STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XL, 3, 1995

ON MEDIAN FOR ONE SPECIAL SPACE

Petru SOLTAN and Chiril PRISĂCARU*

Received: 30.10.1995
AMS subject classification: 52,90C

REZUMAT. - Asupra medianei unui spaţiu special. În lucrare se formulează următoarea problemă: fie $\mathbf{R}^2$ planul vectorial definit peste câmpul numerelor reale cu norma $\|x\| = |x^1| + |x^2|$. Considerăm poligonul $M \subset \mathbf{R}^2$, topologic echivalent cu cercul euclidian şi ale cărui laturi sunt paralele cu una din axele sistemului de coordonate din $\mathbf{R}^2$ şi mulţimea de puncte $S = \{x_1, x_2, ..., x_m\} \subset M$, care are ponderile $p(x_1), ..., p(x_m)$ pozitive. În acest articol ne propunem să formulăm un algoritm, bazat pe d-convexitate, pentru un punct $x_0 \in M$ care minimizează funcţia $f(x) = \sum_{x \in M} p(x_i)d(x,x_i)$.

The following problem is formulated in [1]. Let R be a vector plan on the field of real numbers with norm $\|x\| = |x^1| + |x^2|$, $M \subset R^2$ is a poligon, egual topologically with an euclidian circle, andevery side of it is parallel (see figure) to one axis of coordinate system of R^2, $S = \{x_1, x_2, ..., x_m\} \subset M$ be a set of points having respectively the positive weights $p(x_1), p(x_2), ..., p(x_m)$.

It is required to find a median in M, i.e. such a point $x_0 \in M$ that minimizes the function

$$f(x) = \sum_{x \in M} p(x_i)\, d(x, x_i),$$

where $d(x,x_i)$ represents a distance between points x and x_i calculated following a curve of a minimal length in space $M \subset R^2$ that connects these points.

This problem is solved in [1] using complicated algorithm having however advantage of linear complexity.

In this paper the other algorithm for indicated problem is offtered; it is based on d-

*State University of Moldau, Faculty of Mathematics and Cibernetics, Chişinău, Republic of Moldau

convexity theory and follows from algorithms developped for finding a median in [2]. The maximal parallel to axes segments (by inclusion in M) through any point of local unconvexity [4] (see point x_m on the figure) of a polygon M and set a S are drawn. So, a polygon M is transformed to figure that is divided into parallelograms with sides forming a grid in M; the last is denoted by graph $G = (X, V)$, where X is the set of vertices (nodes), and V is the set of edges. The points $x_1, x_2, ..., x_m$ represent o subset of vertices of G. Denote the other vertices of G by $x_{m+1}, x_{m+2}, ..., x_n$. Hence, we obtain the new set of points X of a polygon M.

Now we assign a new positive weight to every point $x \in X$. If $x_i \in X \setminus S$, then we to x_i the weight $q(x_i) = 1$, ($i = m+1, ..., n$), if $x_i \in S$ then $q(x_i) = p(x_i) + 1$, ($i = 1, ..., m$). The new problem is formulated in the following way: to find such a vertex $x_0 \in X$ of graph G that minimizes the function

$$\varphi(x) = \sum_{x \in X} q(x_i) d(x, x_i).$$

According to norm $\|x\|$ in R^2 and to results of [2], $d(x,x_i)$ of this new problem is the same distance that participates in definition of the function $f(x)$.

Applying the reasoning that was presented in [2], we obtain that graph G satisfied the conditions indicated in this paper which permit to solve a new formulated problem by the same algorithm, since it is not necessary to know distances $d(x,x_i)$, $i = 1, 2, ..., n$.

The essence of this algorithm is the following.

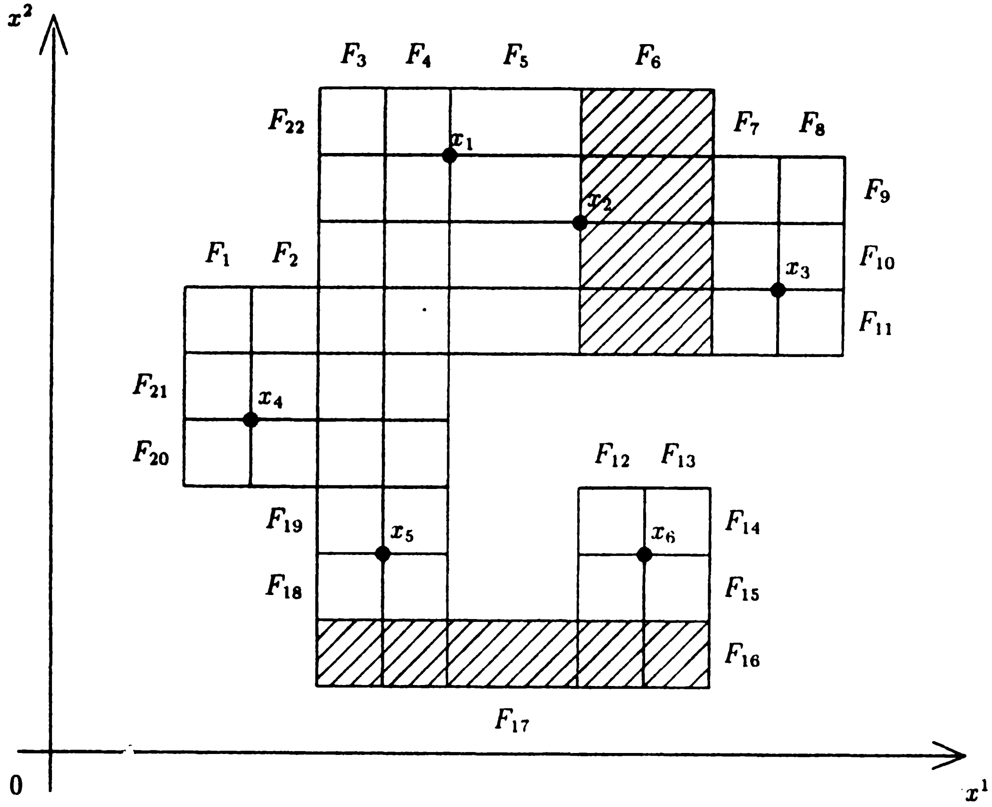

Fig. 1

Let s be the number of horisontal and vertical strips in which polygon M is devided. Denote these strips by F_1, F_2, ..., F_j, ..., F (on the figure the examples of strips are indicated by horisontal sides and $s = 22$).

Put in correspondence to any vertex of graph G a sequence from 0 and 1 of length s according to the following conditions: 1) to point x_1 it is put in correspondence sequence $\varepsilon^1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_j^1, \ldots, \varepsilon_s^1)$, where $\varepsilon_j^1 = 0$, $j = 1, 2, \ldots, s$; 2) to point x_i, $i = 1, 2, \ldots, n$ it is put a sequence $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i, \ldots, \varepsilon_j^i, \ldots, \varepsilon_s^i)$, where $\varepsilon_j^i = 0$, if an arbitrary chain of a graph G connecting the vertices x_1 and x_i intersects a piece F_j even number of time, and $\varepsilon_j^i = 1$, otherwise. So, we construct a function $\alpha : X \to \{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^i, \ldots, \varepsilon^n\}$, which as it follows from [2] is one-to-one.

Further construct a new sequence $r = (r^1, r^2, \ldots, r^j, \ldots, r^s)$ of elements 0 and 1 in correspondence to the conditions:

1) $r_j = 0$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) > \frac{1}{2}\sum_{i=1}^{n} q(x_i),$$

2) $r_j = 1$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) < \frac{1}{2}\sum_{i=1}^{n} q(x_i),$$

3) $r_j = 0$ or $r_j = 1$, if

$$\sum_{i=1}^{n} q(x_i)(1 - \varepsilon_j^i) = \frac{1}{2}\sum_{i=1}^{n} q(x_i),$$

If virtue of [2] there exists an index $i_0, 1 \le i_0 \le n$ such that $r = \varepsilon^{i_0}$. If x_0 is such a vertex of a graph G for which we have $\alpha(x_0) = \varepsilon^o = r$, then $x_0 = \alpha_{-1}\varepsilon^o$ minimizes the function $\varphi(x)$. Moreover the following statement holds:

THEOREM 1. *The vertices of graph $G = (X, U)$ that minimize the function $\varphi(x)$ are the vertices which give the minimal values for the $f(x)$.*

Proof. Consider the function

$$\varphi_\varepsilon(x) = \sum_{i=1}^{n} q^\varepsilon(x_i)\, d(x, x_i),$$

where ε is an arbitrary positive number, and $q^\varepsilon(x_i) = p(x_i)$ if $x_i \in S$ and $q^\varepsilon(x_i) = p(x_i) + \varepsilon$ if $x_i \in X \setminus S$.

It is obvious that the function $\varphi(x)$ satisfies conditions 1)-3) if and only if the function

$\varphi_\varepsilon(x)$ also satisfies these conditions. Therefore by virtue of [2] the vertices of a graph $G = (X,U)$, that minimize functions $\varphi(x)$, $\varphi_\varepsilon(x)$, are the same.

Nowe extend the function $\varphi_\varepsilon(x)$ that is defined on the set X, onto the wholepoligon $M \subset R^2$ preserving the same notation for it. Evidently, it is easy to do this operation by virtue of definitions for distance $d(x,x_i)$ between points x,x_i and norm $\|x\|$ of the space R^2. By the same way transform the function $f(x)$ by

$$f(x) = \sum_{i=1}^{n} p(x_i)\, d(x, x_i),$$

where $p(x_i) = 0$ for $i = m+1, m+2, ..., n$, that is for $x \in X\backslash S$. Observe that for any $x \in M$ we have the relations $\varphi_\varepsilon(x) - f(x) = n\cdot\varepsilon > 0$. Hence if $\varepsilon \to 0$ for any $x \in M$ we obtain $(\varphi_\varepsilon(x) - f(x)) \to 0$. It follows from this that point $x_0 \in M$ minimising the function $\varphi_\varepsilon(x)$ will minimize the function $f(x)$ (the inverse statement generally does not hold). The theorem is proved.

This theorem permits to reduce the solving of the initial problem to finding the median for function $\varphi_\varepsilon(x)$. So, we obtain that in case $\varepsilon = 1$ algorithm of finding the median for the function $\varphi(x)$ is the same for finding the median for the function $f(x)$.

Note. As it is proved in [3], the set of arguments minimizing the function $\varphi(x)$, is d-convexe. Hence applying in addition the results of [2], we oftain that the set of medians of the function $\varphi(x)$ may represent one of the following possibilities: a) case when we have asingle sequence r, respectively, one point $x_0 = \alpha^{-1}(r)$, 2) case when we have two sequences r_1 and r_2, respectively, one segment $u = [x^1 = \alpha^{-1}(r_1), x^2 = \alpha^{-1}(r_2)] \subset U$ that is parallel to one of the axes; 3) case when we have four sequences r_1, r_2, r_3, r_4, we obtain respectively one paralelogram that divides poligonul $M \subset R^2$ having vertices $x^1 = \alpha^{-1}(r_1), x^2 = \alpha^{-1}(r_2), x^3 = \alpha^{-1}(r_3)$, $x^4 = \alpha^{-1}(r_4)$.

One can prove that this property remains valide for the function $f(x)$, but in this case, if M_φ and M_f are the sets of respective median for $\varphi, f(x)$ then $M_\varphi \subset M_f$.

Direct realization of expounded method by one algorithm gives the possibility to obtain the complexity $o(n^2)$, where n is the number of strips, and it is equal to the sum compiled from the number of given points m and the number of the edges of a polygon k. This complexity is determined by the mode of representation of a grid obtained as a result of polygon's division.

Indeed, the further calculations may be reduced to finding the median of two trees: one that is determined by the horisontal strips and the other determined by the vertical strips. To every strips corresponds an edge of a tree. Two edges have one common vertex if the respective strips have the common border. The weigts of this vertex equals to the sum of weights of vertices on grid that belong to this border.

For example for polygon pictured on fig.1 the horizontal tree is H_o, the vertical tree is H_v (fig. 2).

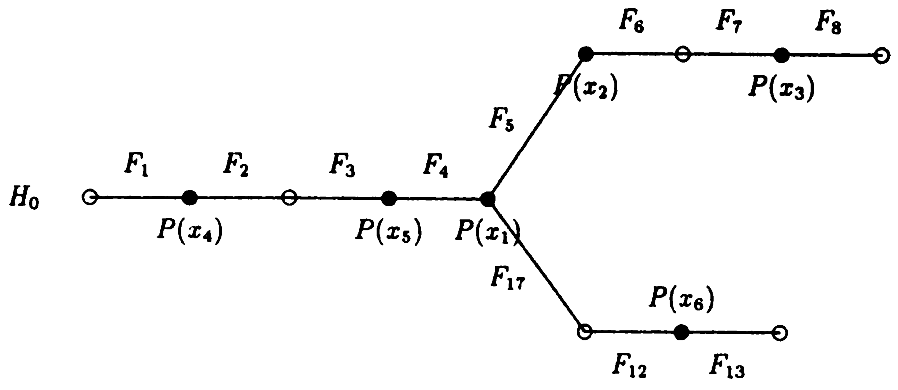

Fig. 2a

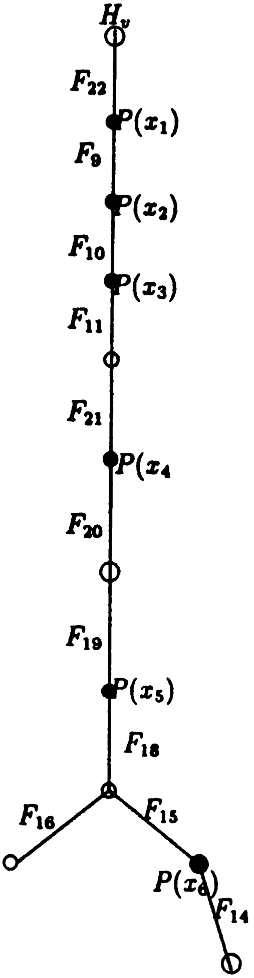

Fig. 2b

Finding the median for every tree maybe executed with the optimal (linear) complexity [2]. Information about the median of every tree determines the median of a polygon. In order to obtain the algorithm with optimal complexity for computing a median it is sufficient to oftain the trees H_0 and H_1 with the optimal complexity.

It is possible to do this job without complete description of a grid obtained from the initial polygon.

It is obvious that the number of edges of constructed trees is $O(n+k)$. For computing the weights of vertices of these trees the optimal method for point locating may be used ([5],[6]).

The median of any tree determines the set of strips. The median of the whole space is determined by the intersection of the union of horisontal strips with the union of vertical strips.

REFERENCES

1. V.Cepoi and F.Dragan, *Computing the median point of a simple rectilinear polygon.* (to appear)
2. P.Soltan, D.Zambitchii, Ch.Prisacaru, *Ekstremalinie zadaci na grafax*, Chisinau, Stiinta, 130 p., 1973.
3. V.Soltan, *"Vvedenie v aksiomaticeskuiu teoriu vîpuklosti"*, Chişinău, Ştiinţa, 223 pagini, 1984.
4. P.Soltan, Ch.Prisăcaru, *O razbienii ploscoi oblasti na d-vipuclosti* DAN SSSR, N2, T.262 1982.
5. H.Edelbrunner, L.J.Guibas and J.Stolfi. *Optimal point location in a monotone subdivizion.* SIAM J.

Comput., 15:317-340, 1985.
6. D.G.Kirkpatrick, *Optimal search in planar subdivizions*, SIAM J. Comput., 12:28-35, 1983.