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CLOSE-TO-CONVEX FUNCTIONS WITH POSITIVE COEFFICIENTS

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REZUMAT. - Funcţii aproape convexe cu coeficienţi pozitivi. În lucrare sunt studiate funcţiile analitice în discul unitate, care satisfac condiţia Re $f'(z) < \beta$, $1 < \beta \leq 2$, $|z| < 1$.

Abstract. Let $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ be analytic in the unit disk $E = \{z: |z| < 1\}$. Coefficient inequalities, distortion Theorems and radius of convexity are determined for functions satisfying Re $f'(z) < \beta$, $1 < \beta \leq 2$, $z \in E$. Further it is shown that such functions are close-to-convex.

1. Introduction. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disk $E = \{z: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions in E. Let S^* be the subclass of S, the members of which are starlike (with respect to the origin) in E. A function $f \in S$ is said to be close-to-convex of order α, denoted by $f \in C(\alpha)$, $0 \leq \alpha < 1$ if there exists a function $g \in S^*$ such that Re $zf'(z)/g(z) > \alpha$ for $z \in E$.

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$C(0) = C$ is the class of close-to-convex functions.

Denote by V the subclass of S, consisting of functions of the form $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$. Further let $P(\alpha)$ be the subclass of V consisting of functions that satisfy $\operatorname{Re} f'(z) > \alpha$, $0 \le \alpha < 1$, $z \in E$.

Such functions are close-to-convex of order α with respect to the identity function z. Thus $P(\alpha) \subset C(\alpha)$, $P(0) = P$ is the subclass of close-to-convex functions in V.

For $1 < \beta \le 3/2$ and $z \in E$, let $U(\beta) = \{f \in V: \operatorname{Re}(1 + zf''(z)/f'(z) < \beta\}$ and for $1 < \beta \le 2$, $z \in E$, let $R(\beta) = \{f \in V: \operatorname{Re} f'(z) < \beta\}$.. In [5] the authors have studied the univalent functions with positive coefficients. Sarangi and Uralegaddi [3] and H.Al-Amiri [1] have studied the functions with negative coefficients that satisfy $\operatorname{Re} f'(z) > \alpha$, $0 \le \alpha < 1$ for $z \in E$. In [2] Ozaki has proved that if $f \in A$, satisfies $\operatorname{Re}(1 + zf''(z)/f'(z)) < 3/2$, then f is univalent. And in [4] R.Singh and S.Singh have shown that such functions are close-to-convex.

In this paper coefficient in equalities, distortion Theorems and radius of convexity are determined for the class $R(\beta)$. Also it is shown that the functions in $R(\beta)$ are close-to-convex. Further it is proved that if $f \in V$, satisfies $\operatorname{Re}(1 + zf''(z)/f'(z)) < 3/2$, then $0 < \operatorname{Re} f'(z) \le 2$, $z \in E$.

We need the following result [5].

THEOREM A. *$f \in U(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)\,|a_n| \le \beta - 1.$$

2. Coefficient inequalities.

THEOREM 1. *If $f \in V$ and $\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$ $(0 \le \alpha < 1)$ then $f \in P(\alpha)$.*

Proof. Let $\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$. It suffices to show that $|f'(z)-1| < 1-\alpha$. We have

$$\begin{aligned} |f'(z)-1| &= \left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \\ &\le \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} \\ &\le \sum_{n=2}^{\infty} n|a_n|. \end{aligned}$$

The last expression is bounded above by $1-\alpha$ by hypothesis. Hence $|f'(z)-1| < 1-\alpha$ and the theorem is proved.

THEOREM 2. *$f \in R(\beta)$ if and only if $\sum_{n=2}^{\infty} n|a_n| \le \beta - 1$.*

Proof. Let $\sum_{n=2}^{\infty} n|a_n| \le \beta - 1$. It suffices to prove that

$$\left|\frac{f'(z)-1}{f'(z)-(2\beta-1)}\right| < 1,\ z \in E.$$

We have

$$\left|\frac{f'(z)-1}{f'(z)-(2\beta-1)}\right| = \left|\frac{\sum_{n=2}^{\infty} n|a_n|z^{n-1}}{2(1-\beta)+\sum_{n=2}^{\infty} n|a_n|z^{n-1}}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} n|a_n|\ |z|^{n-1}}{2(\beta-1)-\sum_{n=2}^{\infty} n|a_n|\ |z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} n|a_n|}{2(\beta-1)-\sum_{n=2}^{\infty} n|a_n|}.$$

The last is bounded above by 1 if $\sum_{n=2}^{\infty} n|a_n| \leq 2(\beta-1)-\sum_{n=2}^{\infty} n|a_n|$ which is equivalent to

$$\sum_{n=2}^{\infty} n|a_n| \leq \beta-1. \tag{1}$$

But (1) is true by hypothesis and the theorem is proved.

Conversely suppose $\operatorname{Re} f'(z) = \operatorname{Re}\{1+\sum_{n=2}^{\infty} n|a_n|z^{n-1}\} < \beta$, $z \in E$. Choose values of z on the real axis. Then letting $z \to 1$ through real values we obtain $1+\sum_{n=2}^{\infty} n|a_n| \leq \beta$. That is $\sum_{n=2}^{\infty} n|a_n| \leq \beta-1$.

COROLLARY. *If $f \in R(\beta)$ then $|a_n| \leq (\beta-1)/n$, with equality only for the functions of the form $f_n(z) = z+((\beta-1)/n)z^n$.*

3. Distortion Theorems. The coefficient bounds enable us to prove

THEOREM 3. *If* $f \in R(\beta)$ *then*

(i) $r - \frac{\beta-1}{2} r^2 \leq |f(z)| \leq r + \frac{\beta-1}{2} r^2$, $(|z| = r)$

with equality for $f(z) = z + \frac{\beta-1}{2} z^2$.

(ii) $1 - (\beta-1)r \leq |f'(z)| \leq 1 + (\beta-1)r \quad (|z| = r)$

with equality for $f(z) = z + \frac{\beta-1}{2} z^2$.

Proof. From Theorem 2 we have

$$2\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n|a_n| \leq \beta - 1.$$

$$|f(z)| < r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{\beta-1}{2} r^2. \tag{2}$$

Similarly

$$|f(z)| > r - \sum_{n=2}^{\infty} |a_n| r^n > r - r^2 \sum_{n=2}^{\infty} |a_n| > r - \frac{\beta-1}{2} r^2. \tag{3}$$

The result (i) follows from (2) and (3). Similarly the result (ii) can be proved.

4. Comparable results.

THEOREM 4. *If* $f \in R(\beta)$ *then* $f \in P(2-\beta)$.

Proof. In view of Theorem 2 and Theorem 1 we must prove $\sum_{n=2}^{\infty} n|a_n| \leq \beta - 1$ implies $\sum_{n=2}^{\infty} n|a_n| \leq 1 - (2-\beta) = \beta - 1$, which is obvious.

COROLLARY. $R(2) \subset P$.

Thus the functions in $R(\beta)$ are close-to-convex.

THEOREM 5. *If* $f \in U(\beta)$ *then* $f \in R(1/(2-\beta))$.

Proof. In view of Theorem A and Theorem 2 we must show that $\sum_{n=2}^{\infty} \frac{n(n-\beta)}{\beta-1} |a_n| \le 1$ implies $\sum_{n=2}^{\infty} n|a_n| < \frac{1}{2-\beta} - 1 = \frac{\beta-1}{2-\beta}$. It is sufficient to show hat

$$\frac{n(n-\beta)}{\beta-1} \ge \frac{(2-\beta)n}{\beta-1}$$

which is equivalent to $n-\beta \ge 2-\beta$, $n = 2, 3,...$ which is obvious.

COROLLARY 1. $U(3/2) \subset P$.

From the above two corollaries we have

COROLLARY 2. *If* $\mathrm{Re}(1 + zf''(z)/f'(z)) < 3/2$ *then* $0 < \mathrm{Re} f'(z) \le 2$.

5. Radius of convexity.

THEOREM 6. *If* $f \in R(\beta)$ *then* $\mathrm{Re}(1 + zf''(z)/f'(z)) > 0$ *in the disk* $|z| < r = r(\beta) = \operatorname{Inf}_n \left(\frac{1}{(\beta-1)n}\right)^{1/(n-1)}$, $n = 2,3,4,...$. *The result is sharp for* $f_n(z) = z + \frac{\beta-1}{n} z^n$ *for some n.*

The proof is similar to that of Theorem 4 in [3]. Observe that $r(2) = 1/2$

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