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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Dedicated to Professor V. Ureche on his 60^{th} anniversary

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REZUMAT. - Asupra unei anumite clase de funcţii analitice. În lucrare este studiată o clasă de funcţii analitice în discul unitate U, notată $T_\lambda(j,\alpha,\beta)$, unde $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $j \in \mathbf{N}_0 = \{0,1,...\}$. Sunt obţinute teoreme de deformare (delimitări ale modulelor funcţiilor şi derivatelor lor) şi estimări ale coeficienţilor dezvoltării în serie Taylor ale funcţiilor din aceste clase.

ABSTRACT. There are many classes of analytic functions in the unit disc U. We shall consider the special class $T_\lambda(j,\alpha,\beta)$ ($0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $j \in \mathbf{N}_0 = \{0,1,...\}$) of analytic functions in the unit disc U. And the purpose of this paper is to show some distorsion theorems for the class $T_\lambda(j,\alpha,\beta)$. Also we show some coefficient estimates for the classes $T_\lambda(j,\alpha,\beta)$, $T_0(j,\alpha,\beta)$, and $T_\lambda(j,\alpha,1)$.

1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic and univalent in the unit disc $U = \{z: |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U satisfing the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For a function $f(z)$ in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^j f(z) = D(D^{j-1} f(z)) \quad (j \in \mathbf{N} = \{1,2,...\}). \tag{1.4}$$

The differential operator D^j was introduced by Salagean [9]. With the help of the differential operator D^j, we say that a function $f(z)$ belonging to S is in the class $S_j(\alpha)$ if and only if

$$\mathrm{Re}\{\frac{D^{j+1} f(z)}{D^j f(z)}\} > \alpha \quad (j \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}) \tag{1.5}$$

for some α $(0 \leq \alpha < 1)$, and for all $z \in U$. The class $S_j(\alpha)$ was defined by Salagean [9].

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1.6}$$

Further, we define the class $T(j,\alpha)$ by

$$T(j,\alpha) = S_j(\alpha) \cap T. \tag{1.7}$$

The class $T(j,\alpha)$ was studied by Hur and Oh [3] and Salagean [10] and [11]. We note that $T(0,\alpha) = T^*(\alpha)$ and $T(1,\alpha) = C(\alpha)$ were studied by Silverman [12]. For this class $T(j,\alpha)$ Salagean [10] and Hur and Oh [3] gave the following lemma.

LEMMA 1. *Let the function* $f(z)$ *be defined by* (1.6). *Then* $f(z)$ *is in the class* $T(j,\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n^j(n-\alpha)\, a_n \le 1-\alpha. \tag{1.8}$$

The result is sharp.

The next lemma may be found in [3].

LEMMA 2. *Let the function* $f(z)$ *defined by* (1.6) *be in the class* $T(j,\alpha)$. *Then we have*

$$|z| - \frac{1-\alpha}{2^j(2-\alpha)}|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{2^j(2-\alpha)}|z|^2 \tag{1.9}$$

and

$$1 - \frac{1-\alpha}{2^{j-1}(2-\alpha)}|z| \le |f'(z)| \le 1 + \frac{1-\alpha}{2^{j-1}(2-\alpha)}|z|. \tag{1.10}$$

The result is sharp.

Let $T_\lambda(j,\alpha,\beta)$ denote the class of functions of the form (1.1) which satisfy the condition

$$\left| \frac{\frac{f(z)}{g(z)} - 1}{\lambda \frac{f(z)}{g(z)} + 1} \right| < \beta \quad (0 \le \lambda \le 1, 0 < \beta \le 1, z \in U) \tag{1.11}$$

where

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \ge 0) \tag{1.12}$$

is in the class $T(j,\alpha)$ ($j \in \mathbb{N}_0$; $0 \le \alpha < 1$).

We note that:

(i) $T_\lambda(0,\alpha,\beta) = \tilde{S}_\lambda(\alpha,\beta)$ (Owa [5,7]):

(ii) $T_\lambda(0,\alpha,1) = S_u(0,\alpha)$, $0 \le u = \lambda \le 1$, (Altintas [1]).

2. Distortion Theorems.

THEOREM 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_\lambda(j,\alpha,\beta)$. Then we have*

$$|f(z)| \ge \frac{(1-\beta|z|)(2^j(2-\alpha)-(1-\alpha)|z|)|z|}{2^j(1+\lambda\beta|z|)(2-\alpha)} \tag{2.1}$$

and

$$|f(z)| \le \frac{(1+\beta|z|)(2^j(2-\alpha)|z|)|z|}{2^j(1-\lambda\beta|z|)(2-\alpha)} \tag{2.2}$$

for $z \in U$. These estimates are sharp.

Proof. We employ the same technique used by Goel and Sohi [2] and

Owa [5,6,7,8]. Since $f(z) \in T_\lambda(j,\alpha,\beta)$, after a simple computation we have

$$\frac{f(z)}{g(z)} = \frac{1-\beta w(z)}{1+\lambda\beta w(z)}, \quad w \in \Omega. \tag{2.3}$$

By using Schwarz's lemma [4], we have $|w(z)| \le |z|$. Hence

$$\frac{1-\beta|z|}{1+\lambda\beta|z|} \le \left|\frac{f(z)}{g(z)}\right| \le \frac{1+\beta|z|}{1-\lambda\beta|z|}. \tag{2.4}$$

Consequently, we have the theorem with the aid of Lemma 2. By taking

$$\frac{f(z)}{g(z)} = \frac{1-\beta z}{1+\lambda\beta z} \tag{2.5}$$

and

$$g(z) = z - \frac{1-\alpha}{2^j(2-\alpha)}z^2 \tag{2.6}$$

we can see the estimates are sharp. This completes the proof of Theorem 1.

COROLLARY 1. *Under the hypotheses of Theorem* 1, $f(z)$ *is included in the disc with center at the origin and radius* r_1 *given by*

$$r_1 = \frac{(1+\beta)(2^j(2-\alpha)+(1-\alpha))}{2^j(1-\lambda\beta)(2-\alpha)}. \tag{2.7}$$

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_\lambda(j,\alpha,\beta)$. *Then we have*

$$|f'(z)| \le \frac{(1+\beta|z|)(2^{j-1}(2-\alpha)+(1-\alpha)|z|)}{2^{j-1}(1-\lambda\beta|z|)(2-\alpha)}$$

$$+\frac{(1+\lambda)\beta(2^j(2-\alpha)+(1-\alpha)|z|)|z|}{2^j(1-\lambda\beta|z|)^2(1-|z|^2)(2-\alpha)} \tag{2.8}$$

for $z \in U$.

Proof. Since $f(z) \in T_\lambda(j,\alpha,\beta)$, by using (2.3), we obtain

$$f'(z) = \frac{1-\beta w(z)}{1+\lambda\beta w(z)} g'(z) - \frac{(1+\lambda)\beta w'(z)}{\{1+\lambda\beta w(z)\}^2} g(z), \quad w \in \Omega. \tag{2.9}$$

Further, we have $|w'(z)| \leq \frac{1}{1-|z|^2}$ by means of Caratheodory's theorem [4].

Hence we obtain the theorem with the aid of Lemma 2.

Remark 1. We have not able to obtain the sharp estimate for $|f'(z)|$ for $f(z) \in T_\lambda(j,\alpha,\beta)$.

3. Coefficient Estimates

THEOREM 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_\lambda(j,\alpha,\beta)$. Then we have*

$$|a_2| \leq \frac{1-\alpha}{2^j(2-\alpha)} + \beta(1+\lambda) \tag{3.1}$$

and

$$|a_3| \leq \frac{1-\alpha}{3^j(3-\alpha)} + \frac{1-\alpha}{2^j(2-\alpha)}\beta(1+\lambda) + \beta(1+\lambda). \tag{3.2}$$

The estimate for $|a_2|$ is sharp.

Proof. Let

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \in \Omega. \tag{3.3}$$

Then we obtain [4]

$$|c_1| \leq 1. \tag{3.4}$$

and

$$|c_2| \le 1 - |c_1|^2. \tag{3.5}$$

Since $f(z) \in T_\lambda(j,\alpha,\beta)$, by using (2.3), we have

$$f(z)\,(1 + \lambda\beta w(z)) = g(z)\,(1 - \beta w(z)), \; w \in \Omega. \tag{3.6}$$

Then, on substituting the power series (1.1), (1.12), and (3.3), for the functions $f(z)$, $g(z)$, and $w(z)$, respectively, in (3.6) we get

$$\left(z + \sum_{n=2}^{\infty} a_n z^n\right)\left(1 + \lambda\beta\sum_{n=1}^{\infty} c_n z^n\right) = \left(z - \sum_{n=2}^{\infty} b_n z^n\right)\left(1 - \beta\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.7}$$

Equating coefficients of z^2 and z^3 on both sides of (3.7), we obtain

$$a_2 = -\beta(1 + \lambda)c_1 - b_2 \tag{3.8}$$

and

$$a_3 = -\beta(1 + \lambda)c_2 + \lambda\beta^2(1 + \lambda)c_1^2 + \beta(1 + \lambda)b_2c_1 - b_3. \tag{3.9}$$

Since $g(z) \in T(j,\alpha)$, by using Lemma 1, we have

$$b_2 \le \frac{1-\alpha}{2^j(2-\alpha)}, \tag{3.10}$$

and

$$b_3 \le \frac{1-\alpha}{3^j(3-\alpha)}. \tag{3.11}$$

Hence we have the theorem. Further we can see that estimate for $|a_2|$ is sharp for $\frac{f(z)}{g(z)}$ defined by (2.5), where $g(z)$ is given by (2.6).

THEOREM 4. *Let the function $f(z)$ defined by* (1.1) *be in the class*

$T_\lambda(j,\alpha,\beta)$. *Then we have*

$$|a_4| \le \frac{1-\alpha}{4^j(4-\alpha)} + \frac{1-\alpha}{3^j(3-\alpha)}\beta(1+\lambda) + \frac{1-\alpha}{2^j(2-\alpha)}\beta(1+\lambda)(1+\lambda\beta)$$
$$+\beta(1+\lambda)+2\lambda\beta^2(1+\lambda). \qquad (3.12)$$

Proof. Equating the coefficients of z^4 on both sides of (3.7), we have

$$a_4 = -b_4 - \beta(1+\lambda)c_3 - \beta(\lambda a_2 - b_2)c_2 - \beta(\lambda a_3 - b_3)c_1. \qquad (3.13)$$

Since

$$c_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)}{z^{n+1}}dz \quad (0<r<1;\ n\in\mathbb{N}) \qquad (3.14)$$

for the coefficients c_n of analytic function $w(z)$ in the unit disc U, we obtain

$$|c_n| \le \frac{1}{r},\ 0<r<1,\ n\in\mathbb{N},\ n\ge 1$$

and then

$$|c_n| \le 1,\ n\in\mathbb{N},\ n\ge 1. \qquad (3.15)$$

From (3.13) and (3.15) we deduce

$$|a_4| \le |b_4| + \beta(1+\lambda) + \beta(\lambda|a_2| + |b_2|) + \beta(\lambda|a_3| + |b_3|). \qquad (3.16)$$

Hence we obtain the theorem by using Lemma 1 and Theorem 3.

Remark 2. We have not been able to obtain sharp estimates for $|a_n|$ ($n \ge 3$) for the function $f(z)$ belonging to the class $T_\lambda(j,\alpha,\beta)$.

THEOREM 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_0(j,\alpha,\beta)$. *Then we have*

$$|a_n| \le \beta\left(\frac{2^j(2-\alpha)+1-\alpha}{2^j(2-\alpha)}\right)+\frac{1-\alpha}{n^j(n-\alpha)} \tag{3.17}$$

for any n ≥ 2.

Proof. Since $f(z)$ belongs to the class $T_0(j,\alpha,\beta)$, from (3.7), we have

$$-\frac{1}{\beta}\sum_{n=2}^{\infty}(a_n+b_n)z^n = \left(z-\sum_{n=2}^{\infty}b_n z^n\right)\left(\sum_{n=1}^{\infty}c_n z^n\right). \tag{3.18}$$

Equating the coefficients of z^n on both sides of (3.18), we have

$$-\frac{1}{\beta}(a_n+b_n) = c_{n-1}-\sum_{m=2}^{n-1}b_m c_{n-m} \tag{3.19}$$

By using that $b_n \ge 0$, $n \ge 2$, from (3.15) and (3.19) we obtain

$$\frac{1}{\beta}|a_n+b_n| \le 1+\sum_{m=2}^{n-1}b_m. \tag{3.20}$$

But

$$\sum_{m=2}^{n-1}2^j(2-\alpha)b_m \le \sum_{m=2}^{n-1}m^j(m-\alpha)b_m \tag{3.21}$$

and by using Lemma 1 we deduce

$$\sum_{m=2}^{\infty}b_m \le \frac{1-\alpha}{2^j(2-\alpha)}. \tag{3.22}$$

From (3.20) and (3.22) we have

$$\frac{1}{\beta}|a_n+b_n| \le 1+\frac{1-\alpha}{2^j(2-\alpha)}. \tag{3.23}$$

Hence we obtain

$$\begin{aligned}|a_n| &\le |a_n+b_n|+|b_n|\\ &\le \beta\left(\frac{2^j(2-\alpha)+1-\alpha}{2^j(2-\alpha)}\right)+\frac{1-\alpha}{n^j(n-\alpha)},\end{aligned} \tag{3.24}$$

because

$$b_n \le \frac{1-\alpha}{n^j(n-\alpha)} \tag{3.25}$$

for any $n \ge 2$ by Lemma 1.

Remark 3. We have not been able to obtain sharp estimates for $|a_n|$ ($n \ge 2$) for the function $f(z)$ belonging to the class $T_0(j,\alpha,\beta)$.

THEOREM 6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_\lambda(j,\alpha,1)$ *and* $\mathrm{Re}(a_k) \ge 0$ ($k = 2,3,\ldots,(n-1)$), *then*

$$|a_n| \le 1 + \lambda + \frac{1-\alpha}{n^j(n-\alpha)} \tag{3.26}$$

for any $n \ge 2$.

Proof. Since $f(z) \in T_\lambda(j,\alpha,1)$, from (3.7), we have

$$\sum_{n=2}^{\infty}(a_n+b_n)z^n = -\left[(1+\lambda)z + \sum_{n=2}^{\infty}(\lambda a_n - b_n)z^n\right]\left(\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.27}$$

Eqũating the coefficients of z^n ($n \ge 2$) on both sides of (3.27) we get

$$a_2 + b_2 = -(1+\lambda)c_1 \text{ and}$$

$$a_n + b_n = -\left[(1+\lambda)c_{n-1} + (\lambda a_2 - b_2)c_{n-2} + \ldots\right.$$

$$\left. + (\lambda a_{n-1} - b_{n-1})c_1\right], \ (n \ge 3). \tag{3.28}$$

From (3.27) and (3.28) we obtain

$$\sum_{k=2}^{n}(a_k+b_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k = -\left[(1+\lambda)z + \sum_{k=2}^{n-1}(\lambda a_k - b_k)z^k\right]w(z), \tag{3.29}$$

where $\sum_{k=n+1}^{\infty} d_k z^k$ converges in U. Using $|w(z)| < 1$ for $z \in U$ and Parseval's identity [4] on both sides of (3.29), we obtain

$$\sum_{k=2}^{n} |a_k+b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k}$$
$$\leq (1+\lambda)^2 r^2 + \sum_{k=2}^{n-1} |\lambda a_k - b_k|^2 r^{2k}. \tag{3.30}$$

Since (3.30) holds for all r in the interval $0 < r < 1$, it follows that

$$\sum_{k=2}^{n} |a_k+b_k|^2 \leq (1+\lambda)^2 + \sum_{k=2}^{n-1} |\lambda a_k - b_k|^2 \tag{3.31}$$

and from (3.31) it follows that

$$|a_n+b_n|^2 \leq (1+\lambda)^2 - (1-\lambda^2)\sum_{k=2}^{n-1} |a_k|^2 - 2(1+\lambda)\sum_{k=2}^{n-1} \text{Re}(a_k)b_k. \tag{3.32}$$

Since $b_n \geq 0$ for all $n \geq 2$ and $\text{Re}(a_k) \geq 0$ ($k = 2,3,\ldots,(n-1)$), it follows that

$$|a_n + b_n| \leq (1+\lambda). \tag{3.33}$$

Hence, by using (3.25) and (3.33), we obtain

$$|a_n| \leq |a_n+b_n| + |b_n| \leq 1+\lambda+\frac{1-\alpha}{n^j(n-\alpha)} \quad (n \geq 2). \tag{3.34}$$

This completes the proof of Theorem 6.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_\lambda(j,0,1)$ *and* $\text{Re}(a_k) \geq 0$ ($k = 2,3,\ldots,(n-1)$), *then*

$$|a_n| \leq 1+\lambda+\frac{1}{n^{j+1}} \quad (n \geq 2). \tag{3.35}$$

Remark 4. Since $\dfrac{1-\alpha}{n^j(n-\alpha)}$ is decreasing on $n(n \geq 2)$, Theorem 6 gives

$$|a_n| \leq 1+\lambda+\frac{1-\alpha}{2^j(2-\alpha)} \quad (n \geq 2) \tag{3.36}$$

for $f(z) \in T_\lambda(j,\alpha,1)$ satisfying $\text{Re}(a_k) \geq 0$ ($k = 2,3,\ldots,(n-1)$).

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