U_1^n$ and m, t_k positive numbers, for each k, with $m \geq 1$, and $\sum_{|\zeta_0^k|=1} t_k = 1$.

$$\text{Let us } H_n^m(g) = \left\{ (u, v) \in \mathbf{C}^{2n} : u = g(\zeta_0), \sum_{|\zeta_0^k|=1} t_k \frac{[D\tilde{g}_k(w_0)]' v}{\zeta_0^k} = m \right\},$$

where $w_0 = g(\zeta_0)$, $g^{-1}(w_0) = \begin{pmatrix} \tilde{g}_1(w_0) \\ \cdots \\ \tilde{g}_n(w_0) \end{pmatrix}$ and $\zeta_0 = \begin{pmatrix} \zeta_0^1 \\ \cdots \\ \zeta_0^n \end{pmatrix}$. Also, let

$H_n(g) = \bigcup_{\substack{m \geq 1 \\ \zeta_0 \in \partial U_1^n}} H_n^m(g)$, and suppose that $H_n(g) \subseteq \Omega$ and $(g(0), 0) \in \Omega$, respectively. The admissible class $\psi_n^n(D, \Omega, g)$ consist of those continuous mappings ψ_n: $\Omega \times U_1^n \to \mathbf{C}'$ which satisfy

$$\psi_n(g(0), 0; 0) \in D$$

and

$$\psi_n(u, v; z) \notin D,$$

for all $(u, v) \in H_n(g)$ and $z \in U_1^n$.

Using this definition and from Theorem 3.3 we obtain the following result.

THEOREM 3.4. *Let $f \in H(U_1^n)$, g be a biholomorphic in U_ρ^n for some $\rho > 1$ and $f(0) = g(0)$. Suppose that there exists $\psi_n \in \psi_n^n(D, \Omega, g)$ such that*

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \in D,$$

for all $z \in U_1^n$. Then f is subordinate to g.

Proof. If we suppose that f is not subordinate to g, then from Theorem 3.3, we can get points $z_0 \in U_1^n$, $\zeta_0 \in \partial U_1^n$, and the real numbers $t_k \geq 0$, for each k, $\sum_{|\zeta_0^k|=1} t_k = 1$, $m \geq 1$ such that at $z = z_0$ the conditions (i) and (ii) are satisfied. Let us $u = f(z_0)$ and $v = Df(z_0)(z_0)$, then it is clear that $(u, v) \in H_n^m(g) \subseteq H_n(g)$, hence using the Definition 3.2 we conclude that $\psi_n(u, v; z_0) \notin D$, but this is a contradiction with the hypothesis. So, f is subordinate to g.

Furthermore we suppose that D is a special domain in $\mathbf{C}^n$, such that there exists h a biholomorphic mapping in U_1^n, with $h(U_1^n) = D$. But, it is clear that this assertion is not true for all domains in $\mathbf{C}^n$.

We denote the class $\psi_n^n(D, \Omega, g)$ by $\psi_n^n(h, \Omega, g)$ and following the result of Theorem 3.4 we obtain:

COROLLARY 3.3. *Let Ω be a domain in $\mathbf{C}^{2n}$, let g,h biholomorphic mappings in U_ρ^n for some $\rho > 1$ and let $f \in H(U_1^n)$ with $f(0) = g(0)$. Suppose that there exists a holomorphic mapping $\psi_n \in \psi_n^n(h, \Omega, g)$ such that*

$\psi_n(g(0), 0; 0) = h(0)$.

If

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \prec h(z),\ z \in U_1^n, \tag{3.2}$$

then $f(z) \prec g(z),\ z \in U_1^n$.

Remark 3.3. The biholomorphic mapping g is said to be a dominant of the differential subordination (3.2) if $f(z) \prec g(z)$ for all $f(z)$ satisfying (3.2). If $\tilde{g}$ is a dominant of (3.2) and $\tilde{g}(z) \prec g(z)$ for all dominants g of (3.2), then $\tilde{g}$ is said to be the best dominant of (3.2).

The following result gives a sufficient condition that g to be the best dominant of the subordination (3.2).

THEOREM 3.5. *Let* Ω *be a domain in* $\mathbf{C}^{2n}$, *let* g, h *biholomorphic mappings in* U_ρ^n, *for some* $\rho > 1$ *and let* $f \in H(U_1^n)$ *such that* $f(0) = g(0)$. *Suppose that there exists a holomorphic mapping* $\psi_n \in \psi_n^n(h, \Omega, g)$ *such that* $\psi_n(g(0), 0; 0) = h(0)$ *and* g *is a solution of differential equation*

$$\psi_n(g(z), Dg(z)(z); z) = h(z),\ z \in U_1^n. \tag{3.3}$$

If

$$\psi_n(f(z), Df(z)(z); z) \prec h(z),$$

then $f(z) \prec g(z)$, $z \in U_1^n$ *and* g *is the best dominant.*

Proof. Using Corollary 3.3, we conclude that f is subordinate to g and because g is a solution of differential equation (3.3), then from Remark 3.3 easily we deduce that g is the best dominant of (3.2).

4. Examples. Finally we obtain some applications which point out the usefulness of the above results.

If M is a positive number, let g: $U_1^n \to \mathbf{C}^n$, given by $g(z) = Mz$, for all $z \in U_1^n$, then g is biholomorphic in U_1^n and is easy to show that in this case the class $H_n^m(g)$ consist of those $(u,v) \in \mathbf{C}^{2n}$, with $u = M\zeta_0$, $u_k = Me^{i\theta_k}$, for $|\zeta_0^k| = 1$, $\sum_{|\zeta_0^k|=1} t_k v_k e^{-i\theta_k} = mM$, where $\zeta_0 \in \partial U_1^n$, $t_k \geq 0$, $\theta_k \in \mathbf{R}$ for all k, $\sum_{|\zeta_0^k|=1} t_k = 1$ and $m \geq 1$. Let $H_n(0) = \bigcup_{\substack{m \geq 1 \\ \zeta_0 \in \partial U_1^n}} H_n^m(g)$, with $g(z) = Mz$, $z \in U_1^n$. Let D and Ω be domains in $\mathbf{C}^n$ and $\mathbf{C}^{2n}$, respectively and suppose that $H_n(0) \subseteq \Omega$. Let us $\psi_n^n(0)$ the class of those continuous mappings ψ_n: $\Omega \times U_1^n \to \mathbf{C}^n$ such that $\psi_n(0, 0; 0) \in D$ and $\psi_n(u, v; z) \notin D$, for all $(u,v) \in H_n(0)$ and $z \in U_1^n$.

An immediate application of Theorem 3.4 is given in the next result:

THEOREM 4.1. *Let* $f \in (U_1^n)$ *with* $f(0) = 0$ *and suppose that there exists a mapping* $\psi_n \in \psi_n^n(0)$ *such that*

$$(f(z), Df(z)(z)) \in \Omega$$

and

$$\psi_n(f(z), Df(z)(z); z) \in D, \ z \in U_1^n.$$

Then $\|f(z)\| < M, \ z \in U_1^n$.

The following theorem consists a direct application of Theorem 4.1.

THEOREM 4.2. *Let M, N be positive real numbers, let a and b functions defined in* U_1^n *which satisfy*

$$|a(z) + mb(z)| \geq \frac{N}{M},$$

for all $m \geq 1$ *and* $z \in U_1^n$. *Let* $f \in H(U_1^n)$ *such that* $f(0) = 0$ *and*

$$\|a(z)f(z) + b(z)Df(z)(z)\| < N, \ z \in U_1^n,$$

then $\|f(z)\| < M, \ z \in U_1^n$.

COROLLARY 4.1. *Let* α *be a function defined in* U_1^n, *which satisfies* $\mathrm{Re}\left[\dfrac{1}{\alpha(z)}\right] \geq -\dfrac{1}{2}, \ z \in U_1^n$. *Let* $f \in H(U_1^n)$ *with* $f(0) = 0$ *and suppose that*

$$\|f(z) + \alpha(z)Df(z)(z)\| < 1, \ z \in U_1^n,$$

then $\|f(z)\| < 1, \ z \in U_1^n$.

For $a(z) \equiv 0$ in Theorem 4.2, we deduce:

COROLLARY 4.2. *Let M, N be positive real numbers, let b*: $U_1^n \to \mathbf{C}$ *be a function which satisfies in* U_1^n *the following condition* $|b(z)| \geq \frac{N}{M}$, $z \in U_1^n$. *Let* $f \in H(U_1^n)$ *such that* $f(0) = 0$ *and*

$$\|b(z)Df(z)(z)\| < N, \ z \in U_1^n,$$

then $\|f(z)\| < M$, $z \in U_1^n$.

The final result is as follows:

THEOREM 4.3. *Let* $f \in H(U_1^n)$ *and let g be a biholomorphic convex in* U_1^n *with* $f(0) = g(0) = 0$ *and* $Dg(0) = I$. *Let B, C be holomorphic functions in* U_1^n *and* $E \in H(U_1^n)$, $E(0) = 0$, *which satisfy*

$$\operatorname{Re} B(z) \geq |C(z)-1| - \operatorname{Re}[C(z)-1] + \alpha \|E(z)\|, \ z \in U_1^n,$$

where α *is a positive real number, with* $\alpha > 4$. *Suppose that*

$$B(z)Df(z)(z) + C(z)f(z) + E(z) \prec g(z), \ z \in U_1^n, \tag{4.1}$$

then $f \prec g$.

The proof is based on the following T.J. Suffridge's result [10].

LEMMA 4.1. *Suppose that* h: $U_1^n \to \mathbf{C}^n$ *is a convex biholomorphic mapping, with* $h(0) = 0$, *then there exists a nonsingular mapping* $T \in L(\mathbf{C}^n)$ *and analytic convex univalent functions* f_j, $j \in \{1, \dots, n\}$ *of one variable such that* $h(z) = T\begin{pmatrix} f_1(z_1) \\ \dots \\ f_n(z_n) \end{pmatrix}$, *for all* $z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} \in U_1^n$.

Proof of Theorem 4.3. Using Lemma 4.1 by easily computation we deduce that there exists analytic and convex functions g_j , $j\in\{1, ..., n\}$ of one variable such that $g_j'(0) = 1$, $j\in\{1, ..., n\}$ and

$$g(z) = \begin{pmatrix} g_1(z_1) \\ \dots \\ g_n(z_n) \end{pmatrix}, \ z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} \in U_1^n .$$

Since $\alpha > 4$, there exists $r_0 \in (0,1)$ such that $\alpha = \dfrac{(1+r_0)^2}{r_0}$ and $\alpha > \dfrac{(1+r)^2}{r} > 4$, for $r_0 < r < 1$.

If we set $f^r(z) = f(rz)$ and $g^r(z) = g(rz)$, for $r_0 < r < 1$, then from (4.1) we obtain that

$$B^r(z)Df^r(z)(z) + C^r(z)f^r(z) + E^r(z) \prec g^r(z),\ z \in U_1^n \tag{4.2}$$

and $r_0 < r < 1$.

If we suppose that f^r is not subordinate to g^r for some $r \in (r_0,1)$, then there exists an integer $k\in\{1, ..., n\}$ such that $f_k^r(U_1^n)\not\subset g_k^r(U)$. Using Lemma 2.2, we can get points $z_0 \in U_1^n$, $\zeta_k \in \partial U$, and a real number $m_k \geq 1$ such that $f_k^r(z_0) = g_k^r(\zeta_k)$ and $[Df_k^r(z_0)]'(z_0) = m_k\zeta_k g_k^{r\prime}(\zeta_k)$.

If we denote by $\psi_k(z) = B^r(z)[Df_k^r(z)]'(z) + C^r(z)f_k^r(z) + E_k^r(z),\ z \in U_1^n$ and $\lambda_k = \dfrac{\psi_k(z_0) - g_k^r(\zeta_k)}{\zeta_k g_k^{r\prime}(\zeta_k)}$, then $\operatorname{Re}\lambda_k = m_k \operatorname{Re} B^r(z_0) + \operatorname{Re}\left\{[C^r(z_0) - 1]\dfrac{g_k^r(\zeta_k)}{\zeta_k g_k^{r\prime}(\zeta_k)}\right\} +$

$$+\operatorname{Re}\left[\frac{E_k^r(z_0)}{\zeta_k g_k^{r\prime}(\zeta)}\right] \geq m_k \operatorname{Re} B^r(z_0) + \operatorname{Re}\left[C^r(z_0) - 1\right] - |C^r(z_0) - 1| - 4|E_k^r(z_0)| \geq 0,$$

using the inequality from the hypothesis and also, from very known relations for convex univalent functions in the unit disc U:

$$\operatorname{Re}\left[\frac{z g_k'(z)}{g_k(z)}\right] > \frac{1}{2} \text{ and } |g_k'(z)| > \frac{1}{(1+|z|)^2},\ z \in U.$$

Now, using the fact that $\zeta_k g_k^{r\prime}(\zeta_k)$ is the outward normal to the boundary of the convex domain $g_k^r(U)$, we obtain that $\psi_k(z_0) \notin g_k^r(U)$, but this is a contradiction with (4.2). So, we must have f^r subordinate to g^r, for all $r_0 < r < 1$, hence letting $r \to 1^-$, we deduce f subordinate to g.

If $n = 1$ in Theorem 4.3, then this result was obtained by S.S. Miller and P.T. Mocanu [8].

If $C(z) \equiv 1$ in Theorem 4.3, we obtain

COROLLARY 4.3. *Let $f \in H(U_1^n)$ and let g be a biholomorphic convex in U_1^n with $f(0) = g(0) = 0$ and $Dg(0) = I$. Let B be holomorphic function in U_1^n and $E \in H(U_1^n)$ with $E(0) = 0$ and suppose that*

$$\operatorname{Re} B(z) \geq \alpha \|E(z)\|,\ z \in U_1^n,$$

where α is a positive number, with $\alpha > 4$.

If

$$B(z)\,Df(z)\,(z) + f(z) + E(z) \prec g(z),\ z \in U_1^n,$$

then $f \prec g$.

REFERENCES

1. V. Aubuchelvi, R. Parvatham, *Jack's Lemma for holomorphic vector valued functions*, Serdica Bulgaricae Math. Publ., **17**(1991), 241-242.
2. B. Chabat, *Introduction à l'analyse complexe*, tome 2, Ed. MIR, Moscou, 1990.
3. K. Dobrowolska, P. Liczberski, *On some differential inequalities for holomorphic functions of many variables*, Dem. Mathematica, vol. XIX, no. **2**(1981), 383-397.
4. Sh. Gong, S.S. Miller, *Partial differential subordinaions and inequalities defined on complete circular domain,* Comm. in partial differential equations, **11**(11)(1985), 1243-1255.
5. P. Liczberski, *Jack's Lemma for holomorphic mappings*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, vol. XLXIII, 1986, 131-139.
6. S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 289-305.
7. S.S. Miller, P.T. Mocanu, *Differential subordinations and univalent functions,* Michigan Math. J., **20**(1981), no. 2, 157-171.
8. S.S. Miller, P.T. Mocanu, *Averaging Operators and a Generalized Robinson Differential Inequality*, J. of Math. Anal. Appl., vol. 173, no. **2**(1993), 459-469
9. Gr. Şt. Sălăgean, H. Wiesler, *Jack's Lemma for holomorphic vector valued functions*, Mathematica (Cluj-Napoca), **23**(46), **1**(1986), 85-90.
10. T.J. Suffridge, *The principle of subordination applied to functions of several variables*,

Pacific J. of Math., **33**(1970), 241-248.

T.J. Suffridge, *Starlike and convex maps in Banach spaces*, Pacific Journal of Mathematics, vol. 46, **2**(1973), 575-589.

T.J. Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Mathematics, vol. **599**(1976), 146-159.