OPTIMIZATION OF d-CONVEX FUNCTIONS ON NETWORKS

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REZUMAT. - Optimizarea funcţiilor d-convexe pe reţele. În acest articol sunt introduse şi studiate funcţiile d-convexe definite pe spatiul metric al unei reţele. Sunt discutate unele proprietăţi ale acestui tip de funcţii şi o metodă de rezolvare a problemei:
P: $f(z) \to \min$,
unde $f: N \to \mathbb{R}$ este o funcţie d-convexă.

1. Introduction. The actual period in the development of metric convexity is connected with investigations of discrete structures and of some extreme problems on them ([2], [14], [15], [13], [10], [22]). At the same time a considerable part of the results on convexity in discrete spaces is concentrated around metric convexity in graphs ([12], [16], [18], [20], [21]). It is interesting to mention that notions like convex set and convex function in graphs appeared previously in connection with some location problems ([3], [4], [5], [9], [23]). Another concept which was the direct result of location problems is the network (see [4], [5], [9]). In this article we deal with metric convexity (see [6], [7]) in networks and our aim is to define and study convex functions for these kind of spaces. We also give a method to solve the minimization of d-convex functions on networks. As we shall see, networks are closely related to graph, although they are not discrete metric spaces.

For convenience, we define here networks as metric spaces and some necessary notions related to them. Notice that we adopt definitions used in [3], [4], [5], [6], [7], [9].

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We start with a undirected, connected graph $G=(W,A)$, without loops or multiple edges. To each vertex i in $W=\{1,...,n\}$ we associate a point v_i from X. Thus yields a finite subset $V=\{v_1,...,v_n\}$ of X, called the **vertex set** of the network. We also associate to each edge (i,j) in A a rectifiable arc $[v_i,v_j]\subset X$, called **edge** of the network. Let assume that $[v_i,v_j]$ has the positive length e_{ij} and denote by U the set of all edges. We define the **network** $N=(V,U)$ by the union $N = \bigcup_{(i,j)\in A} [v_i, v_j]$. It is obvious now that N is a geometric image of G, which follows naturally from an embedding of G in X. Let us suppose that for each $[v_i,v_j]$ in U there exists a continuous one-one mapping $Q_{ij}:[v_i,v_j]\longrightarrow[0,1]$ with $Q_{ij}(v_i)=0$, $Q_{ij}(v_j)=1$, $Q_{ij}([v_i,v_j])=[0,1]$ and if $x,y\in[v_i,v_j]$ such that $x\in[v_i,y)$ then $Q_{ij}(x)<Q_{ij}(y)$. It is obvious that to each point x from $[v_i,v_j]$ corresponds a unique point, namely $Q_{ij}(x)$, in [0,1]. Any connected and closed subset of an edge bounded by two points x and y of $[v_i,v_j]$ is called a **closed subedge** and is denoted by [x,y]. If one or both of x,y miss we say that the subedge is open in x (or in y) or is open and we denote this by [x,y) or (x,y] or (x,y), respectively. Using Q_{ij}, it is possible to compute the length of [x,y] as $e([x,y])=|Q_{ij}(x)-Q_{ij}(y)|e_{ij}$. Particularly we have $e([v_i,v_j])=e_{ij}$, $e([v_i,x])=Q_{ij}(x)e_{ij}$ and $e([x,v_j])=(1-Q_{ij}(x))e_{ij}$.

By analogy with graphs we introduce the notions:

The **degree** $g_N(v)$ of $v\in V$ in N is the number of closed edges in N which contain v.

A **path** D(x,y) linking two points x and y in N is a sequence of edges and at most two subedges at extremities. If x=y then the path is called **cycle. The length of a path (cycle)** is the lengths sum of all its component edges and subedges and will be denoted by e(D(x,y)). If a path (cycle) contains only distinct vertices then we call it **elementary**.

A network N is **connected** if for any points x,y in N there exists a path $D(x,y)\subset N$.

An edge $[v_i,v_j]$ in U is called **isthmus** if $N\setminus(v_i,v_j)$ isn't connected.

Any connected subset $N'\subset N$ is called **subnetwork** of N. Any network $N'(V')=(V',U')$, where $V'\subset V$ and U′ is the set of all edges from U having the extremities in V′, is an **induced**

network.

A connected network without cycles is called **tree**.

Let $D^*(x,y)$ be the shortest path between the points x,y in N. We define a **distance** on N as follows: $d(x,y)=e(D^*(x,y))$ for any x,y in N. It is obvious that (N,d) is a metric space.

The **metric segment** between the points $x,y \in N$ is the set

$$\langle x,y\rangle = \{z \in N \mid d(x,z)+d(z,y)=d(x,y)\}.$$

It is clear that the metric segment $\langle x,y\rangle$ coincides with the union of all the shortest paths between x and y.

A set $M \subset N$ is **d-convex** ([11]) if for any two points x,y in M we have $\langle x,y\rangle \subset M$.

By **neighborhood** of the point $x \in N$ with radius r we mean the set

$$B(x,r)= \{z \in N \mid d(x,z)<r\}.$$

We also use as neighborhoods the sets $B_M(x,r)=\{z \in M \mid d(x,z)<r\}$, where $x \in M$ and M is some connected subset of N.

2. d-Convex Functions Our purpose in this section is to introduce the class of d-convex functions defined on the metric space (N,d) of a network N=(V,U). This approach was inspired by the papers [12], [17], [20], [21].

Let us consider a connected network N and a real valued function $f:N \longrightarrow R$.

Definition 2.1. ([16]) f is called d-convex on N if for any points $x,y \in N$ and any $z \in \langle x,y\rangle$ the inequality

$$f(z) \le \frac{d(x,z)}{d(x,y)}f(y)+\frac{d(y,z)}{d(x,y)}f(x)$$

holds.

One can state the following simple properties of d-convex functions on N. Note that this results was already proved for the more general case of metric spaces ([17], theorems 1-4).

Theorem 2.2. 1) For any d-convex functions f,g and any real number $\lambda \geq 0$, the functions f+g and λf are also d-convex.

2) The pointwise supremum of any family of d-convex functions is also a d-convex function.

3) The limit of any punctually convergent sequence of d-convex functions is also a d-convex function.

4) For any d-convex function f and any real number λ, the sets $\{z \in N | f(z) \leq \lambda\}$ and $\{z \in N | f(z) < \lambda\}$ are d-convex.

It will be needed the following preliminary results, which will establish links between d-convex functions and constants. Further on we denote by d-C and I the family of d-convex and respectively constant functions on N.

Lemma 2.3. If $C \subset N$ is an elementary cycle, then any d-convex function, $f: N \longrightarrow R$, is constant on C.

Proof. Let us consider the d-convex function $f: C \longrightarrow R$. What we have to prove is that for any $x,y \in C$, $f(x)=f(y)$. It is easy to see that there exists the points $z_1,...,z_n \in C$, $n \geq 5$, satisfying the properties:

1. $z_i \in \langle z_{i-1}, z_{i+1} \rangle$, $i=2,...,n$, where $z_{n+1}=z_1$.
2. $z_1=x$ and there exists $k \in 2,...,n$, such that $z_k=y$.

Let us assume that $f(z_p)=\max\{f(z_i) | i=1,...,n\}$. From the d-convexity of f results

$$f(z_p) \leq \frac{d(z_{p+1}, z_p)}{d(z_{p-1}, z_{p+1})} f(z_{p-1}) + \frac{d(z_{p-1}, z_p)}{d(z_{p-1}, z_{p+1})} f(z_{p+1}) \leq$$

$$\leq \frac{d(z_{p+1}, z_p)}{d(z_{p-1}, z_{p+1})} f(z_p) + \frac{d(z_{p-1}, z_p)}{d(z_{p-1}, z_{p+1})} f(z_p) = f(z_p),$$

This leads to $f(z_p)=f(z_{p-1})=f(z_{p+1})$. By iterating this method we obtain $f(z_1)=...=f(z_n)$, thus $f(x)=f(y)$. Since $x,y \in C$ were arbitrarily chosen, we conclude that f is constant on C. ■

The following definition refers to a class of networks, closely connected with d-convex functions.

Definition 2.4. A connected network $N=(V,U)$, is called quasitree if there exists at least one vertex $v\in V$ such that $g(v)=1$.

We mention below some simple properties regarding quasitrees that will be of use later.

Lemma 2.5. If $N=(V,U)$ is a quasitree with at least a cycle, then there is a connected subnetwork $R=(V',U')\subset N$, $V'\subset V$, $U'\subset U$, maximal with respect to inclusion, such that any vertex $v\in V'$ has $g_R(v)\geq 2$ and all cycles in N are contained in R.

Proof. Let us consider the set

$$V' = V'' \cup \left(\bigcup_{v,v'\in V''} (D(v,v') \cap V) \right),$$

where V″ is the set of all vertices in V, which lie on some cycle of N. The subnetwork generated by V′, $R=N(V')$ is that one we are looking for. Indeed, from the way we define V′, any $v\in V'$ has $g_R(v)\geq 2$. Consider now a cycle C in N. Then his vertices will be in V″ and hence $C\subset N(V')=R$. Now, let us prove that R is maximal. We assume that there exists a subnetwork $R_1=(V_1', U_1')\subset N$, having the same properties as R and $R\subset R'$. Consequently $|V_1\setminus V'|\geq 1$. We consider $v\in V_1\setminus V$ and $v_1,v_2\in V'$. From the way we define R it follows that there exists a path $D(v_1,v_2)\subset R$. On the other hand, since R′ is connected, we deduce that there exists the paths $D(v,v_1)$ and $D(v,v_2)$, which are not contained in R.

It follows that the union $D(v_1,v_2)\cup D(v_2,v)\cup D(v,v_1)$ is a cycle of N not contained in R, which is a contradiction. ■

Remark. 1) Further on we refer to R as the root of the quasitree N.

2) If N does not contains any cycle, then any point from N can be viewed as R.

Lemma 2.6. The closure of N\R, cl(N\R) is a not empty forest and each tree T from this forest satisfies $|T\cap R|=1$ and $T\cap R\subset V$.

Proof. This is the direct consequence of the previous lemma. Indeed cl(N\R) is a not necessarily connected network, without cycles, that is, a forest. From the definition of quasitrees results that N\R contains at least the vertex v of degree 1 and the edge incident to v, and therefore is not empty.

The fact that for any $T \subset N \setminus R$ holds $|T \cap R| = 1$ is also clear, since $|T \cap R| \geq 2$ implies the existence of a cycle not included in R. ■

Lemma 2.7. If $N=(V,U)$ is a quasitree and $f \in d\text{-}C$, then f is constant on the root R.

Proof. Considering Lemma 2.3 we can affirm that f is constant on any cycle in N. Consequently if two cycles C_1, C_2 have at least one common point then f is constant on $C_1 \cup C_2$. Taking into account the way we define the root of a quasitree and Lemma 2.5 it is clear that any two cycles in R either has notempty intersection or there exists a linking path between them. Our aim is to show that in this last case any $f \in d\text{-}C$ is constant on the union of this two cycles with the linking path. In order to get this consider two cycles C_1,C_2 and a path $D(x,y)$ such that $x \in C_1$, $y \in C_2$, $D(x,y) \cap C_1 = \{x\}$, $D(x,y) \cap C_2 = \{y\}$. If there exists another path $D(x',y')$ linking C_1 and C_2 then $C_1 \cup D(x,y) \cup D(x',y') \cup C_2$ will form a sequence of three or more cycles that can be ordered such that each two consecutive cycles have notempty intersection. This provides us that f is constant on $C_1 \cup D(x,y) \cup D(x',y') \cup C_2$.

Suppose now that $D(x,y)$ is the unique path between C_1 and C_2. Then $D(x,y) = \langle x,y \rangle$. Assume that $f(z) = \alpha_1$, for all $z \in C_1$, $f(z) = \alpha_2$, for all $z \in C_2$ and $\alpha_1 > \alpha_2$. Then by d-convexity of f for any $z \in D(x,y) \setminus \{x,y\}$ we have

$$f(z) \leq \frac{d(x,z)}{d(x,y)} f(y) + \frac{d(y,z)}{d(x,y)} f(x) = \alpha_2 \frac{d(x,z)}{d(x,y)} + \alpha_1 \frac{d(y,z)}{d(x,y)} < \alpha_1$$

On the other hand, for $r>0$, small enough, the set $cl(B(x,r))$ is a d-convex star. Let us consider $z_1 \in C_1 \cap cl(B(x,r)) \setminus \{x\})$ and $z_2 \in D(x,y) \cap cl(B(x,r)) \setminus \{x\})$. Obviously $x \in \langle z_1,z_2 \rangle$ and $f(z_2) < \alpha_1$. We have

$$\alpha_1 = f(x) \leq \frac{d(x,z_1)}{d(z_1,z_2)} f(z_2) + \frac{d(x,z_2)}{d(z_1,z_2)} f(z_1) =$$

$$= \frac{d(x,z_1)}{d(z_1,z_2)} f(z_2) + \alpha_1 \frac{d(x,z_2)}{d(z_1,z_2)} < \alpha_1 \frac{d(x,z_1)}{d(z_1,z_2)} + \alpha_1 \frac{d(x,z_2)}{d(z_1,z_2)} \alpha_1 = \alpha_1,$$

which is impossible. The same conclusion can be drawn for $\alpha_1 < \alpha_2$. Thus $\alpha_1 = \alpha_2 = f(z)$, for any

$z\in D(x,y)$. Thus $f\in$d-C is constant on R. ■

Summing up the above lemmas we conclude this part with

Theorem 2.8. d-C$\neq$I if and only if N is a quasitree.

Proof. Consider a quasitree N=(V,U) and denote by R its root. Then any function $f:N\longrightarrow R$, $f(x)=\alpha+d(x,R)$, with $\alpha\in R_+$ is d-convex and obviously not constant. Let us prove that f is d-convex. We consider the points $x,y\in N$. The proof falls naturally in three parts.

1) $x,y\in R$. From Lemma 2.7 we have $f(z)=\alpha$, for any $z\in R$ and the inequality in Definition 2.1 holds.

2) $x\in R$ and $y\in N\setminus R$. Then for any $z\in<x,y>$ we have

$$\alpha+d(z,R)=f(z)\le\frac{d(x,z)}{d(x,y)}f(y)+\frac{d(y,z)}{d(x,y)}f(x)=$$

$$=(\alpha+d(y,R))\frac{d(x,z)}{d(x,y)}+\alpha\frac{d(y,z)}{d(x,y)}=\alpha+d(y,R)\frac{d(x,z)}{d(x,y)}\Leftrightarrow$$

$$\Leftrightarrow d(z,R)\le\frac{d(x,z)}{d(x,y)}d(y,R)$$

If $z\in R$, then the previous inequality is true. If $z\in N\setminus R$, then because z and y lies on the same tree T from $cl(N\setminus R)$ and $T\cap R=\{v\}$ (see Lemma 2.6) we have

$$d(y,R)=d(y,z)+d(z,R),\ d(x,z)=d(z,R)+d(v,x).$$

Therefore

$$d(z,R)\le\frac{d(x,z)}{d(x,y)}d(y,z)+\frac{d(x,z)}{d(x,y)}d(z,R)\Leftrightarrow$$

$$\Leftrightarrow d(z,R)\left(1-\frac{d(x,z)}{d(x,y)}\right)=d(z,R)\frac{d(y,z)}{d(x,y)}\le\frac{d(y,z)}{d(x,y)}d(x,z)\Leftrightarrow$$

$$\Leftrightarrow d(z,R)\le d(x,z)=d(z,R)+d(z,x),$$

which is obvious.

3) $x,y\in N\setminus R$. Then for any $z\in<x,y>$ we have

$$f(z) \le \frac{d(x,z)}{d(x,y)} f(y) + \frac{d(y,z)}{d(x,y)} f(x) \Leftrightarrow$$

$$\Leftrightarrow \alpha + d(z,R) \le \frac{d(x,z)}{d(x,y)}(\alpha + d(y,R)) + \frac{d(y,z)}{d(x,y)}(\alpha + d(x,R)) \Leftrightarrow$$

$$\Leftrightarrow d(z,R) \le \frac{d(x,z)}{d(x,y)} d(y,R) + \frac{d(y,z)}{d(x,y)} d(x,R). \tag{1}$$

At this point we have to analyze two cases:

i. $\langle x,y\rangle \cap R = \emptyset$. Then x,y lies on the same tree from cl(N\R). There exists $t \in \langle x,y\rangle$ such that for all $z \in \langle x,y\rangle$, $d(z,R) = d(z,t) + d(t,R)$.

Using this relation in (1) we have the sequence of equivalencies

$$(1) \Leftrightarrow d(z,t) + d(t,R) \le \frac{d(x,z)}{d(x,y)}(d(y,t) + d(t,R)) +$$
$$+ \frac{d(y,z)}{d(x,y)}(d(x,t) + d(t,R)) \Leftrightarrow$$
$$\Leftrightarrow d(z,t) \le \frac{d(x,z)}{d(x,y)} d(y,t) + \frac{d(y,z)}{d(x,y)} d(x,t) \Leftrightarrow$$

$$d(x,y)d(z,t) \le d(x,z)d(y,t) + d(y,z)d(x,t) \tag{2}$$

Now we have to consider the possibilities:

a) $t = x$ ($t = y$): (2) is equivalent with $d(x,y)d(z,t) \le d(x,y)d(z,t)$;

b) $z \in \langle x,t\rangle$: (2) is equivalent with $2d(x,z)d(y,z) \ge 0$;

c) $z \in \langle t,y\rangle$: (2) is equivalent with $2d(y,z)d(t,x) \ge 0$.

All these inequalities are true.

ii. $\langle x,y\rangle \cap R \ne \emptyset$. For any $z \in \langle x,y\rangle$ we have

$$f(z) \le \frac{d(x,z)}{d(x,y)} f(y) + \frac{d(y,z)}{d(x,y)} f(x) \Leftrightarrow$$

$$\Leftrightarrow d(z,R) \le \frac{d(x,z)}{d(x,y)} d(y,R) + \frac{d(y,z)}{d(x,y)} d(x,R). \tag{3}$$

Since x and y lie on different trees from cl(N\R), the following relations hold:

If $z \in R$, then $d(z,R) = 0$ and (3) is true.

If z lies on the same tree with x (respectively y) then $d(x,R) = d(x,z) + d(z,R)$ ($d(y,R) = d(y,z) + d(z,R)$) and therefore

$$(3) \Rightarrow d(z,R) \le \frac{d(x,z)}{d(x,y)}d(y,R)+\frac{d(y,z)}{d(x,y)}d(x,z)+\frac{d(y,z)}{d(x,y)}d(z,R) \Rightarrow$$

$$\Rightarrow d(x,z)d(z,R) \le d(x,z)(d(y,R)+d(y,z)) \Rightarrow$$

$$\Rightarrow d(x,z)(d(y,R)+d(y,z)-d(z,R)) \ge 0,$$

which is true, because $d(z,y) \ge d(z,R)$.

In order to prove the reverse implication we start by assuming that N is not a quasitree. Then all vertices in N are of degree at least 2. This allows us to affirm that any vertex is either on a cycle or on some path linking two cycles. But any d-convex function is constant on this kind of networks (see proof of Lemma 2.7) and hence d-C=I. ■

3. Optimization of d-convex functions

In this section we give a method to solve the problem of minimization without constraints, of a d-convex not constant function on a network. Many concrete problems are of this type. This becomes obvious if we refer to important location problems as the determination of centers and medians in networks (see [9]). On the other hand there are many problems where the constraints either do not influence the solution or are equalities and therefore can be reduced to problems or sequences of problems without constraints.

First we have to introduce two basic notions.

Definition 3.1. We said that a function $f:N\longrightarrow\mathbb{R}$ has a global minimum on N at the point $z\in N$ if for any point $y\in N$ we have $f(z)\le f(y)$.

Definition 3.2. We said that a function $f:N\longrightarrow\mathbb{R}$ has a local minimum at the point $z\in N$ if there exists a number $r>0$ such that $f(z)\le f(y)$, for any point $y\in B(z,r)$.

Let us recall (see [24]), that a metric space X is called Λ-convex, where $\Lambda\subset[0,1]$, if for every $x,y\in X$ and every $\lambda\in\Lambda$, there exists a point $z\in X$ such that $d(x,z)=\lambda d(x,y)$ and $d(z,y)=(1-\lambda)d(x,y)$. The following theorem is proved in [1] (also see [17], theorem 10).

Theorem 3.3. Let the space X be Λ-convex and $\lambda\in\Lambda$. If a d-convex function $f:X\longrightarrow\mathbb{R}$

has a local minimum on the d-convex set $A \subset X$, this minimum is also global.

It is easily seen that a network is a Λ-convex space and therefore Theorem 3.3 stands also for d-convex functions on networks.

On the other hand, because of Theorem 2.8 we have to consider only the case of quasitree, since this type of network is the only one which could be domain for a not constant d-convex function. Considering also the fact that any $f \in d\text{-}C$ is constant on the root of a quasitree (see Lemma 2.7) we can state the following

Lemma 3.4. If N is a quasitree containing at least one cycle and $f: N \longrightarrow \mathbf{R}$ is d-convex, then any point from the root is a global minimum on N.

Proof. If N contains a contains a cycle then the root of N contains this cycle and therefore at least three edges. Taking in account Theorem 2.8, we can assume that $f(z)=\alpha$, for any $z \in R$.

Suppose now that there exists a point $x \in N \setminus R$ such that $f(x)<\alpha$. We denote by T that tree from $cl(N \setminus R)$, which contains x. We also consider an interior point z, of some edge included in R. If $\{v\}=\langle x,z \rangle \cap R \cap cl(N \setminus R)$ then by the d-convexity of f we have

$$\alpha = f(v) \le \frac{d(x,v)}{d(x,z)} f(z) + \frac{d(z,v)}{d(x,z)} f(x) = \alpha \frac{d(x,v)}{d(x,z)} + f(x) \frac{d(z,v)}{d(x,z)} < \alpha,$$

which is impossible. ■

It is easy to see that any minimization of a d-convex function f, on a quasitree, can be reduced to the minimization of a function f' on the tree obtained from N by contracting the root R into a single point z_R. The function f' has the same values as f on the points from $N \setminus R$ and $f'(z_R)=\alpha=f(z)$, where $z \in R$. Then if S is the set of solutions for $f(z) \longrightarrow \min$ and S' is the set of solutions for $f'(z) \longrightarrow \min$ then clearly $S' \cup R \setminus \{z_R\} = S$. It is also important to observe that f' is also d-convex.

Thus we can conclude that it will be enough to find a method to solve the problem $f(z) \longrightarrow \min$, when N is a tree. In order to get such a method we need the following result

Theorem 3.5. If N is a tree, $f: N \longrightarrow \mathbf{R}$ is d-convex and S is the set of solutions for the problem

P: $$f(z) \longrightarrow \min$$

then S contains either a single point, or S is a subtree of N.

Proof. Let us assume that $|S|>1$ and consider two points $x,y \in S$. Then

$$\alpha = \min\{f(z) \mid z \in N\} = f(x) = f(y).$$

On the other hand from the d-convexity of f, for any $z \in \langle x,y \rangle$ we have

$$f(z) \leq \frac{d(x,z)}{d(x,y)} f(y) + \frac{d(y,z)}{d(x,y)} f(x) \leq \alpha.$$

It follows that $f(z)=\alpha$, for any $z \in \langle x,y \rangle$ and thus $\langle x,y \rangle \subset S$. Clearly, for any two points from S the metric segment between them is also contained in S. Thus S is a connected d-convex set of N, namely a subtree. ■

Remark. If we recall the previous proof it follows that the global minimum points set of is d-convex. Thus we recover a basic property of convex functions.

We are now able to give an algorithm to solve P.

Algorithm 3.6.

Step 1. Determine the set $VM=\min\{f(v) \mid v \in V\}$. Let $S=\emptyset$ and $\alpha=f(v)$, where $v \in VM$.

Step 2. Determine the set $UM=\{[v,v'] \in U \mid v \in VM\}$ and if $|UM|=k$, denote the elements from UM by $UM=\{u_1,\ldots,u_k\}$.

Step 3. For $j=1$ to k perform Step 4.

Step 4. Solve the problem:

P_j: $$\min\{f(T_{u_j}(x)) \mid x \in [0,1]\},$$

where $T_{u_j}=Q_{u_j}^{-1}$. Let $\alpha_j = \min\{f(T_{u_j}(x)) \mid x \in [0,1]\}$ and S_j be the set of solutions for P_j.

If $\alpha > \alpha_j$ then $\alpha := \alpha_j$, $S := T_{u_j}(S_j)$ and go to Step 5.

If $\alpha = \alpha j$ then $S := S \cup T_{u_j}(S_j)$.

Step 5. End algorithm with α as minimal value of f and S set of solutions for P.

Remark. 1) The problem P_j from Step 4, in the previous algorithm is a classic minimization problem of a convex function on [0,1]. Indeed for any $\lambda \in (0,1)$ and any $x,y \in [0,1]$ we have

$$f(T_{u_j}(\lambda x+(1-\lambda)y)=f(z_\lambda)\le\frac{d(z_x,z_\lambda)}{d(z_x,z_y)}f(z_y)+\frac{d(z_y,z_\lambda)}{d(z_x,z_y)}f(z_x) =$$

$$\frac{(Q_{u_j}(z_\lambda)-Q_{u_j}(z_x))e(u_j)}{(Q_{u_j}(y)-Q_{u_j}(x))e(u_j))}f(T(y))+\frac{(Q_{u_j}(z_y)-Q_{u_j}(z_\lambda))e(u_j)}{(Q_{u_j}(z_y)-Q_{u_j}(z_x))e(u_j)}f(T(x) =$$

$$\frac{\lambda x+(1-\lambda)y-x}{y-x}f(T_{u_j}(y))+\frac{y-\lambda x-(1-\lambda)y}{y-x}f(T_{u_j}(x) =$$

$$= (1-\lambda)f(T_{u_j}(y))+\lambda f(T_{u_j}(x)).$$

Taking also into account the analytic expression of $f\circ T_{u_j}$, we can use an appropriate technique of one dimensional minimization (see [8], p. 117-130).

2) The complexity of Algorithm 3.6. is $O(nO_1)$, where O_1 is the complexity of the method used to solve P_j.

3) There are situations when the difficulty of the problem will be increased by the determination of $f\circ T_{u_j}$, or this determination is technically impossible. In this case we propose the substitution of Step 4 with

Step 4'. Solve the problem:

P_j': $$\min\{f(z))\mid z\in u_j\}.$$

Let $\alpha_j=\min\{f(z)\mid z\in u_j\}$ and S_j' be the set of solutions for P_j'.

If $\alpha>\alpha_j$ then $\alpha:=\alpha_j$, $S:=S_j'$ and go to Step 5.

If $\alpha=\alpha_j$ then $S:=S\cup S_j'$.

For solving P_j' we propose the following approximation algorithm.

First we make the assumptions that $u_j=[v_j,v_j']$, $f(v_j)\le f(v_j')$ and $\epsilon=e(u_j)/p$, where p is fixed in order to obtain a satisfactory diminution of the error ϵ in finding the solution.

Algorithm 3.7.

Step 1. Set $x:=v_j$, $y:=v_j'$; xold:=x; yold:=y; $S_j':=\emptyset$; z is the middle point of u_j. If

$f(x)=f(y)=f(z)$ then $S_j' := u_j$ and go to Step 6.

Step 2. Repeat:

If $f(z)>f(x)$ then yold:=y and y:=z.

Otherwise, if $f(z)<f(x)$ then perform Step 3 and if $f(z)=f(x)$ then perform Step 4.

until $((d(x,xold)\leq\epsilon)$ and $(d(y,yold)\leq\epsilon))$ or $(d(x,y)\leq\epsilon)$.

Go to Step 5.

Step 3. $z':=z$; $z'':=z$;

Repeat:

Assign the middle point of $[x,z']$ to z' and the middle point of $[y,z'']$ to z'',

until $(f(z')>f(z))$ and $(f(z'')>f(z))$.

xold:=x; x:=z'; yold:=y; y:=z''.

Step 4. Assign the middle point of $[x,z]$ to z'.

If $f(z')=f(z)$ then $S_j':=S_j'\cup[x,z]$; xold:=x; x:=z.

Otherwise yold:=y; y:=z.

Step 5. $S_j':=S_j'\cup[x,y]$;

Step 6. Stop algorithm.

Remark. The previous algorithm is a combination of the bisection method and Fibonacci's technique.

REFERENCES

[1] T.T. Arkhipova, I.V. Sergienko, On the formalization and solution of some problems of organizing the computing process in data processing systems, Kibernetica 1973, 5, 11-18; Englsh transl. in Cybernetics 9 (1973).

[2] V. Boltyanskii, P. Soltan, Combinatorial geometry of various classes of convex sets, Ştiinţa, Chişinău, 1978. (Russian).

[3] P.M. Dearing, R.L. Francis,T.J. Lowe, A minimax location problem on a network, Transportation Sci., 1974, 8, 333-343.

[4] P. M. Dearing, R. L. Francis, T. J. Lowe, Convex location problems on tree networks, Oper. Res. 24

(1976), 628-624.

[5] J. Hooker, Nonlinear Network location Models, Ph.D. Thesis, Univ. of Microfilms Int., Ann Arbor, 1984.

[6] M.E. Iacob, V. Soltan, On geodetic Characteristic of Networks, Studii şi Cercetări Matematice, 5, 46, 1994, 521-527.

[7] M.E. Iacob, Divided differences and convex functions of higher order on networks, Studia Univ. Babeş-Bolyai Math., 3, 1994, 43-56.

[8] V. Karmanov, Programmation Matematique, Editions MIR, Moscou, 1977.

[9] M. Labbe, Essay in network location theory, Cahiers de Centre d'Etude et de Recherche Oper., 1985, 27(1-2), 7-130.

[10] T.T. Lebedeva, I.V. Sergienko, V.P. Soltan, On conditions for the coincidence of a local minimum and the global minimum in discrete optimization problems, Sov. Math. Dokl., 1985, 32(1), 78-81.

[11] K. Menger, Metrische Untersuchungen, Ergebnisse eines math. Kolloq. Wien. 1 (1931), 2-27.

[12] A. Sochircă, V. Soltan, d-Convex functions on graphs, Mat. Issled. No. 11(1988), 93-106. (Russian)

[13] P. Soltan, V. Cepoi, Solution of Weber's problem for discrete median metric spaces, Trans. Inst. Math. Tbilisi 85 (1987), 52-76. (Russian)

[14] P. Soltan, Ch. Prisăcaru, Steiner's problem on graphs, Dokl. Akad. Nauk SSSR 198 (1971), 46-49. (Russian)

[15] P. Soltan, D. Zambiţchi, Ch. Prisăcaru, Extremal problems on graphs and algorithms of their solution, Ştiinţa, Chişinău, 1973. (Russian)

[16] P. Soltan, V. Soltan, d-Convex functions, Dokl. Akad. Nauk SSSR 249 (1979), 555-558. (Russian)

[17] V. Soltan, Some properties of d-convex functions. I. II, Bull. Acad. Sci. Moldova. Ser. Phis.- Techn. Math. Sci. No. 2 (1980), 27--31; No. 1 (1981), 21-26. (Russian)

[18] V. Soltan, d-Convexity in graphs, Dokl. Akad. Nauk SSSR 272 (1983), 535-537. (Russian)

[19] V. Soltan, Introduction to the axiomatic convexity theory, Ştiinţa, Chişinău, 1984. (Russian)

[20] V. Soltan, Metric convexity in graphs, Studia Univ. Babeş-Bolyai. Mathematica (1991), No. 4, 3-43.

[21] V. Soltan, V. Cepoi, Some classes of d-convex functions in graphs, Dokl. Akad. Nauk SSSR 273 (1983), 1314-1317. (Russian)

[22] V. Soltan, V. Cepoi, d-Convexity and Steiner functions on a graph, Dokl. Akad. Nauk Belorussian SSR 29 (1985), 407-408. (Russian).

[23] B. C. Tansel, R. L. Francis, T. J. Lowe, Location on networks: a survey, Manag. Sci. 29 (1983), 482-511.

[24] J.S.W. Wong, Remarks on metric spaces, Nederl. Akad. Wetensch. Proc. Soc. A 69=Indag. Math. 28 (1966), 70-73.