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FIRST AND SECOND DIFFERENTIAL SUBORDINATIONS IN SEVERAL COMPLEX VARIABLES

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Dedicated to Professor V. Ureche on his 60^{th} anniversary

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REZUMAT. - Subordonări diferenţiale de ordinul întâi şi doi în mai multe variabile complexe. În această lucrare sunt prezentate rezultate privind teoria subordonărilor diferenţiale de ordinul întâi şi doi pentru aplicaţii olomorfe de mai multe variabile complexe precum şi două aplicaţii referitoare la funcţii mărginite şi respectiv funcţii convexe.

1. Introduction. In several papers S.S.Miller and P.T.Mocanu [2,3] have considered analytic functions defined in the unit disc U which satisfy certain differential conditions and they determined properties of the functions themselves. Here, we consider similar relationships for mappings of several complex variables. We shall show that if a holomorphic mapping of several complex variables satisfies certain differential inequalities, then the function itself must satisfy an associated subordination.

In Section 2 we obtain several results concerning first and second order

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differential subordinations for holomorphic mappings defined in the unit ball. Section 3 contains applications of these results to bounded functions and convex functions.

We let $\mathbf{C}^n$ denote the space of n-complex variables $z = (z_1,\dots,z_n)'$, with the euclidian inner product $<z,w> = \sum_{j=1}^{n} z_j\bar{w}_j$ and the norm $\|z\| = (<z,z>)^{1/2}$. The ball $\{z \in \mathbf{C}^n: \|z\| < r\}$ will be denoted by B_r^n. For short we write B^n for B_1^n.

Vector and matrices marked with the symbol ' and * denote the transposed and the transposed conjugate vector or matrix, respectively.

We denote by $\mathcal{L}(\mathbf{C}^n)$ the space of continuous linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm:

$$\|A\| = \sup\{\|Az\|: \|z\| \le 1\},\ A \in \mathcal{L}(\mathbf{C}^n).$$

The class of holomorphic mappings $f(z) = (f_1(z),\dots,f_n(z))'$ from B^n into $\mathbf{C}^n$ is denoted by $\mathcal{H}(B^n)$

We denote by $Df(z)$ and $D^2f(z)$ the first and the second Fréchet derivatives of f at z.

We say that $f \in \mathcal{H}(B^n)$ is locally biholomorphic (locally univalent) if f has a local holomorphic inverse at each point in B^n, or equivalently, derivative $Df(z) = \left(\dfrac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$ is nonsingular at each point $z \in B^n$.

If $f, g \in \mathcal{H}(B^n)$, we say that f is subordinate to g (in B^n) if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B^n$, and we shall write $f \prec g$ to indicate that f is subordinate to g.

Now we present a previous generalization of Jack's, Miller's and Mocanu's lemma [1] which will be the basic tool in obtaining our main results.

THEOREM 1. *Let f be holomorphic in B^n with $f(0) = 0$ and $f \not\equiv 0$.*

If $\|f(\dot{z})\| = \max\limits_{\|z\| \le \|\dot{z}\|} \|f(z)\|$, $\dot{z} \in B^n$ *and if* $Df(\dot{z})$ *is nonsingular then there exists a real positive number* m *which satisfies* $m \le \dfrac{\|\dot{z}\|^2}{\|f(\dot{z})\|^2}$ *such that:*

$$((Df(\dot{z}))^*)^{-1}(\dot{z}) = m f(\dot{z}) \tag{1}$$

and

$$\frac{\|w\|^2 - \operatorname{Re}(\dot{z}^*(Df(\dot{z}))^{-1} D^2 f(\dot{z})(w,w))}{\|Df(\dot{z})w\|^2} \ge m \tag{2}$$

for all $w \in \mathbb{C}^n \setminus \{0\}$ *which satisfy* $\operatorname{Re} < w, \dot{z} > = 0$.

2. First and second order differential inequalities.

LEMMA 1. *Let q be a holomorphic and univalent mapping defined on $\overline{B}^n$. Let p be holomorphic in B^n, with $p(0) = q(0)$. Suppose that there exists a point $\dot{z} \in B^n$, $\|\dot{z}\| = r_0 < 1$ such that $Dp(\dot{z})$ is nonsingular and*

$$p(\dot{z}) \in \delta q(B^n),\ p(B^n_{r_0}) \subset q(B^n). \tag{3}$$

If $\zeta_0 = q^{-1}(p(\mathring{z}))$ *then there exists a real positive number which satisfies* $m < \|\mathring{z}\|^2$ *such that:*

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = m((Dq(\mathring{\zeta}))^*)^{-1}(\mathring{\zeta}) \tag{4}$$

and

$$\|w\|^2 - \mathrm{Re}\,\mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) \geq m(\|u\|^2 - \mathrm{Re}\,\mathring{\zeta}^*(Dq(\mathring{\zeta}))^{-1}D^2q(\mathring{\zeta})(u,u)) \tag{5}$$

for each $w \in \mathbf{C}^n\backslash\{0\}$ *with* $\mathrm{Re} < w, \mathring{z} > = 0$, *where* u *is defined by* $u = (Dq(\mathring{\zeta}))^{-1}Dp(\mathring{z})w$.

Proof. Let $f: q^{-1} \circ p: \bar{B}^n_{r_0} \to \bar{B}^n$. The function $f(z) = q^{-1}(p(z))$ is holomorphic in $\bar{B}^n_{r_0}$ and satisfies $f(0) = q^{-1}(p(0)) = 0$, $\|f(\mathring{z})\| = \|q^{-1}(p(\mathring{z}))\| = \|\mathring{\zeta}\| = 1$. Since $p(B^n_{r_0}) \subset q(B^n)$ we have:

$$1 = \|f(\mathring{z})\| = \max_{\|z\| \leq r_0} \|f(z)\|.$$

Also, since p is locally univalent at $\mathring{z}$ and q^{-1} is univalent on $q(B^n)$ we have that $Df(\mathring{z})$ is nonsingular. Thus f satisfies the conditions of Theorem 1. By Theorem 1 we obtain that there exists a real positive number m such that $m \leq \|\mathring{z}\|^2$ and

$$((Df(\mathring{z}))^*)^{-1}(\mathring{z}) = mf(\mathring{z}). \tag{6}$$

Since $p(z) = q(f(z))$ we have:

$$Dp(z) = Dq(f(z))Df(z) \tag{7}$$

and

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = ((Dq(f(\mathring{z})))^*)^{-1}((Df(\mathring{z}))^*)^{-1}(\mathring{z}). \tag{8}$$

By using (6) and the fact that $f(\mathring{z}) = \xi$ we get

$$((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = m((Dq(\mathring{\xi}))^*)^{-1}(\mathring{\xi})$$

so part (i) of Lemma 1 is satisfied.

Differentiating (8) at $z = \mathring{z}$ we get

$$D^2p(\mathring{z})(w,w) = D^2q(f(\mathring{z}))(Df(\mathring{z})w, Df(\mathring{z})w) + Dq(f(\mathring{z}))D^2f(\mathring{z})(w,w)$$

for each $w \in \mathbf{C}^n$.

By multiplying the above equality to the left with

$$\mathring{z}(Dp(\mathring{z}))^{-1} = \mathring{z}^*(Df(\mathring{z}))^{-1}(Dq(f(\mathring{z})))^{-1}$$

we obtain:

$$\begin{aligned}\mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) = {}& \mathring{z}^*(Df(\mathring{z}))^{-1}(Dq(f(\mathring{z})))^{-1}D^2q(f(\mathring{z}))\,(Df(\mathring{z})w, Df(\mathring{z})w) + \\ & + \mathring{z}^*(Df(\mathring{z}))^{-1}D^2f(\mathring{z})(w,w).\end{aligned}$$

Substituting (6) into the above equality, using the fact that $f(\mathring{z}) = \mathring{\xi}$ and noting $u = Df(\mathring{z})w$ we get

$$\begin{aligned}\|w\|^2 - \mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) = {}& \|w\|^2 - \mathring{z}^*(Df(\mathring{z}))^{-1}D^2f(\mathring{z})(w,w) - \\ & - m\mathring{\xi}^*(Dq(\mathring{\xi}))^{-1}D^2q(\mathring{\xi})(u,u).\end{aligned} \tag{9}$$

Next, we shall use that part (ii) of Theorem 1. If we take the real parts in (9) we have

$$\|w\|^2 - \mathring{z}^*(Dp(\mathring{z}))^{-1}D^2p(\mathring{z})(w,w) \geq m(\|u\|^2 - \mathrm{Re}\,\mathring{\xi}^*(Dq(\mathring{\xi}))^{-1}D^2q(\mathring{\xi})(u,u))$$

for each $w \in \mathbf{C}^n \setminus \{0\}$ which satisfy $\operatorname{Re} < w, \mathring{z} > = 0$.

Also, we note that the condition $\operatorname{Re} < w, \mathring{z} > = 0$ implies $\operatorname{Re} < u, \mathring{\zeta} > = 0$ where by u we denoted $u = Df(\mathring{z})w$.

Indeed:

$$\begin{aligned} \operatorname{Re} < Df(\mathring{z})w, \mathring{\zeta} > &= \operatorname{Re} < (Dq(\mathring{\zeta}))^{-1} Dp(\mathring{z})w, \mathring{\zeta} > = \\ &= \operatorname{Re} < Dp(\mathring{z})w, ((Dq(\mathring{\zeta}))^{-1})^*(\mathring{\zeta}) > = \\ &= \operatorname{Re} < Dp(\mathring{z})w, m((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) > = \\ &= m \operatorname{Re} < w, \mathring{z} > = 0. \end{aligned}$$

LEMMA 2. *Let q be a holomorphic and univalent mapping defined $\bar{B}^n$. Let p be holomorphic in B^n, locally univalent on B^n, with $p(0) = q(0)$.*

If $p \not\prec q$ then there exists $\mathring{z} \in B^n$, $\mathring{\zeta} \in \bar{B}^n$ with $\|\mathring{\zeta}\| = 1$ and a real positive number $m \leq 1$ such that

$$\text{(i)} \qquad p(\mathring{z}) = q(\mathring{\zeta}) \tag{10}$$

$$\text{(ii)} \qquad ((Dp(\mathring{z}))^*)^{-1}(\mathring{z}) = m((Dq(\mathring{\zeta}))^*)^{-1}(\mathring{\zeta}) \tag{11}$$

and

$$\text{(iii)} \qquad \begin{aligned} &\|w\|^2 - \operatorname{Re} \mathring{z}^*(Dp(\mathring{z}))^{-1} D^2 p(\mathring{z})(w,w) \geq \\ &\geq m(\|u\|^2 - \operatorname{Re} \mathring{\zeta}^*(Dq(\mathring{\zeta}))^{-1} D^2 q(\mathring{\zeta})(u,u)), \end{aligned}$$

for each $w \in \mathbf{C}^n \setminus \{0\}$ with $\operatorname{Re} < w, \mathring{z} > = 0$, where for each w, u is defined by $u = (Dq(\mathring{\zeta}))^{-1} Dp(\mathring{z})w$.

Proof. Since $p(0) = q(0)$ and $p(B^n) \not\subset q(B^n)$ there exists $0 < r_0 < 1$ such that $p(B_{r_0}^n) \subset q(B^n)$ and $p(\bar{B}_{r_0}^n) \cap q(\delta B^n) \neq \emptyset$.

Hence there exists $z_0 \in \bar{B}_{r_0}$ and $\zeta_0 \in \delta B^n$ such that $p(\mathring{z}) = q(\mathring{\zeta})$ and $p(B_{r_0}^n) \subset q(B^n)$.

The functions p,q satisfy the conditions of Lemma 1 and hence Lemma 2 is proved.

Before obtaining the main result of this section we need to specify the class of functions for which the differential inequalities will hold.

DEFINITION. Let D be a set in $\mathbf{C}^n$ and let q be holomorphic and univalent on $\bar{B}^n$.

We define $\psi[D,q]$ to be the class of mappings $\psi\colon \mathbf{C}^n \times \mathbf{C}^n \times B^n \to \mathbf{C}^n$ for which $\psi(r,s,z) \notin D$ when $r = q(\zeta)$, $s = m((Dq(\zeta))^*)^{-1}(\zeta)$, $z \in B^n$, $0 \leq m \leq 1$ and $\|\zeta\| = 1$.

We now present the main differential subordination result:

THEOREM 2. *Let $D \subseteq \mathbf{C}^n$, q be a holomorphic and univalent mapping defined on $\bar{B}^n$ and let $\psi \in \Psi[D,q]$. If $p \in \mathcal{H}(B^n)$ is locally univalent on B^n with $p(0) = q(0)$ and if p satisfies*

$$\psi(p(z),((Dp(z))^*)^{-1}(z),z) \in D \tag{13}$$

when $z \in B^n$, then $p \prec q$.

Proof. If $p(B^n) \not\subset q(B^n)$ there exists points $\dot{z} \in B^n$ and $\dot{\zeta} \in bB^n$ that satisfy (i) and (ii) of Lemma 2. Using these conditions with $r = p(\dot{z})$, $s = ((Dp(\dot{z}))^*)^{-1}(\dot{z})$ in Definition we obtain

$$\psi(p(\dot{z}),((Dp(\dot{z}))^*)^{-1}(\dot{z}),\dot{z}) \notin D.$$

Since this contradicts (13) we must have $p(B^n) \subset q(B^n)$ and $p \prec q$.

The definition of $\psi[D,q]$ requires that the function q behave very nicely on δB^n. If this is not the case, or the behavior of q on δB^n is unknown, it may still be possible to prove $p \prec q$ by the following result:

THEOREM 3. *Let* $0 < \rho < 1$, *let* q *be holomorphic and univalent in* B^n *and suppose that:*

$$\psi(r,s;z) \in \psi[D,q_\rho] \text{ where } q_\rho(z) = q(\rho z). \tag{14}$$

If $p \in \mathcal{H}(B^n)$ *is locally univalent on* B^n *with* $p(0) = q(0)$ *and if* p *satisfies* (13), *when* $z \in B^n$ *then* $p \prec q$.

Proof. Since q_ρ is holomorphic and univalent on $\bar{B}^n$, the class $\psi[D,q_\rho]$ is well defined from (14) and Theorem 2 we obtain $p(B^n) \subset q_\rho(B^n)$, which implies $p(B^n) \subset q(B^n)$.

3. Bounded and convex functions on B^n. If we take $q(z) = Mz$ (where $M > 0$) in Definition and Theorem 2 we obtain the following result:

COROLLARY 1. *Let D be a set in $\mathbb{C}^n$ and let be $\psi: \mathbb{C}^n \times \mathbb{C}^n \times B^n \to \mathbb{C}^n$ be such that*

$$\psi(M\zeta, \frac{m}{M}\zeta, z) \notin D \tag{15}$$

where $\zeta \in \mathbb{C}^n$ with $\|\zeta\| = 1$, $m \in \mathbb{R}_+^*$ with $m \leq 1$ and $z \in B^n$.

If $p \in \mathcal{H}(B^n)$ with $p(0) = 0$ and if p satisfies

$$\psi(p(z), ((Dp(z))^*)^{-1}(z), z) \in D \tag{16}$$

where $z \in B^n$, then $\|p(z)\| < M$.

We show the applicability of this result by an example.

Example. For $D = B_M^n$, $\psi(r,s,z) = r + \lambda s$ where $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ and $p \in \mathcal{H}(B^n)$ with $p(0) = 0$

$$\|p(z) + \lambda((Dp(z))^*)^{-1}(z)\| < M \text{ implies } \|p(z)\| < M.$$

Our final result is an application concerning subordination to convex mappings.

COROLLARY 2. *Let $h \in \mathcal{H}(B^n)$ be a convex mapping with $h(0) = 0$ and let $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\,\lambda \geq 0$.

Let $p \in \mathcal{H}(B^n)$ locally univalent on B^n with $p(0) = 0$.

If

$$p(z) + \lambda((Dp(z))^{*})^{-1}(z) \in h(B^{n}) \tag{17}$$

for every $z \in B^n$ then $p \prec h$.

Proof. Suppose that $p \nprec h$.

By using Lemma 2 we obtain that there exists $\dot{z} \in B^{n}$, $\mathring{\zeta} \in \bar{B}^{n}$ with $\|\mathring{\zeta}\| = 1$ such that:

$$p(\dot{z}) = h(\mathring{\zeta})$$

and

$$((Dp(\dot{z}))^{*})^{-1}(\dot{z}) = m((Dh(\dot{z}))^{*})^{-1}(\dot{z})$$

where $m \in \mathbf{R}_{+}$, $m \leq 1$.

If we note

$$\psi_0 = h(\mathring{\zeta}) + \lambda((Dp(\dot{z}))^{*})^{-1}(\dot{z})$$

we obtain:

$$\psi_0 = h(\mathring{\zeta}) + m\lambda((Dh(\dot{z}))^{*})^{-1}(\dot{z}).$$

Straightforward calcultion gives us:

$$< \psi_0 - h(\zeta), ((Dh(\dot{z}))^{*})^{-1}(\dot{z}) > = m\lambda \|((Dh(\dot{z}))^{*})^{-1}(\dot{z})\|^2 .$$

Since $m > 0$ and $\operatorname{Re} \lambda \geq 0$ we have

$$\operatorname{Re} < \psi_0 - h(\zeta), (Dh(\dot{z}))^{*})^{-1}(\dot{z}) \geq 0$$

and taking account that $h(B^n)$ is a convex set we obtain $\psi_0 \notin h(B^n)$ which contradicts (17).

Hence our supposition was false, which means that $p \prec h$.

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