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SOME ASPECTS OF GRAPHS PLANARITY

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REZUMAT: - Câteva aspecte ale planarităţii grafelor. Prezentăm un algoritm de testare a planarităţii grafelor hamiltonienc. Slgoritmul este dat şi pentru testarea grafelor 4-convexe şi 3-convexe.

1. Introduction. The aim of this paper is to present an algorithm (and its Pascal language version) for testing the planarity of hamiltonian graphs. For a certain planar hamiltonian graph we will build a planar representation of it.

In the first part some theoretical results will be presented. These results will lead us to the fundamental idea applied in the algorithm. As some known results show us, the algorithm may be used in order to test the planarity of 4-connected graphs and 3-connected graphs with at most an articulation set having the cardinal 3.

Among different theoretical results, the notion of bridge of a subgraph is widely used. Thus, the concept of overlapped bridges plays an important role for the theory of plane representation of graphs [3].

The bridges were very used for the investigation of planar graph cycles. Important results have been obtained by Tutte, Thomasson, Nelson.

The concept of bridge was also successfully used for studying the properties of graphs with respect to connectedity.

In the scientific literature the "bridge method" is used as a method to prove different graph theory theorems.

2. Basic concepts. In this paper we will only talk about finite, undirected graphs, with no loops and with no multiple edges.

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Let us denote MxM by $M^{(2)}$, and let G=(V,E) such a graph. The graph G'=(V',E') is a subgraph of the graph G=(V,E) if $V'\subseteq V$ and $E'\subseteq E$. Thus, we will write $G'\subseteq G$. We remark that $M\subset N$ means $M\subseteq N$ and $M\neq N$, M and N being finite sets.

If $W\subseteq V$, then the graph $(W,E\cap W^{(2)})$ is a subgraph of G=(V,E), namely the subgraph induced by W. We denote this subgraph by G(W).

If $E'\subseteq E(G)$, then G-E' denotes the graph produced from G by eliminating the edges from E'. So, G-E'=(V(G),E(G)-E').

If $W\subseteq V(G)$, then G-W is the graph produced from G by eliminating the vertices from W (obviously, if x is eliminated from W, so will be the edges incident to x). So, $G-W=(V-W,E\cap(V-W)^{(2)})$.

If W={x} we write G-x instead of G-{x}. Similary, if $E'=\{e\}\subseteq E$ we write G-e instead of G-{e}.

If $H\subseteq G$ (H is a subgraph of G) we may write G-H instead of G-V(H).

If $e\in V^{(2)}-E$, then the graph produced from G by adding the edge e is $G\cup\{e\}$ or $G\cup e$.

Let x and y be two vertices of G, not necessarily distinct. By x,y-chain we mean an alternant sequence of vertices and edges $x_1,e_1,x_2,e_2,\ldots,x_k,e_k,x_{k+1}$, where $x=x_1$, $y=x_{k+1}$ and $e_i=(x_i, x_{i+1})\in E(G)$, $1\le i\le k$. The x,y-chain may also be denoted by $W=x_1x_2\ldots x_{k+1}$, or $W=[x_1,x_2,\ldots,x_{k+1}]$.

The chain above is an elementary x,y-chain if all its vertices are distinct. The elementary x,x-chain has the length equal to zero and is made up only by the vertex x.

In what follows, when we will say cycle, we will mean an elementary cycle.

If $\mu=x_1\ldots x_k$ is an elementary chain, $P=x_i$, $Q=x_j$ and $1\le i\le j\le k$, then the P,Q-segment of μ is the elementary P,Q-chain $x_i\ldots x_j$, and we will approach it as a partial chain of μ, with the limits P and Q. This segment will be denoted by $\mu[P,Q]$, and $\mu[P,Q]-\{P,Q\}$ will be denoted by $\mu(P,Q)$.

3. The bridges of a subgraph H. Let H be a subgraph of the graph G=(V(G),E(G)).

Definition 3.1 An H-bridge in G is a subgraph of G-E(H) which is either an edge (with its limits), linking the two vertices of H, or a connected component K of G-H, with all the edges (and their limits) of G incident to K.

The H-bridges from the former category will be named singular or diagonal bridges of H, and the others (non-singular) will be named regular bridges.

In the figure 3.1, B_i, $i \in \{1,2,3,4\}$, are regular H-bridges, and B_5 is a singular H-bridge, where H is the cycle represented by the dotted line.

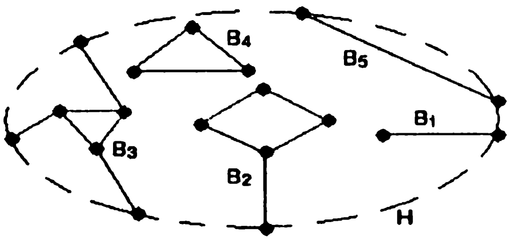

Figura 3.1

Definition 3.2 If B is an H-bridge, the vertices from $V(B) \cap V(H)$ are named supporting vertices of B, and the vertices of $V(B) \setminus V(H)$ are named inner vertices of B.

We notice that the H-bridges B_i, $i=1,\ldots,5$ from figure 3.1 have 1,1,3,0,2 supporting vertices respectively, and 1,4,3,3,0 inner vertices respectively.

Definition 3.3 The kernel of an H-bridge is the subgraph induced by the inner vertices of B.

We remark that the kernel of a singular bridge is the empty graph.

Lemma 3.1 If B_1 and B_2 are two H-bridges, $B_1 \neq B_2$, then the kernels of the two bridges do not have common vertices, i.e.

$$(V(B_1) \setminus V(B_2)) \cap (V(B_2) \setminus V(B_1)) = \varnothing.$$

Proof: If B_1 or B_2 are singular, the lemma holds because in the former case $V(B_1) \setminus V(B_2) = \varnothing$ and in the latter $V(B_1) \setminus V(B_2) = \varnothing$.

Let us suppose that B_i is a regular H-bridge, $i=1,2$, and that the kernels of the two bridges have at least a common vertex. From the definition 3.1, the kernel of B_i is a connected component K_i of G-H, $i=1,2$. Then it would result that $K_1=K_2$, and, moreover, $B_1=B_2$. This contradicts the hypothesis and concludes the proof.

Remark 3.1 Two H-bridges may have in common only supporting vertices.

4. Overlap graphs and circle graphs. In what follows we will consider only the bridges of the cycles.

Let B be a bridge of a cycle C in the graph G, having the supporting vertices $a_1,\dots,a_s$, $s\geq 2$, which are on C in this cyclic order. The s segments of C, denoted by $C_B[a_i,a_{i+1}]$, $1\leq i\leq s-1$ and $C_B[a_s,a_1]$, are named segments of C with respect to the C-bridge B.

Definition 4.1 Let B and B' be two C-bridges in the graph G. B and B' are parallel if and only if there exist two vertices x and y on C so that all the supporting vertices of B are included in the segment C[x,y] and all the supporting vertices of B' are included in the segment C[y,x].

In figure 4.1, the C-bridges B_1 and B_3 are parallel, where C is the cycle represented by the dotted line.

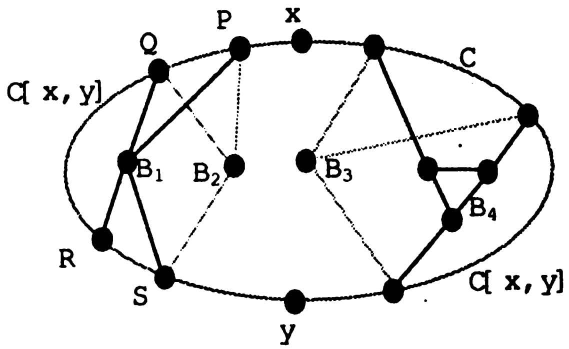

Figura 4.1

Remark 4.1 If B or B' has at most a supporting vertex, then B and B' are parallel.

Remark 4.2 Let us suppose that the C-bridge B has at least two supporting vertices. These vertices subdivide C into segments with respect to B. Then, B and B' are parallel if and only if either B' does not have supporting vertices or there exists a segment of C with respect to B which includes all the supporting vertices of B'.

Definition 4.2 Two C-bridges B and B' overlap if and only if B and B' are not parallel.

The pairs (B_1, B_2) and (B_3,B_4) from figure 4.1 are examples of overlapping C-bridges.

Definition 4.3 Two C-bridges B and B' are crossed if and only if there exist four vertices P, Q, R, S in this cyclic order on the cycle C, so that P and R are in V(B) and Q and S are in V(R').

An example of crossed bridges is the pair (B_1, B_2) of C-bridges from figure 4.1.

Lemma 4.1 Let C be a cycle of the graph G and B and B' two C-bridges. The following statements are equivalent:

(1) B and B' overlap;

(2) B and B' are crossed or they have exactly three supporting vertices each, and those are identical.

Proof. (1) ==> (2)

Let $a_1, \ldots, a_s$, s>=2 the supporting vertices of B_1, on this cyclic order on C (because B and B' overlap, each of them has at least two supporting vertices). We denote $a_{s+1}=a_1$. Let $C_B[a_i,a_{i+1}]$, 1<=i<=s the segments of C with respect to the C-bridge B.

Case (a): Let us suppose that there exists a number k in {1,...,s} so that $V(B')\cap(C_B(a_k,a_{k+1})\neq\emptyset$.

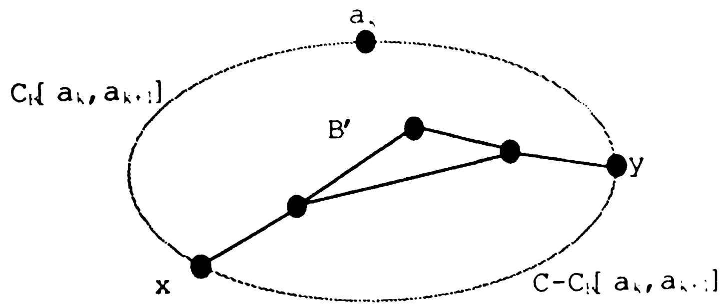

Figura 4.2

Let $x\in V(B')\cap(C_B(a_k,a_{k+1})$. Because B and B' overlap, there exists a supporting vertex y of B' in $C-C_B[a_k,a_{k+1}]$ (otherwise, all the supporting vertices of B' would be in $C_B[a_k, a_{k+1}]$, and thus B and B' would be parallel). Then, a_k, x, a_{k+1}, y are on C in this cyclic order (see fig. 4.2) and a_k, $a_{k+1} \in V(B)$ and $x,y \in V(B')$. From definition 4.3 it results that B and B' are crossed.

Case (b): Let us suppose that there is no k in {1,...,s} so that $V(B')\cap(C_B(a_k,a_{k+1})\neq\emptyset$, i.e. for every k in {1,...,s}, $V(B')\cap(C_B(a_k,a_{k+1})=\emptyset$, Let us denote by S_B the set of supporting vertices of B and by $S_{B'}$ the set of supporting vertices of B'. From our supposition we have $S_{B'}\subseteq S_B$.

If $S_{B'} \subset S_B$ then there exists an l in {1,...,s} so that $a_l \notin S_{B'}$. With no lack of generality we may take l = 1. Then let i be the lowest index for which $a_i \in S_{B'}$, and let j be the greatest index for which $a_j \in S_{B'}$, with i and j in {2,...,s}. Thus, there exists a segment of C with respect to B', namely $C_{B'}[a_j,a_i]$, that fulfills $V(B)\cap(C_{B'}(a_k,a_{k+1})\neq\emptyset$, and we are in the case (a). So, B and B' are crossed.

Let us now suppose that $S_B=S_{B'}$. Let us denote $p = |S_B|$. If p=2 then B and B' are parallel. It results p >= 3. If p=3 we may apply the lemma. If p >= 4, then B and B' are crossed.

(2) ==> (1) Let us firstly consider the case when B and B' have exactly three supporting vertices each, and these are identical.

It is obvious that there exists no segment of C with respect to B that should contain all the supporting vertices of B'. From the remark 4.2 we deduce that B and B' are not parallel, so B and B' overlap.

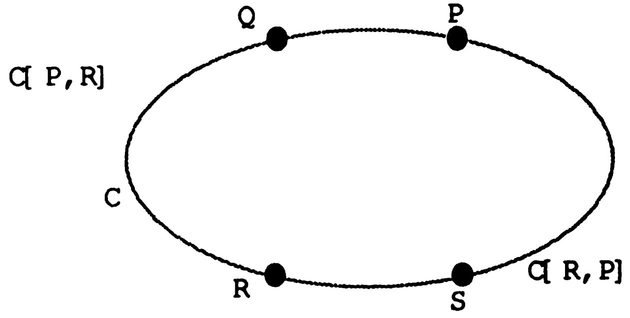

Figura 4.3

In the other case B and B' are crossed, i.e. there exist four vertices P,Q,R,S in that cyclic order on C so that P and R are in V(B) and Q and S are in V(R') (fig. 4.3). Let us suppose that B and B' are parallel. So, there exists a segment of C with respect to B that contains all the supporting vertices of B'. Let $C_B[x,y]$ be this segment.

So, $Q,S \in C_B[x,y]$. It results that $C[Q,S] \subseteq C_B[x,y]$ or $C[S,Q] \subseteq C_B[x,y]$. But, C[Q,S] contains the vertex R of B and C[S,Q] contains the vertex P of B. So, between the supporting vertices of B there exists at least a supporting vertex of B, and this contradicts the definition of the segments of C with respect to the C-bridge B.

Definition 4.4 Let C be a cycle of the graph G. We consider the C-bridges as the vertices of a new graph O(G:C), that will be called overlap graph of G with respect to C.

There exists an edge between two vertices B and B' of O(G:C) if and only if B and B' overlap.

Definition 4.5 If C is a hamiltonian cycle, the graph O(G:C) is called overlap graph or circle graph of G with respect to G.

Remark 4.3 If C is a hamiltonian cycle all the C-bridges are diagonals of C.

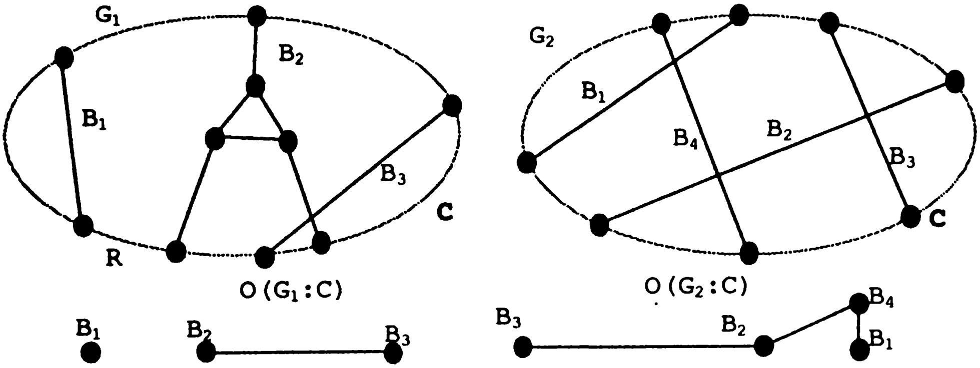

Figura 4.4

In fig. 4.4 are represented two graphs with their overlap graphs. The overlap graph $O(G_2{:}C)$ is an example of circle graph.

Let us consider a hamiltonian graph G and a hamiltonian cycle C of it. Let G' be a geometrical representation of G in the plane such that C is represented by a simple closed Jordan curve C' and all the diagonals of C are represented by simple Jordan curves situated in the finite region of C'. The number of the intersection points of the Jordan curves that represent the diagonals of C equals the number of edges in the circle graph O(G:C). If C' is chosen as a geometrical circle, and its diagonals are represented by straight line segments, then we have a second definition of a circle graph:

Definition 4.6 The vertices of a circle graph are chords of the geometrical circle, and two chords are linked by an edge if and only if they intersect in a point interior to the circle.

4.1 Parallel bridges.

Lemma 4.1.1 Let C be a cycle of the graph G and B, B' two C-bridges. If B and B' are parallel then $|V(B) \cap V(B')| <= 2$ and the common vertices, if they exist, are supporting vertices.

Proof: From the remark 3.1, two C-bridges may have in common only supporting vertices. If $|V(B) \cap V(C)| <= 1$ or $|V(B') \cap V(C)| <= 1$, the statement in the lemma is true.

Now we suppose that both B and B' have at least two supporting vertices. From remark 4.2 it results that there exists a segment of C with respect to B that contains all the supporting vertices of B'. Let this segment be $C_B[x,y]$, where $x, y \in S_B$. Then, we have $V(B) \cap V(B') \subsetneq \{x,y\}$ and so $|V(B) \cap V(B')| <= 2$ and that concludes the proof.

Proposition 4.1.1 Let C be a cycle of the graph G and B and B' two diagonals of C. Obviously, $|V(B) \cap V(B')| <= 1$. If $|V(B) \cap V(B')| = 1$, B and B' are parallel.

Proposition 4.1.2 Let C be a cycle of the graph G and B and B' two diagonals of C. If B and B' are crossed

then $|V(B) \cap V(B')| = 0$.

Because $|V(B)|=|V(B')|=2$, the statements above are (almost) obvious.

We state the following remark:

Remark 4.1.1 Let us denote by $\mathcal{B}$ the set of C-bridges from a graph G. We define the binary relation r as:

B r B' <==> B is parallel with B', for all B and B' in $\mathcal{B}$

The relation r is not an equivalence relation on $\mathcal{B}$. Even if r is reflexive and symmetrical, it is not transitive.

As we see in fig. 4.1.1., B_1 r B_2 and B_2 r B_3, but B_1 r B_3 fails (B_1 and B_3 are crossed C-bridges).

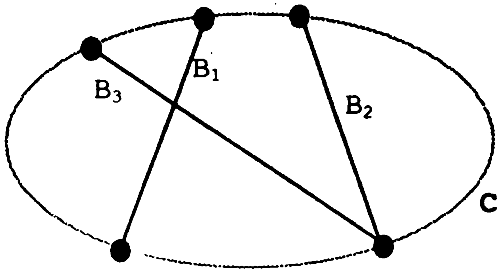

Figura 4.1.1.

4.2 Connected overlap graphs. We know that any C-bridge with at most a supporting vertex is parallel with any other bridge. This implies that any bridge with this property is an isolated vertex of the overlap graph O(G:C). Consequently, if O(G:C) is connected with |V(O(G:C))| >= 2, then every C-bridge has at least two supporting vertices.

Lemma 4.2.1 Let C be a cycle of the graph G, K a connected subgraph of O(G:C) and x and y two vertices of C. Let us suppose that K has a supporting vertex on each of the segments C(x,y) and C(y,x) (1). Then there exists a bridge B of K with a supporting vertex on each of the segments (2).

Remark A rigorous statement would be:

(1) C-bridges from G, considered as vertices of K, have supporting vertices on both of the segments C(x,y) and C(Y,x).

(2) There exists a C-bridge B, $B \subseteq G$, $B \in V(K)$, with at least a vertex on each of the segments above.

In what follows we will use for simplicity, the same style of statement.

Proof of the Lemma 4.2.1: Let a', a" be the supporting vertices of K on C(x,y) and C(y,x). Let us suppose that a' and a" are supporting vertices of the C-bridges B' and B", where B' and B" are from V(K). But, K is connected, so there exists an elementary B',B"-chain [$B'=B_1,\dots,B_q=B''$] in O(G:C). Because B" has a supporting vertex on C(y,x), there exists an index i such that B_i has a supporting vertex in C(y, x). Let us denote by i the smallest index with the above property.

If i=1 then with B := B' the lemma is true.

If i>1 then B_{i-1} has no supporting vertex in C(y,x). Because B_{i-1} and B_i overlap (they are succesive vertices in the elementary B',B"-chain from O(G:C)), there results that B_i has a supporting vertex in C(y,x) (otherwise B_i and B_{i-1} are parallel). With B := B_i, the lemma is true.

For sets K of C-bridges (or subgraphs of O(G:C)), we define the segments of C with respect to K.

Let K be a set of C-bridges with at least two distinct supporting vertices, $a_1, \dots, a_s$, s >= 2, on C in this cyclic order. Then the s segments denoted $C_K[a_i, a_{i+1}]$, 1<=i<=s-1 and $C_K[a_s,a_1]$ are called the segments of C with respect to K.

Let K and K' be two disjoint sets of C-bridges.

K and K' are parallel if and only if either one of them has at most a supporting vertex, or both of them have at least two supporting vertices, and a segment of C with respect to K contains the set of supporting vertices of K'.

K and K' overlap if and only if they are not parallel.

K and K' are crossed if and only if there exist four vertices P, Q, R, S in this cyclic order on C, so that P and R are in K and Q and S are in K'.

Lemma 4.2.2 Let C be a cycle of the graph G. If K and K' are disjoint sets of C-bridges, the following statements are equivalent:

(i) K and K' overlap;

(ii) K and K' are crossed or they have exactly three supporting vertices each, and these coincide.

The proof is analogous to that given for the lemma 4.1.

Theorem 4.2.1 Let C be a cycle of the graph G. Let K and K' be two connected subgraphs of O(G:C), not connected by any edge. Then, K and K' are parallel.

Proof. We prove the theorem by reduction to absurd. We suppose that K and K' overlap. From the lemma 4.2.2 we have two cases:

(i) K and K' are crossed;

(ii) K and K' have exactly three supproting vertices each, and those coincide.

Case (i): K and K' are crossed. Then from definition, there exist four vertices P, Q, R, S in this cyclic order on C, so that P and R are in K and Q and S are in K'. From lemma 4.2.1 there exists a bridge B in K having a vertex P' on C(S,Q) and a vertex R' on C(Q,S) (see fig. 4.2.1).

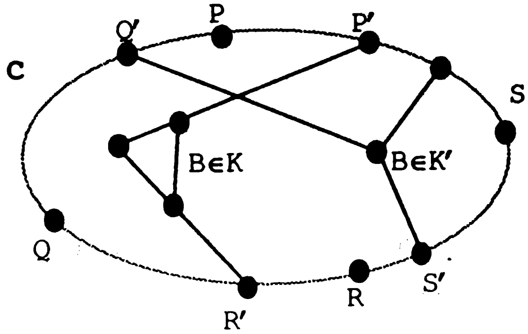

Figura 4.2.1.

From here it results that K' has the vertex Q on C(P',R'), and the vertex S on C(R',P'). From lemma 4.2.1 there exists a bridge B' in K' having a vertex Q' on C(P',R') and a vertex S' on C(R',P'). So, P',Q',R',S' are four vertices in this cyclic order on C, with P' and R' in B and Q' and S' in B'. It follows that the C-bridges B and B' are crossed, and so they overlap. Then, (B,B') is an edge that connects K with K' in O(G:C), and this contradicts the hypothesis.

Case (ii): K and K' have the same supporting set with three distinct vertices, x, y and z.

If K is an isolated vertex B of O(G:C), then x, y and z will be the supporting vertices of B.

If K has q >= 2 C-bridges $B_1, \ldots, B_q$, because K is connected every bridge B in K has the property that there exists a B^* in K, $B^* \neq B$, so that B and B^* overlap. But they cannot be crossed, so they have the same set of three supporting vertices, namely {x, y, z}.

Using the same logic for K', we have proved that all the bridges from K∪K'have the same set of three supporting vertices, {x, y, z}. But then the subgraph of O(G:C) induced by the bridges from K∪K' is complete. There results that K and K' are connected by an edge of O(G:C), and that contradicts the hypothesis.

In order to give a characterization of connected overlap graphs, we need the following definition:

Definition 4.2.1 A 2-separation (G_1, G_2) of the graph G contains two subgraphs G_1 and G_2 of G having at least three vertices and fulfilling the following conditions:

(i) $G = G_1 \cup G_2$;

(ii) $|V(G_1) \cap V(G_2)| = 2$;

(iii) $|E(G_1) \cap E(G_2)| = 0$.

Theorem 4.2.2 Let C be a cycle of the graph G. Then the overlap graph O(G:C) is not connected if and only if there exists a 2-separation (G_1, G_2) of G so that:

(i) $\{x,y\} := V(G_1) \cap V(G_2) \subseteq V(C)$;

(ii) $C[x,y] \subseteq G_1$, $C[y,x] \subseteq G_2$;

(iii) neither G_1, nor G_2 is a segment of C.

Proof: (<==) We suppose that there exists a 2-separation (G_1, G_2) of G satisfying the properites (i), (ii), (iii). Because the subgraph G_i is not a segment of C, with i=1,2, there results that G_i contains a C-bridge B_i, i=1,2.

Let us suppose that

$$S_{B1} \cap V(C(x,y)) = \varnothing. \tag{*}$$

Then $S_{B1} \subseteq \{x,y\}$. Let us prove this. If B_1 is singular, this is obvious. Otherwise, B_1 is a regular C-bridge. Let us suppose that there exists u in S_{B1} such that

$$u \notin \{x,y\} \tag{**}$$

From (*) and (**) we have that $u \in V(C(x,y))$. But, from (22) we have $C[x,y] \subseteq G_2$. So, $u \in V(G_1)$. Let v be an interior vertex of B_1. Obviously, $v \in V(G_1)$. From (1) we have that $v \notin V(G_2)$. We have obtained so far that

$$v \in V(G_1) \text{ and } v \notin V(G_2)$$

$$u \notin V(G_1) \text{ and } u \in V(G_2)$$

Since B_1 is connected there exists an elementary v,u-chain in B_1. From the definition of the 2-separation, $|E(G_1) \cap E(G_2)| = 0$. So, the edges of the elementary v,u-chain are either edges of G_1, or edges of G_2. In both of the cases there exists an edge in $E(G_i)$ (of the elementary chain) with the property that an extremity of it is not in $V(G_i)$, i=1,2, and that's impossible. So, $S_{B1} \subseteq \{x,y\}$. But then B_1 is parallel with any other C-bridge in G, so it is an isolated vertex of O(G:C). From the hypothesis (*), the statement to be proven is valid.

Let us now suppose that

$$S_{B2} \cap V(C(x,y)) = \varnothing.$$

By rationing in the same way we will conclude that O(G:C) is not connected.

Let us now suppose that

$$S_{B1} \cap V(C(x,y)) \neq \varnothing,$$

and

$$S_{B2} \cap V(C(x,y)) \neq \varnothing.$$

If B_1 and B_2 are members of the same component K of O(G:C), then from lemma 4.2.1 it results that the component K contains a C-bridge having at least a supporting vertex on each of the segments C(x,y) and C(x,y). Similarly we have that there exists an elementary chain with an extremity in $V(G_1)$ - $\{x,y\}$ and the other extremity in $V(G_2)$ - $\{x,y\}$, with the edges are either edges of G_1, or edges of G_2, and that is impossible.

Consequently, B_1 and B_2 are in different connected compenents of O(G:C), and so O(G:C) is not connected.

(==>)Let us suppose that O(G:C) is not connected. We consider two cases:

(a) G contains a C-bridge B with at most a supporting vertex. Then B is a component of O(G:C), namely an isolated vertex. (B is parallel with any other C-bridge).

Let us consider that B does not have supporting vertices. Let us denote

$$G_1=(V(B),E(B)\cup\{(x,y)\})$$

and

$$G_2=G-\{(x,y)\}-B,$$

where (x,y) is an edge of the cycle C. Then, (G_1,G_2) is a 2-separation of G that satisfies (1), (2), (3).

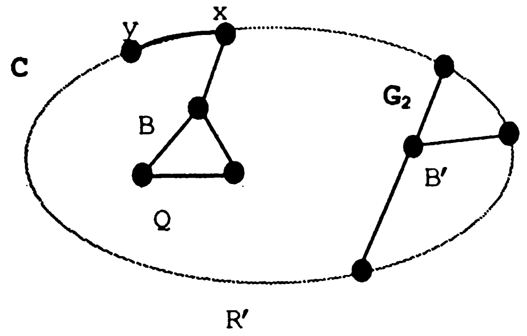

Figura 4.2.2.

Remark: The property (3), especially the fact that G_2 is not a segment of C, is less obvious. But O(G:C) is not connected, B being an isolated vertex in O(G:C). So, there exists a C-bridge B' in G, B'≠B. From definition of G_2 we have that B' is a subgraph of G_2, and so G_2 is not a segment of C (see figure 4.2.2).

If B has a single supporting vertex, namely a, we chose an edge (x,y) of C so that x=a.

Let us denote

$$G_1=(V(B)\cup\{x,y\},E(B)\cup\{(x,y)\})$$

and

$$G_2=G-\{(x,y)\}-(B-\{x\}).$$

Then, (G_1, G_2) is a 2-separation of G satisfying (1),(2),(3).

(b) G contains only C-bridges with at least two supporting vertices. Let us suppose that O(G:C) has p connected components, with p >= 2. Let these be $K_1, ..., K_p$. From Theorem 4.2.1 we deduce that K_1 and K_2 are parallel. There results that all the supporting vertices of K_2 are on a segment of C with respect of K_1. Let this segment be $C_{K1}[x,y]$.

Case (1). $(x,y) \in E(G)$. We denote with B the diagonal of C determined by (x,y).

(1.a) B is in K_1. Since K_1 is a connected component of O(G:C), we have that $K_1=\{B\}$, seen as a set of C-bridges, or K_1 is an isolated vertex (B) in O(G:C). So, we chose $G_1=C[x,y] \cup \{ K_i \mid K_i$ has all the supporting vertices on $C[x,y]$, $i=2,...,p\}$ and $G_2=G(A)$, where $A=(V(G)\setminus V(G_1))\cup\{x,y\}$. Thus, we assured that in G_2 there exists at least a C-bridge B.

(1.b) B is not in K_1. We will show that B is an isolated vertex in O(G:C). We will supose that there exists a C-bridge B_1 so that B_1 and B are crossed. Let x_1, y_1 supporting vertices of B_1, with $x_1 \in C(x,y)$ and $y_1 \in C(y,x)$. But, then $x \in C(y_1,x_1)$ and $y \in C(x_1,y_1)$. From the lemma 4.2.1 there exists a bridge B_2 of K_1 having at least a supporting vertex on each of the segments $C(x_1,y_1)$, $C(y_1,x_1)$. So, B_1 and B_2 are crossed, and thus B_1 is in K_1, and x_1 is supporting vertex of K_1 on the segment $C_{k1}[x,y]$. This is a contradiction, and from here we have that B is parallel with any other C-bridge, and thus it is an isolated vertex in O(G:C). We chose $G_1=C[x,y] \cup \{ K_i \mid K_i$ has all the supporting vertices on $C[x,y]$, $i=1,...,p\}$ and $G_2=G(A)\setminus(x,y)$, where $A=(V(G)\setminus V(G_1))\cup\{x,y\}$. Case (2). $(x,y) \notin E(G)$. We chose G_1 the same way as in the case (1.b) and $G_2=G(A)$, where $A=(V(G)\setminus V(G_1))\cup\{x,y\}$.

Remark. We denote by S_{ki} the set of supporting vertices of the component K_1, $i=1,...,p$. From the lemma 4.2.1 we have that $S_{Ki} \subseteq C[x,y]$ or $S_{Ki} \subsetneq C[y,x]$, $i=1,...,p$. This assures that $G=G_1\cup G_2$ (if $K_i \notin G_1$, then it is certain that $K_i \in G_2$, $i=1,...,p$).

5. A planarity criterion for hamiltonian graphs. Any graph G=(V,E) may be represented in a plane in the following way:

(1) For each vertex x in V a point $\phi(x)$ of the plane is assigned, such that distinct points of the plane are assigned to distinct vertices of the graph.

(2) For each edge e-(x,y) a Jordan curve $\phi(e)$ of the plane is assigned, with the limits $\phi(x)$ and $\phi(y)$ such that no interior point $\phi(e)$ of the Jordan curve is the image of a vertex of

G, and two distinct Jordan curves $\phi(e)$ and $\phi(e')$, $e \neq e'$, with e,e' in E, have at most a single common point, and three distinct Jordan curves, $\phi(e)$, $\phi(e')$ and $\phi(e'')$, $e \neq e' \neq e'' \neq e$, with e, e', e'' in E, does not intersect in a common interior point.

An interior point common for two Jordan curves is called intersection vertex of G. The image of G obtained in this way is called the planar representation of G. If G has a planar representation with no intersection points, G is called a planar graph, and its representation is called plane representation, or plane graph.

Let G be a planar graph. We suppose that G contains a cycle C with two overlapped C-bridges, B_1 and B_2. Then, the bridges B_1 and B_2 may not be represented either both inside C, or both outside C, without having some intersection points. Thus, it results that a bridge is represented inside C, and the other outside C. Consequently, in G the interior vertices of B_1 and the interior vertices of B_2 are separated by the cycle C, if B_1 and B_2 are regular bridges.

Remark 5.1 In a plane 2-connected graph, the border of any face is identical with its contour. A hamiltonian graph obviously is 2-connected (it has no articulation points).

Lema 5.1 Let C be a cycle of the graph G, so that O(G:C) has no edge, and $C \cup B$ is planar for each C-bridge B. Then, G has a plane representation so that C is the contour of the infinite face.

Proof. We will do an induction on the number of C-bridges. Let G be a graph which has a cycle C and only one C-bridge B, so that $C \cup B$ is planar (obviously, O(G:C) has no edge). Then, $C \cup B$ is planar and there exists a plane representation of G such that C is the contour of the infinite face.

Let us suppose that the lemma is true for any graph G and any cycle C in G which has k C-bridges, with $1 <= k <= m$. Let G' a graph and C' a cycle of G' having m+1 C'-bridges with the property that $C' \cup B'$ is planar for each C'-bridge B', and O(G':C') has no edge. Since O(G':C') is not connected, from the theorem 4.2.2 there exists a 2-separation (G_1, G_2) of G' such that:

(1) $\{x,y\} = V(G_1) \cap V(G_2) \subseteq V(C')$;

(2) $C'[x,y] \subseteq G_1$ and $C'[y,x] \subseteq G_2$;

(3) neither G_1 nor G_2 is a segment of C'.

From (3) we have that G_i contains at least a C'-bridge, i=1,2. (*). As we saw in the proof of the Theorem 4.2.2, for every C'-bridge B_1 in G_1 and for every C'-bridge B_2 in G_2, we have that $S_{B1} \subseteq C'[x,y]$, $S_{B2} \subseteq C'[y,x]$.

We know that $C_1 := C'[x,y] \cup (x,y)$ is a cycle of $G_1 \cup (x,y)$, and $C_2 := C'[y,x] \cup (y,x)$ is a cycle of $G_2 \cup (y,x)$. Each C'-bridge in G' is either a C_1-bridge in $G_1 \cup (x,y)$, or a C_2-bridge in $G_2 \cup (y,x)$ (see fig. 5.1). (**)

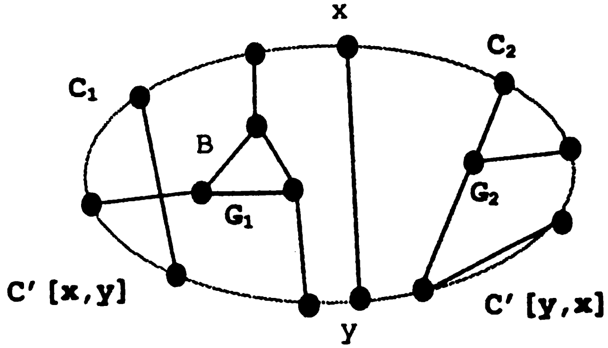

Figura 5.1

From (*) and (**) it results that the cycle C_i has at least a C_i-bridge, i=1,2. So, the cycles C_i have at least m bridges, i=1,2.

We denote $G_1^0 = G_1 \cup (x,y)$ and $G_2^0 = G_2 \cup (y,x)$. For each C_1-bridge B_1^0 from G_1^0, we have that $C' \cup B_1^0$ is planar (from the hypothesis), so $C_1 \cup B_1^0$ is planar.

Since O(G':C') has no edge, we have that $O(G_1^0{:}C_1)$ has no edge.

Similarly, $C_2 \cup B_2^0$ is planar for each C_2-bridge B_2^0 from G_2^0, and $O(G_2^0{:}C_2)$ has no edge.

From the induction hypothesis, we have that G_i^0 has a plane representation so that C_i is the contour of the infinite face, i=1,2.

Let $\underline{G_i^0}$ be the plane representation of G_i^0, i=1,2.

In order to obtain a plane representation of G' with no intersection points, we represent G_2^0 in the infinite face of $\underline{G_1^0}$ and then we erase the curve $\phi(e)$ that represents the edge e=(x,y)

(see fig. 5.2).

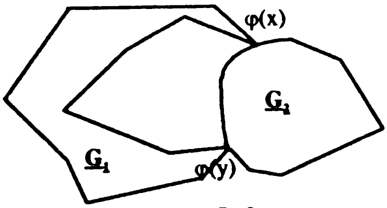

Figura 5.2.

Lemma 5.2 Let C be a cycle of the 2-connected graph G so that O(G:C) is bipartit and C∪B is planar for each C-bridge B. Then G is a planar graph.

In order to prove this lemma we mention here the following theorem, presented in [3]:

Theorem 5.1 If J and J' are closed polygonal lines from R^2, there exists a homeomorphisn ϕ: $R^2 \to R^2$ so that $\phi(J) = J'$ and the image through ϕ of a segment is also a segment.

Proof of Lemma 5.2. We will use the Theorem 5.1, with the aim of placing the plane representation of some planar bridges into a well-determined region of R^2. The fact that the image through the homeomorphism is also a planar graph may be deduced from the injectivity of the homeomorphism.

We denote by $M_1 \cup M_2 = V(O(G:C))$ the parts of the bipartit graph O(G:C). Let G_i be made by the cycle C and the bridges from the set M_i, i=1,2. Since it does not exist any edge between the vertices from M_i of O(G:C), any bridge of M_i is parallel with any other bridge of M_i. So, $O(G_i:C)$ has no edge, i=1,2. From the Lemma 5.1, the graph G_1 has a plane representation $\underline{G_1'}$ so that C is the contour of the infinite face. Is known that there exists a plane representation $\underline{G}_1$ of G_1 where any edge is represented by a line segment, C being the contour of the infinite face. (*)

Let B_1, ..., B_p the C-bridges of M_2. We know that $C \cup B_1$ is plane, and from the definition of C-bridge, and having in mind that G is 2-connected, C is the contour of a certain face in any plane representation of $C \cup B_1$. We represent B_1 in the exterior of the representation of the cycle C.

Let us suppose that we have represented B_i, $1 <= i <= k-1 <= p-1$, in the exterior of the representation of C, with no intersection points. Let us prove that $C \cup B_1 \cup B_2 \cup ... \cup B_k$ is planar.

Let us first show that $C \cup B_1 \cup B_2 \cup ... \cup B_k$ is 2-connected. We suppose that $C \cup B_1 \cup B_2 \cup ... \cup B_k$ is not 2-connected. There results that this graph has at least an articulation point. We have two cases:

(a) There exists an articulation point x on the cycle C. There results that there exists l, 1 <= l <= k-1, so that B_l has only one supporting vertex, namely x. But then x is an articulation point in G, which leads to a contradiction.

(b) There exists an articulation point x in the kernels of the C-bridges B_i, i = 1, ..., k-1. From the lemma 3.1 the kernels of the C-bridges B_i, i = 1, ..., k-1 have no common vertices, and thus x is a member of only one C-bridge. By eliminating this articulation point we do not affect the other C-bridges, and the cycle C. So, it is an articulation point in G, contradiction.

Let $K=\{B_1,...,B_{k-1}\}$ the set of C-bridges already represented in the plane, with no intersection vertices. Since B_k is parallel with any other bridge in K, there exists $C_k[x,y]$ (a segment of C with respect to K, with the limits x and y) containing all the supporting vertices of B_k.

From the hypothesis of the lemma we have that $C \cup B_k$ is planar. From lemma 5.1, $C \cup B_k$ has a plane representation such that C is the contour of the infinite face.

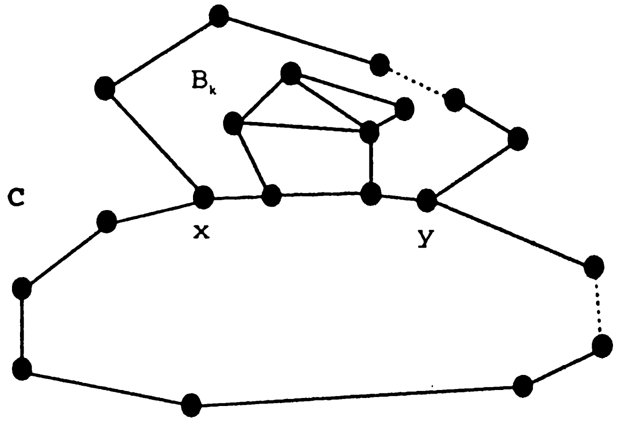

Figura 5.3.

Because $C \cup B_1 \cup ... \cup B_{k-1}$ is 2-connected planar, we may conclude that the border of the finite face that touches $C_k[x,y]$ is a cycle. So, from the theorem 5.1 we may represent B_k in the finite face of $C \cup B_1 \cup ... \cup B_{k-1}$ which touches $C_k[x,y]$, without having some intersection points (see fig. 5.3), and thus $C \cup B_1 \cup ... \cup B_{k-1}$ is planar.

To conclude, $C \cup B_1 \cup \ldots \cup B_{k-1} = G_2$ is plane, and there exists a plane representation $\underline{G_2}$ of G_2 so that all the bridges of M_2 are represented in the exterior of the cycle C. (**)

From (*) and (**) we conclude that $G=G_1 \cup G_2$ is a planar graph.

The following result is intuitively obvious:

Proposition 5.1 Let G be a planar graph with the cycle C and let B_1 and B_2 be two C-bridges. If B_1 and B_2 overlap then B_1 and B_2 cannot be represented in the same region of C.

Lemma 5.3 Let G be a 2-connected graph. If G has a subgraph H so that O(H:C) is not bipartit, then G is not planar.

Proof. By reduction to absurd. We suppose that G is planar. Then H is also planar. Because O(H:C) is not bipartit,, there exist two C-bridges both interior or exterior to C, connected by an edge of O(H:C). Thus there results that two overlapped C-bridges are represented in the same region of C, and that contradicts the proposition 5.1.

Before formulating the planarity criterion for hamiltonian graphs, we make the following remark:

Remark 5.2. We notice from fig. 5.4 that B is a planar C-bridge, but $C \cup B$ is not planar. The condition "$C \cup B$ is planar for each C-bridge B" is present in the hypotheses of the lemmas 5.1 and 5.2.

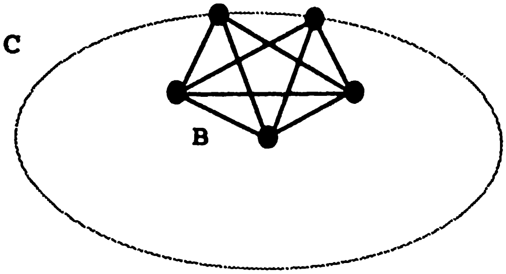

Figura 5.4.

Theorem 5.2 (Planarity criterion for hamiltonian graphs)

Let G be a hamiltonian graph and C a hamiltonian cycle of G. Then, G is planar if and only if O(G:C) is bipartit.

Proof. (==>) We prove by reduction to absurd. We suppose that O(G:C) is not bi-partit. Since G is hamiltonian, G is a graph at least 2-connected. From lemma 5.3, taking H=G there results that G is not planar, and this contradicts the hypothesis.

(<==) We have that O(G:C) is bi-partit. C being a hamiltonian cycle, all the C-bridges are diagonals of C and O(G:C) is a circle graph.

Let G' the plane representation of G, and C' the plane representation of C, so that C' is a geometrical circle and the diagonals of C are represented by line segments (see section 4). Then, C∪B is planar for each C-bridge B. From lemma 5.2 there results that G is a planar graph.

6. An algorithm for testing the planarity of a hamiltonian graph. Building of a plane representation.. The Theorem 5.2 suggests a simple algorithm for testing the planarity of a hamiltonian graph. If the given graph is planar, the algorithm will allow us to build a plane representation of it.

Algorithm 6.1:

S1 Determine a hamiltonian cycle C in the graph G.
If there are no hamiltonian cycles in G, then
  Message: "the graph G is not hamiltonian"
  STOP
EndIf

S2 Buid the circle graph O(G:C)

S3 If O(G:C) is not bipartit then
  Message: "the graph G is not planar"
  STOP
EndIf

S4 Let M_1 and M_2 the parts of the bipartit graph O(G:C). The vertices of G are represented on a circle in the plane, in their order in the cycle C. The diagonals from M_1 are represented by line segments, inside the circle. The diagonals from M_2 are represented by circle arcs, outside the circle.

6.1 Testing the planarity of 4-connected graphs and 3-connected graphs. We give here two known results that will help us to extend the applicability domain of the given algorithm.

Theorem 6.1.1 (W.T. Tuttle) Each 4-connected planar graph has a hamiltonian cycle.

Theorem 6.1.2 (C. Thomassen) Let G be a 3-connected planar graph with an articulation set of cardinal at most 3. Then G has a hamiltonian cycle.

From the results above, the next consequences are straightforward:

Consequence 6.1.1 Let G be a 4-connected graph. If there are no hamiltonian cycles in G, then G is not planar.

Consequence 6.1.2 Let G be a 3-connected planar graph with an articulation set of cardinal at most 3. If there are no hamiltonian cycles in G, then G is not planar.

So, we may use the algorithm 6.1 in order to test the planarity of 4-connected graphs and of 3-connected graphs with an articulation set of cardinal at nost 3. The only modification in the algorithm is the message from the step S1: "the graph G is not planar".

7. An implementation of the algorithm 6.1. Even if in this paper we only discuss about undirected graphs, in order to implement the algorithm 6.1 we chose a representation useful for oriented graphs, either.

Let G=(V,E) be an undirected graph, with |V|=n and |E|=m. We suppose that V={1,...,n} and E={1,...,m}. We establish an arbitrary orientation for the edges of the graph G and we obtain an oriented graph G'=(V,E,alpha,omega), where

alpha: E --> V, omega: E --> V and

for each e in E, alpha(e) represents the initial extremity of the arc e and omega(e) represents the final extremity of the arc e.

We define the enumeration types edge and vertex in this way:

```
edge   = -m .. m;
vertex =  1 .. n;
```

We use the array NODE: array [edge] of vertex;

in.order to store alpha(e) in NODE[-e] and omega(e) in NODE[e], for every e in E [1].

We will also use the arrays:

FIRST: array [vertex] of edge;

NEXT : array [edge] of edge;

in order to chain all the arcs incident to a vertex of the graph, the sign of an arc playing the role of direction indicator.

Remark 7.1 Any supplementary information concerning the vertices and the edges (labels, markers, and so on) may be memorized in arrays having the lenght n and m, respectively. This is one of the main advantages of representing a graph by adjacency lists, as compared with a representation by using incidence matrix. The great majority of the algorithms based on incidence matrix request that this matrix must be inspected element by element. From here it arises the imposibility to reduce the complexity of the algorithms at a value smaller that $O(n^2)$. [3]

Remark 7.2 The introduction of the graph in the memory of the computer may be simply realized by stating the extremities of each arc. Then, in order to complete the arrays FIRST and NEXT, we use the following algorithm (linear time):

```
for v:= 1 to n do.
   FIRST[v]:=0;
for e:=m downto 1 do
  begin
    NEXT[e]:=FIRST[NODE[-e]];
    FIRST[NODE[-e]]:=e;
    NEXT[-e]:=FIRST[NODE[e]];
    FIRST[NODE[e]]:=-e;
  end;
```

Considerations regarding the source text of the program

1. The following result is well known:

Let G be an undirected graph, with no loops and no multiple edges, with $|V|=n$ and $|E|=m$. If $n >= 3$ and G is planar, then $m<=3n-6$. [5]

In the function Introduction we verify if the inequality mentioned above is fulfilled. If $m > 3n-6$, we display a message concerning the nonplanarity of G, and the execution of the program stops.

2. Let G=(V,E) be a hamiltonian graph, with |V|=n and |E|=m, and be C a hamiltonian cycle of G. Then O(G:C) has exactly m-n vertices.

It is easy to show that a bipartit graph with p vertices has at most $p^2/4$ edges.

In the function GraphCon, which builds the overlap graph O(G:C), each time a new edge is added we verify that the number of edges of O(G:C) is not bypassing $[(m-n)^2/4]$. In the case of an affirmative answer, the graph O(G:C) cannot be bipartit, and thus G is not planar.

Remark 7.3 At the function GraphCerc a variable of the type set of byte has been used. This is the cause of the restriction for the order of the graph to be n <= 87. So, for the given graph with more than 87 vertices is necessary to rewrite the code of the function GraphCerc where the variable CycleHamilt is used, and, obviously, its type will be changed.

Remark 7.4 Obviously, the program may be optimized. For example, the variable succ from the procedure Colouring may be put as global variable, in order to avoid the repeated allocation in the Heap, caused by the recursive call.

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