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Fixed point theorems for operators with a contractive iterate in b-metric spaces

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Dedicated to Professor Ioan A. Rus on the occasion of his 80th anniversary

Abstract. We consider, in this paper, mappings with a contractive iterate at a point, which are not contractions, and prove some uniqueness and existence results in the case of $b-$metric spaces. A data dependence result and an Ulam-Hyers stability result are also proved.

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1. Introduction

The well known Banach contraction's principle states that in a complete metric space each contraction has a unique fixed point and the sequence of successive approximations converges to the fixed point. We consider, in this paper, mappings with a contractive iterate at a point, which are not contractions, and prove some uniqueness and existence results in the case of $b-$metric spaces. Some related results for the case of metric spaces can be found in [12, 4, 17, 19] The starting point of this theory is the article of V.M. Sehgal [22], where the author proves the following result:

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ a continuous mapping satisfying the condition: there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(y), f^{n(x)}(x)) \leq kd(y, x).$$

Then f has a unique fixed point u and $f^n(x_0) \to u$, for each $x_0) \in X$.

We investigate mappings that are not necessary continuous and extend the previous result to the case of $b-$metric spaces. The data dependence of the fixed points is also considered. In the second part of the paper we prove an Ulam-Hyers stability result. For more results regarding this concepts see [8, 13, 20, 21].

2. Preliminaries

The b-metric space is a generalization of a usual metric space, which was introduced by Czerwik [15, 14]. In fact, such general setting of metric spaces were considered earlier, for example, by Bourbaki [11], Bakhtin [3], Heinonen [18]. Following these initial papers, b-metric spaces and related fixed point theorems have been investigated by a number of authors, see e.g. Boriceanu et al.[9], Bota [10], Aydi et al. [1, 2].

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. We recollect some essential definitions and fundamental results. We begin with the definition of a b-metric space.

Definition 2.1. *(Bakhtin* [3]*, Czerwik* [15]*) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:*

1. *$d(x, y) = 0$ if and only if $x = y$,*
2. *$d(x, y) = d(y, x)$,*
3. *$d(x, z) \leq s[d(x, y) + d(y, z)]$,*

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

It is clear that a b-metric is a usual metric if we take $s = 1$. Hence, we conclude that the class of b-metric spaces is larger than the class of usual metric spaces. For more details and examples on b-metric spaces, see e.g. [3, 5, 11, 14, 15, 18].

For the sake of completeness we state the following examples, see [5, 6].

Example 2.2. Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to [0, \infty)$ defined by:

$$d(x,y) := \begin{cases} 0, & x = y \\ 2s, & x, y \in X_1 \\ 1, & \text{otherwise.} \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

Example 2.3. The set $l^p(\mathbb{R})$ (with $0 < p < 1$), where

$$l^p(\mathbb{R}) := \left\{ (x_n) \subset \mathbb{R} \middle| \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$,

$$d(x,y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

(where $x = (x_n), y = (y_n) \in l^p(\mathbb{R})$) is a b-metric space with coefficient $s = 2^{1/p} > 1$. Notice that the above result holds for the general case $l^p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 2.4. The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p}, \text{ for each } x, y \in L^p[0,1],$$

is a b-metric space. Notice that $s = 2^{1/p}$.

We will present now the notions of convergence, compactness, closedness and completeness in a b-metric space.

Definition 2.5. *Let (X,d) be a b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called:*

(a) *convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.*

(b) *Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to +\infty$.*

Remark 2.6. Notice that in a b-metric space (X,d) the following assertions hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) $(X, \overset{d}{\to})$ is an L-space (see Fréchet [16], Blumenthal [7]);

(iv) in general, a b-metric is not continuous;

Taking into account of (iii), we have the following concepts.

Definition 2.7. *Let (X,d) be a b-metric space. Then a subset $Y \subset X$ is called:*

(i) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$;

(ii) compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

Definition 2.8. *The b-metric space (X,d) is complete if every Cauchy sequence in X converges.*

Lemma 2.9. *(Czerwik [15]) Let (X,d) be a b-metric space. Then and let $\{x_k\}_{k=0}^n \subset X$. Then $d(x_n, x_0) \le sd(x_0, x_1) + \ldots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n)$.*

3. Main results

In order to prove the first main result we need the following Lemma:

Lemma 3.1. *Let (X,d) be a complete b-metric space with $s \ge 1$ and $f : X \to X$ a mapping which satisfies the condition: there exists an $a \in (0, \frac{1}{s})$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(x), f^{n(x)}(y)) \le ad(x,y).$$

Then for each $x \in X$, $r(x) = sup_n d(f^n(x), x)$ is finite.

Proof. Let $x \in X$ and let $l(x) = max\{d(f^k(x), x), k = 1, 2, \dots, n(x)\}$.

If $n \in \mathbb{N}$ there exists $k \geq 0$ such that

$$k \cdot n(x) < n \leq (k+1) \cdot n(x).$$

We have:

$$\begin{aligned} d(f^n(x), x) &\leq s[d(f^{n(x)}(f^{n-n(x)}(x)), f^{n(x)}(x)) + d(f^{n(x)}(x), x)] \\ &\leq s \cdot a \cdot d(f^{n-n(x)}(x), x) + s \cdot l(x) \\ &\leq s \cdot l(x) + a \cdot s^2 \cdot l(x) + a^2 \cdot s^3 \cdot l(x) + \ldots + a^k \cdot s^{k+1} \cdot l(x) \\ &= s \cdot l(x)[1 + s \cdot a + s^2 \cdot a^2 + \ldots + s^k \cdot a^k] \\ &= s \cdot l(x) \cdot \frac{1 - (s \cdot a)^{k+1}}{1 - s \cdot a} \leq s \cdot l(x) \cdot \frac{1}{1 - sa}. \end{aligned}$$

Hence $r(x) = sup_n d(f^n(x), x)$ is finite. □

The next result presents a fixed point theorem for a mapping with a contractive iterate. A data dependence result is also proved.

Theorem 3.2. *Let (X, d) be a complete b-metric space with $s \geq 1$ and $f : X \to X$ a mapping which satisfies the condition: there exists an $a \in (0, \frac{1}{s})$ such that for each $x \in X$ there is a positive integer $n(x)$ such that for all $y \in X$*

$$d(f^{n(x)}(x), f^{n(x)}(y)) \leq ad(x, y).$$

Then:

(i) *f has a unique fixed point $x^* \in X$ and $f^n(x_0) \to x^*$, for each $x_0 \in X$, as $n \to \infty$.*
If, in addition, the b−metric is continuous we have:

(ii) *$d(x_0, x^*) \leq sd(x_0, f^{n(x_0)}(x_0)) + \frac{s^2}{1-sa} r(x_0)$, for each $x_0 \in X$.*

(iii) *Let $g : X \to X$ such that there exists $\eta > 0$ with*

$$d(f^{n(x)}(x), g(x)) \leq \eta, \ \forall x \in X.$$

Then

$$d(x^*, y^*) \leq s \cdot \eta + \frac{s}{1 - sa} \cdot r(y^*),$$

for all $y^* \in Fix(g)$.

Proof. (i) Let $x_0 \in X$ be arbitrary. Let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and inductively $m_i = n(x_i)$, $x_{i+1} = f^{m_i}(x_i)$. We show that the sequence $\{x_n\}$ is convergent. By routine calculation we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f^{m_{n-1}}(f^{m_n}(x_{n-1})), f^{m_{n-1}}(x_{n-1})) \\ &\leq a \cdot d(f^{m_n}(x_{n-1}), x_{n-1}) \leq \ldots \leq a^n \cdot d(f^{m_n}(x_0), x_0). \end{aligned}$$

Estimating $d(x_n, x_{n+p})$ we obtain

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \ldots + s^{p-1} \cdot d(x_{n+p-1}, x_{n+p}) \\
&\leq s \cdot a^n \cdot d(f^{m_n}(x_0), x_0) + s^2 \cdot a^{n+1} \cdot d(f^{m_n}(x_0), x_0) + \ldots \\
&+ s^p \cdot a^{n+p-1} \cdot d(f^{m_n}(x_0), x_0) \\
&\leq s \cdot a^n \cdot r(x_0) + s^2 \cdot a^{n+1} \cdot r(x_0) + \ldots + s^p \cdot a^{n+p-1} r(x_0) \\
&= s \cdot a^n \cdot r(x_0)[1 + s \cdot a + \ldots + (s \cdot a)^{p-1}] \\
&= s \cdot a^n \cdot r(x_0) \cdot \frac{1 - (sa)^p}{1 - sa} \to 0, n \to \infty.
\end{aligned}$$

Hence $\{x_n\}$ is Cauchy. Let $x_n \to x^* \in X$. We want to show that $f(x^*) = x^*$. First we show that

$$f^{n(x^*)}(x_m) = y_m \to f^{n(x^*)}(x^*), \text{ as } m \to \infty.$$

We have

$$d(f^{n(x^*)}(x_m), f^{n(x^*)}(x^*)) \leq a d(x_m, x^*) \to 0, \text{as } m \to \infty.$$

On the other side we can write

$$d(f^{n(x^*)}(x^*), x^*) \leq s \cdot [d(f^{n(x^*)}(x^*), f^{n(x^*)}(x_i)) + d(f^{n(x^*)}(x_i), x^*)]$$

where for i sufficiently large we have

$$d(f^{n(x^*)}(x^*), f^{n(x^*)}(x_i)) < \frac{\varepsilon}{3s}.$$

We also have that

$$\begin{aligned}
d(f^{n(x^*)}(x_i), x_i) &= d(f^{n(x^*)}(f^{m_{i-1}}(x_{i-1})), f^{m_{i-1}}(x_{i-1})) \\
&= d(f^{m_{i-1}}(f^{n(x^*)}(x_{i-1})), f^{m_{i-1}}(x_{i-1})) \\
&\leq a \cdot d(f^{n(x^*)}(x_{i-1}), x_{i-1}) \leq a^i \cdot d(f^{n(x^*)}(x^*), x^*) < \frac{\varepsilon}{3s^2}
\end{aligned}$$

for i sufficiently large.
We also have

$$d(f^{n(x^*)}(x_i), x^*) \leq s \cdot [d(f^{n(x^*)}(x_i), x_i) + d(x_i, x^*)] < s\frac{\varepsilon}{3s^2} + s\frac{\varepsilon}{3s^2} = \frac{2\varepsilon}{3s}$$

Hence

$$d(f^{n(x^*)}(x_i), x^*) \leq s\left[s\frac{\varepsilon}{3s^2} + s\frac{\varepsilon}{3s^2}\right] + \frac{\varepsilon}{3s} = \varepsilon.$$

Thus $f^{n(x^*)}(x^*) = x^*$ which gives us the existence of a fixed point for $g = f^{n(x^*)}$.

In order to prove the uniqueness of the fixed point let us consider x^* and y^* two fixed points with $x^* \neq y^*$. We have

$$d(x^*, y^*) = d(g(x^*), g(y^*)) = d(f^{n(x^*)}(x^*), f^{n(x^*)}(y^*)) \leq a \cdot d(x^*, y^*),$$

which is a contradiction with $a \in (0, 1)$.

From the uniqueness of the fixed point and from $f^{n(x^*)} = x^*$ we can conclude that x^* is a fixed point for f too. Indeed we have

$$f(x^*) = f(f^{n(x^*)}(x^*)) = f^{n(x^*)}(f(x^*)),$$

so $f(x^*)$ is a fixed point for $f^{n(x^*)}$. But $f^{n(x^*)}$ has a unique fixed point x^*. Hence $f(x^*) = x^*$.

To show that $f^n(x_0) \to x^*$ let us consider the set

$$\rho_* = \max\{d(f^m(x_0), x^*) : m = 0, 1, 2, \dots, (n(x^*) - 1)\}.$$

For $n \in \mathbb{N}$ sufficiently large we have: $n = r \cdot n(x^*) + q,\ 0 \le q < n(u), r > 0$ and

$$\begin{aligned} d(f^n(x_0), x^*) &= d(f^{rn(x^*)+q}(x_0), f^{n(x^*)(x^*)}) \\ &\le ad(f^{(r-1)n(x^*)+q}(x_0), x^*) \le \dots \\ &\le a^r d(f^q(x_0), x^*) \le a^r \rho_* \end{aligned}$$

Since $n \to \infty$ implies $r \to \infty$, we have $d(f^n(x_0), x^*) \to 0$, as $n \to \infty$. This establish the theorem.

(ii) In order to prove the second assertion we consider the following inequality obtained above:

$$d(x_n, x_{n+p}) \le s \cdot a^n \cdot r(x_0) \cdot \frac{1 - (sa)^p}{1 - sa}.$$

Since the $b-$metric is continuous and letting $p \to \infty$ we obtain:

$$d(x_n, x^*) \le \frac{sa^n}{1 - sa} \cdot r(x_0).$$

For $n = 1$ we have

$$d(x_1, x^*) = d(f^{n(x_0)}(x_0), x^*) \le \frac{s}{1 - sa} r(x_0).$$

Taking into account the previous inequalities we have:

$$\begin{aligned} d(x_0, x^*) &\le s(d(x_0, x_1) + d(x_1, x^*)) \\ &\le sd(x_0, x_1) + \frac{s^2}{1 - sa} r(x_0) \\ &= s \cdot d(x_0, f^{n(x_0)}(x_0)) + \frac{s^2}{1 - sa} r(x_0) \end{aligned}$$

(iii) For the data dependence of the fixed points, using the result from (ii) for $x_0 = y^*$, we obtain:

$$\begin{aligned} d(x^*, y^*) &\le sd(y^*, f^{n(y^*)}(y^*)) + \frac{s^2}{1 - sa} r(y^*) \\ &= s \cdot d(g(y^*), f^{n(y^*)}(y^*)) + \frac{s^2}{1 - sa} r(y^*) \\ &\le s \cdot \eta + \frac{s^2}{1 - sa} r(y^*) \end{aligned}$$

□

In the second part of the paper is presented an Ulam-Hyers stability result. We begin with the definition of the Ulam-Hyers stability for a fixed point equation.

Definition 3.3. *Let (X, d) be a b-metric space with $s \geq 1$ and $f : X \to X$ a mapping. The fixed point equation*

$$x = f(x), \; x \in X \tag{3.1}$$

is called Ulam-Hyers stable if $\forall \varepsilon > 0$ and $\forall x \in X$ there exists $n(x) \in \mathbb{N}^*$ such that $\forall y^*$ a solution of the inequality

$$d(y, f^{n(y)}(y)) \leq \varepsilon \tag{3.2}$$

there exist $c > 0$ and $x^* \in X$ a solution of (3.1) such that

$$d(y^*, x^*) \leq \varepsilon. \tag{3.3}$$

Theorem 3.4. *Let (X, d) be a complete b-metric space with $s \geq 1$. Suppose that all the hypothesis of Theorem 3.2 hold.*

Then the fixed point problem (3.1) is Ulam-Hyers stable.

Proof. Let us estimate the following:

$$\begin{aligned} d(y^*, x^*) &\leq s(d(y^*, f^{n(y^*)}(y^*)) + d(f^{n(y^*)}(y^*), x^*) \\ &= s(\varepsilon + d(f^{n(y^*)}(y^*), f^{n(y^*)}(x^*))) \\ &\leq s\varepsilon + s \cdot a \cdot d(y^*, x^*) \end{aligned}$$

Hence:

$$d(y^*, x^*) \leq \frac{s\varepsilon}{1 - sa}$$

□

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