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Global nonexistence of solution for coupled nonlinear Klein-Gordon with degenerate damping and source terms

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Abstract. In this article we consider a coupled system of nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solutions by concavity method.

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1. Introduction

We consider the following system

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) \\ \qquad +a_1 |u_t|^{m-2} u_t + m_1^2 u = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) \\ \qquad +a_2 |v_t|^{r-2} v_t + m_2^2 v = f_2(u,v), \end{cases} \tag{1.1}$$

where $u = u(t,x)$, $v = v(t,x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and a_1, a_2, b_1, b_2, m_1, $m_2 > 0$ and β_1, β_2, m, $r \geq 2$, $\alpha > 2$, and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^\rho u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^\rho v. \end{aligned} \tag{1.2}$$

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The system (1.1) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1),\ x \in \Omega \\ u(x) = v(x) = 0 \qquad x \in \partial\Omega. \end{cases} \tag{1.3}$$

Originally the interaction between the source term and the damping term in the wave equation is given by

$$u_{tt} - \Delta u + a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u, \text{ in } \Omega \times (0, T), \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history. It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters m, p and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $\left|u\right|^{p-2} u$ then a uniform estimate of the form

$$\left\|u_t(t)\right\|_2 + \left\|\nabla u(t)\right\|_2 \leq C, \tag{1.5}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1(\Omega) \times L^2(\Omega)$, where C is a positive constant independent of t. The estimate (1.5) shows that any local solution u of problem (1.4) can be continued in time as long as (1.5) is verified. This result has been proved by several authors. See for example [2, 5, 7, 15, 20, 3]. On the other hand in the absence of the damping term $\left|u_t\right|^{m-2} u_t$, the solution of (1.4) ceases to exist and there exists a finite value T^* such that

$$\lim_{t \to T^*} \left\|u(t)\right\|_p = +\infty, \tag{1.6}$$

the reader is refereed to Ball [1] and Kalantarov & Ladyzhenskaya [6] for more details. When both terms are present in equation (1.4), the situation is more delicate. This case has been considered by Levine in [8, 9], where he investigated problem (1.4) in the linear damping case ($m = 2$) and showed that any local solution u of (1.4) cannot be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [8] and [9] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim_{t \to T^*} \theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [4] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (1.4) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function L which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{1.7}$$

In $[0,\infty)$, where $\nu > 0$. Inequality (1.7) leads to a blow up of the solutions in finite tim $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [4] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [4] (see for example [10, 11, 12, 18]. In particular, Vitillaro in [18] combined the arguments in [4] and [11] to extend the result in [4] to situations where the damping is nonlinear and the solution has positive initial energy.

In [19], Yang, studied the problem

$$\begin{aligned} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ +a|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{aligned} \tag{1.8}$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [19][Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$. ([19], [Remark 2]).

Messaoudi and Said-Houari [13] improved the result in [19] and showed that the blow up of solutions of problem (1.8) takes place for negative initial data only regardless of the size of Ω.

The absence of the terms m_1u^2 and m_2v^2, equations (1.1) take the form:

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) \\ \quad +a_1|u_t|^{m-2}u_t = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) \\ \quad +a_2|v_t|^{r-2}v_t = f_2(u,v), \end{cases}$$

In [16] Rahmoun. A and Ouchenane. D proved the global nonexistence result, Under an appropriate assumptions on the initial data and under some restrictions on the parameter ; β_1;β_2; m; r and on the nonlinear functions f_1 and f_2.

2. Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm. The constants $C, c, c_1, c_2, \dots,$ used throughout this paper are positive generic constants, which may be different in various occurrences. We define

$$F(u,v) = \frac{1}{2(\rho+2)}\left[b_1|u+v|^{2(\rho+2)} + 2b_2|uv|^{\rho+2}\right].$$

Then, it is clear that, from (1.2), we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v). \tag{2.1}$$

The following lemma was introduced and proved in [14]

Lemma 2.1. *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right)\leq F(u,v)\leq\frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right). \quad (2.2)$$

The energy functional is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ &\quad +m_1^2\|u\|_2^2+m_2^2\|v\|_2^2-\int_\Omega F(u,v)\,dx. \end{aligned} \quad (2.3)$$

Let us define the constant r_α as follows

$$r_\alpha=\frac{N\alpha}{N-\alpha},\quad \text{if } N>\alpha,\ r_\alpha>\alpha \text{ if } N=\alpha,\ \text{and } r_\alpha=\infty \text{ if } N<\alpha. \quad (2.4)$$

The inequality below is the key to prove the global nonexistence of solution. A similar version of this lemma was first introduced in [17]

Lemma 2.2. *Suppose that $\alpha>2$, and $2<2(\rho+2)<r_\alpha$. Then there exists $\eta>0$ such that the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\leq\eta\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}}, \quad (2.5)$$

holds.

Proof. It is clear that by using the Minkowski's inequality, we get

$$\|u+v\|_{2(\rho+2)}^2\leq 2(\|u\|_{2(\rho+2)}^2+\|v\|_{2(\rho+2)}^2),$$

the embedding $W_0^{1,\alpha}\hookrightarrow L^{2(\rho+2)}(\Omega)$ gives

$$\|u\|_{2(\rho+2)}^2\leq C\|\nabla u\|_\alpha^2\leq C(\|\nabla u\|_\alpha^\alpha)^{\frac{2}{\alpha}}\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}},$$

and similary, we have

$$\|v\|_{2(\rho+2)}^2\leq C\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}.$$

Thus, we deduce from the above estimates that

$$\|u+v\|_{2(\rho+2)}^2\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}, \quad (2.6)$$

also, Hölder and Young's inequalities give

$$\begin{aligned} \|uv\|_{(\rho+2)} &\leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \\ &\leq C(\|\nabla u\|_{2(\rho+2)}^2+\|\nabla v\|_{2(\rho+2)}^2) \\ &\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}. \end{aligned} \quad (2.7)$$

Collecting the estimates (2.6) and (2.7), then (2.5) holds. This completes the proof of Lemma 2.2 □

Lemma 2.3. *Let $v > 0$ be a real positive number and L be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+v}(t), \tag{2.8}$$

defined in $[0, \infty)$.
If $L(0) > 0$, then the solution ceacesto exist for $t \geq L(0)^{-v} \xi^{-1} v^{-1}$.

Proof. Direct integration of (2.8) gives

$$L^{-v}(0) - L^{-v}(t) \geq \xi vt.$$

Thus we obtain the following estimate

$$L^{v}(t) \geq \left[L^{-v}(0) - \xi vt\right]^{-1}. \tag{2.9}$$

It is clear that the right-hand side of (2.9) is unbounded when

$$\xi vt = L^{-v}(0).$$

This completes the proof. □

In the following lemma, we show that the total energy of our system is a nonincreasing function of t.

Lemma 2.4. *Let (u, v) be the solution of system* (1.1)-(1.3)*, then the energy functional is a non-increasing function for all $t \geq 0$*

$$\begin{aligned}\frac{dE(t)}{dt} &= -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|\nabla u_t\|_{\beta_1}^{\beta_1} - \|\nabla v_t\|_{\beta_2}^{\beta_2} \\ &\quad -a_1\|u_t\|_m^m - a_2\|v_t\|_r^r - m_1^2\|u\|_2^2 - m_2^2\|v\|_2^2.\end{aligned} \tag{2.10}$$

Proof. We multiply the first equation in (1.1) by u_t and second equation by v_t and integrate over Ω, using integration by parts, we obtain (2.10). □

3. Global nonexistence result

In this section, we prove that, under some restrictions on the initial data and under som restrictions on the parameter $\alpha, \beta_1, \beta_2, m, r$, then the lifespan of solution of problem (1.1)- (1.3) is finite

Theorem 3.1. *Suppose that β_1, β_2, m, $r \geq 2$, $\alpha > 2$, $\rho > -1$ such that β_1, $\beta_2 < \alpha$, and* $\max\{m, r\} < 2(\rho + 2) < r_\alpha$, *where r_α is the Sobolev critical exponent of $W_0^{1,\alpha}(\Omega)$. defined in* (2.4). *Assume further that*

$$E(0) < E_1, \qquad (\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1.$$

Then, any weak solutions of (1.1)-(1.3) *cannot exist for all time. Here the constants E_1 and ζ_1 are defined in* (3.1).

In order to prove our result and for the sake of simplicity, we take $b_1 = b_2 = 1$ and introduce the following

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \zeta_1 = B^{\frac{-2(\rho+2)}{2(\rho+2)-\alpha}}, \qquad E_1 = \left(\frac{1}{\alpha} - \frac{1}{2(\rho+2)}\right)\zeta_1^{\alpha}, \tag{3.1}$$

where η is the optimal constant in (2.5).
The following lemma allows us to prove a blow up result for a large class of initial data. This lemma is similar to the one in [17] and has its origin in [18]

Lemma 3.2. *Let (u, v) be a solution of (1.1)-(1.3). Assume that $\alpha > 2$, $\rho > -1$. Assume further that $E(0) < E_1$ and*

$$\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1. \tag{3.2}$$

Then there exists a constant $\zeta_2 > \zeta_1$ such that

$$\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 > \zeta_2, \tag{3.3}$$

and

$$\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\zeta_2, \ \forall t \geq 0. \tag{3.4}$$

Proof. We first note, by (2.3) and the definition of B, that

$$\begin{aligned} E(t) &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &\quad - \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \\ &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &\quad - \frac{\eta}{2(\rho+2)}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}} \\ &\geq \frac{1}{\alpha}\zeta^\alpha - \frac{\eta}{2(\rho+2)}\zeta^{2(\rho+2)}, \end{aligned} \tag{3.5}$$

where $\zeta = \left[\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + m_1^2\|u\|_\alpha^\alpha + m_2^2\|v\|_\alpha^\alpha\right]^{\frac{1}{\alpha}}$. It is not hard to verify that g is increasing for $0 < \zeta < \zeta_1$, decreasing for $\zeta > \zeta_1$, $g(\zeta) \to -\infty$ as $\zeta \to +\infty$, and

$$g(\zeta_1) = \frac{1}{\alpha}\zeta_1^\alpha - \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_1^{2(\rho+2)} = E_1,$$

where ζ_1 is given in (3.1). Therefore, since $E(0) < E_1$, there exists $\zeta_2 > \zeta_1$ such that $g(\zeta_2) = E(0)$.
If we set $\zeta_0 = \left[\|\nabla u(0)\|_\alpha^\alpha + \|\nabla v(0)\|_\alpha^\alpha\right]^{\frac{1}{\alpha}} + m_1^2\|u(0)\|_2^2 + m_2^2\|v(0)\|_2^2$, then by (3.5) we have $g(\zeta_0) \leq E(0) = g(\zeta_2)$, which implies that $\zeta_0 \geq \zeta_2$.
Now, establish (3.3), we suppose by contradiction that

$$\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 < \zeta_2,$$

for some $t_0 > 0$; by the continuity of $\|\nabla u(.)\|_\alpha^\alpha + \|\nabla v(.)\|_\alpha^\alpha + m_1^2\|u(.)\|_2^2 + m_2^2\|v(.)\|_2^2$ we can choose t_0 such that

$$(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2\|u(t_0)\|_2^2 + m_2^2\|v(t_0)\|_2^2 > \zeta_1.$$

Again, the use of (3.5) leads to

$$E(t_0) \geq g(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha) + m_1^2\|u(t_0)\|_2^2 + m_2^2\|v(t_0)\|_2^2 > g(\zeta_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0,T)$. Hence, (3.3) is established. To prove (3.4), we make use of (2.3) to get

$$\begin{aligned}&\frac{1}{\alpha}(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha) + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2\\ &\leq E(0) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].\end{aligned}$$

Consequently, (3.3) yields

$$\begin{aligned}\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] &\geq \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) - E(0)\\ &\geq \frac{1}{\alpha}\zeta_2^\alpha - E(0)\\ &\geq \frac{1}{\alpha}\zeta_2^\alpha - g(\zeta_2) \qquad (3.6)\\ &= \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_2^{2(\rho+2)}.\end{aligned}$$

Therefore, (3.6) and (3.1) yield the desired result. □

Proof. (of Theorem 3.1). We suppose that the solution exists for all time and set

$$H(t) = E_1 - E(t). \qquad (3.7)$$

By using (2.3) and (3.7) we get

$$\begin{aligned}H'(t) &= \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u_t\|_{\beta_1}^{\beta_1} + \|\nabla v_t\|_{\beta_2}^{\beta_2}\\ &\quad + a_1\|u_t\|_m^m + a_2\|v_t\|_r^r + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2.\end{aligned}$$

From (2.10), It is clear that for all $t \geq 0$, $H'(t) > 0$. Therefore, we have

$$\begin{aligned}0 < H(0) \leq H(t) &= E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right)\\ &\quad - \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)\\ &\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \qquad (3.8)\end{aligned}$$

From (2.3) and (3.3), we obtain, for all $t \geq 0$,

$$\begin{aligned}&E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)\\ &< E_1 - \frac{1}{\alpha}\zeta_1^\alpha = -\frac{1}{2(\rho+2)}\zeta_1^\alpha < 0.\end{aligned}$$

Hence,

$$0<H(0)\leq H(t)\leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right],\ \forall t\geq 0.$$

Then by (2.2), we have

$$0<H(0)\leq H(t)\leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right],\ \forall t\geq 0. \tag{3.9}$$

We then define

$$L(t)=H^{1-\sigma}(t)+\varepsilon\int_\Omega (uu_t+vv_t))dx, \tag{3.10}$$

for ε small to be chosen later and

$$\begin{aligned} 0\quad<\quad \sigma\leq \min\left\{\frac{1}{2},\ \frac{\alpha-m}{2(\rho+2)(m-1)},\ \frac{\alpha-r}{2(\rho+2)(r-1)},\right.\\ \left.\frac{(\alpha-2)}{2(\rho+2)},\ \frac{\alpha-\beta_1}{2(\rho+2)(\beta_1-1)},\ \frac{\alpha-\beta_2}{2(\rho+2)(\beta_2-1)}\right\}. \end{aligned} \tag{3.11}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (1.7). Indeed, taking the derivative of (3.10), using (1.1) and adding subtracting $\varepsilon kH(t)$, we obtain

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon kH(t)\\ &+\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right)\\ &+\varepsilon(1-k)\int_\Omega F(u,v)-\varepsilon kE_1\\ &-\varepsilon\int_\Omega \nabla u\nabla u_t dx-\varepsilon\int_\Omega \nabla v\nabla v_t dx\\ &+\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)\\ &-\varepsilon\int_\Omega |\nabla u_t|^{\beta_1-2}\nabla u_t\nabla u dx-\varepsilon\int_\Omega |\nabla v_t|^{\beta_2-2}\nabla v_t\nabla v dx\\ &-\varepsilon a_1\int_\Omega |u_t|^{m-2}u_t u dx-\varepsilon a_2\int_\Omega |v_t|^{r-2}v_t v dx. \end{aligned} \tag{3.12}$$

We then exploit Young's inequality to get for $\mu_i,\lambda_i,\delta_i>0\ i=1,2$

$$\begin{aligned} \int_\Omega \nabla u\nabla u_t dx &\leq \frac{1}{4\mu_1}\|\nabla u\|_2^2+\mu_1\|\nabla u_t\|_2^2,\\ \int_\Omega \nabla v\nabla v_t dx &\leq \frac{1}{4\mu_2}\|\nabla v\|_2^2+\mu_2\|\nabla v_t\|_2^2, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \int_\Omega |\nabla u_t|^{\beta_1-1}\nabla u dx &\leq \frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1}+\frac{\beta_1-1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1},\\ \int_\Omega |\nabla v_t|^{\beta_2-1}\nabla v dx &\leq \frac{\lambda_2^{\beta_2}}{\beta_2}\|\nabla v\|_{\beta_2}^{\beta_2}+\frac{\beta_2-1}{\beta_2}\lambda_2^{-\beta_2/(\beta_2-1)}\|\nabla v_t\|_{\beta_1}^{\beta_1}, \end{aligned} \tag{3.14}$$

and also

$$\int_{\Omega} |u_t|^{m-2} u_t u dx \leq \frac{\delta_1^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m ,$$

$$\int_{\Omega} |v_t|^{r-2} v_t v dx \leq \frac{\delta_2^r}{r} \|v\|_r^r + \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_r^r . \tag{3.15}$$

A substitution of (3.13)-(3.15)) in (3.12) and using (2.2) yields

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ & + \varepsilon \left(1 + \frac{k}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ & + \varepsilon \left(\frac{c_0}{2(\rho+2)} - \frac{k c_1}{2(\rho+2)}\right) \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\ & - \frac{\varepsilon}{4\mu_1} \|\nabla u\|_2^2 - \mu_1 \varepsilon \|\nabla u_t\|_2^2 - \frac{\varepsilon}{4\mu_2} \|\nabla v\|_2^2 - \varepsilon \mu_2 \|\nabla v_t\|_2^2 \\ & + \varepsilon \left(\frac{k}{\alpha} - 1\right) \left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & - \varepsilon \frac{\lambda_1^{\beta_1}}{\beta_1} \|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon \frac{\beta_1 - 1}{\beta_1} \lambda_1^{-\beta_1/(\beta_1-1)} \|\nabla u_t\|_{\beta_1}^{\beta_1} \\ & - \varepsilon \frac{\lambda_2^{\beta_2}}{\beta_2} \|\nabla v\|_{\beta_2}^{\beta_2} - \varepsilon \frac{\beta_2 - 1}{\beta_2} \lambda_2^{-\beta_2/(\beta_2-1)} \|\nabla v_t\|_{\beta_1}^{\beta_1} \\ & - a_1 \varepsilon \frac{\delta_1^m}{m} \|u\|_m^m - a_1 \varepsilon \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m \\ & - a_2 \varepsilon \frac{\delta_2^r}{r} \|v\|_r^r - a_2 \varepsilon \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_m^m . \end{aligned} \tag{3.16}$$

Let us choose δ_1, δ_2, μ_1, μ_2, λ_1, and λ_2 such that

$$\begin{cases} \delta_1^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \delta_2^{-r/(r-1)} = M_2 H^{-\sigma}(t) \\ \mu_1 = M_3 H^{-\sigma}(t) \\ \mu_2 = M_4 H^{-\sigma}(t) \\ \lambda_1^{-\beta_1/(\beta_1-1)} = M_5 H^{-\sigma}(t) \\ \lambda_2^{-\beta_2/(\beta_2-1)} = M_6 H^{-\sigma}(t), \end{cases} \tag{3.17}$$

for M_1, M_2, M_3, M_4, M_5 and M_6 large constants to be fixed later. Thus, by using (3.17), and for

$$M = M_3 + M_4 + (\beta_1 - 1) M_5/\beta_1 + (\beta_2 - 1) M_6/\beta_2 + (m-1) M_1/m + (r-1) M_2/r,$$

then, inequality (3.16) takes the form

$$\begin{aligned} L^{'}(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H^{'}(t)+\varepsilon k H(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{k c_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \\ & -\varepsilon k E_1+\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ & -\frac{\varepsilon}{4M_3} H^\sigma(t)\|\nabla u\|_2^2-\frac{\varepsilon}{4M_4} H^\sigma(t)\|\nabla v\|_2^2 \\ & -\frac{a_1\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t)\|u\|_m^m \\ & -\frac{a_2\varepsilon}{r} M_2^{-(r-1)} H^{\sigma(r-1)}(t)\|v\|_r^r \\ & -\varepsilon \frac{M_5^{-(\beta_1-1)}}{\beta_1} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} \\ & -\varepsilon \frac{M_6^{-(\beta_2-1)}}{\beta_2} H^{\sigma(\beta_2-1)}(t)\|\nabla u\|_{\beta_2}^{\beta_2}. \end{aligned} \tag{3.18}$$

We then use the two embedding

$$L^{2(\rho+2)}(\Omega) \hookrightarrow L^m(\Omega), W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega),$$

and (3.9) to get

$$\begin{aligned} H^{\sigma(m-1)}(t)\|u\|_m^m \leq\ & c_2(\|u\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)+m} \\ & +\|v\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)}\|u\|_{2(\rho+2)}^m) \\ \leq\ & c_2(\|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m} \\ & +\|\nabla v\|_\alpha^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_\alpha^m). \end{aligned} \tag{3.19}$$

Similarly, the embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (3.9) give

$$\begin{aligned} H^{\sigma(r-1)}(t)\|v\|_r^r \leq\ & c_3(\|v\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)+r} \\ & +\|u\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)}\|v\|_{2(\rho+2)}^r) \\ \leq\ & c_3(\|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r} \\ & +\|\nabla u\|_\alpha^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_\alpha^r). \end{aligned} \tag{3.20}$$

Furthermore, the two embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$, yields

$$\begin{aligned} H^\sigma(t)\|\nabla u\|_2^2 \leq\ & c_4\left(\|u\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2+\|v\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2\right) \\ \leq\ & c_4\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)+2}+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla u\|_\alpha^2\right), \end{aligned} \tag{3.21}$$

and

$$H^{\sigma}(t)\|\nabla v\|_2^2 \leq c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2\right) = c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)+2}\right). \tag{3.22}$$

Since $max(\beta_1,\beta_2)<\alpha$ then we have

$$\begin{aligned} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} &\leq c_6(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1} \\ &\quad+\|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1}) \\ &= c_6(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \\ &\quad+\|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1}), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} H^{\sigma(\beta_2-1)}(t)\|\nabla v\|_{\beta_2}^{\beta_2} &\leq c_7(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2} \\ &\quad+\|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2}) \\ &= c_7(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2} \\ &\quad+\|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)+\beta_2}), \end{aligned} \tag{3.24}$$

for some positive constants c_2, c_3, c_4, c_5, c_6 and c_7. By using (3.11) and the algebraic inequality

$$z^\nu \leq (z+1) \leq \left(1+\tfrac{1}{a}\right)(z+a), \quad \forall z\geq 0,\ 0<\nu\leq 1,\ a\geq 0. \tag{3.25}$$

We have, for all $t\geq 0$,

$$\begin{cases} \|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m} \leq d\left(\|\nabla u\|_\alpha^\alpha+H(0)\right) \leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r} \leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right) \\ \|\nabla u\|_\alpha^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right) \\ \|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)+\beta_2} \leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right), \end{cases} \tag{3.26}$$

where $d=1+1/H(0)$.
Also keeping in mind the fact that $max(m,r)<\alpha$, using Yong's inequality, the

inequality (3.25) togrther withe (3.11), we conclude

$$\begin{cases} \|\nabla v\|_{\alpha}^{2\sigma(m-1)(\rho+2)} \|\nabla u\|_{\alpha}^{m} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)} \|\nabla v\|_{\alpha}^{r} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right) \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla u\|_{\alpha}^{2} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla v\|_{\alpha}^{2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right) \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \end{cases} \tag{3.27}$$

where C is a generic positive constant. Taking into account (3.19)- (3.27), then (3.18) takes the form

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & +\varepsilon(\left[k/\alpha - 1 - kE_1\zeta_2^{-a}\right] - CM_1^{-(m-1)} - CM_2^{-(r-1)} \\ & -\frac{C}{4}M_3^{-1} - \frac{C}{4}M_4^{-1} - CM_5^{-(\beta_1-1)} \\ & -CM_6^{-(\beta_2-1)} - 1)\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\left(k - CM_1^{-(m-1)} - CM_2^{-(r-1)} - \frac{C}{4}M_3^{-1} - \frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)} - CM_6^{-(\beta_2-1)}\right) H(t) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)} - \frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{3.28}$$

for some constant k. Using $k = c_0/c_1$, we arrive at

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H'(t) \\ & +\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & +\varepsilon\left(\overline{c} - CM_1^{-(m-1)} - CM_2^{-(r-1)} - \frac{C}{4}M_3^{-1} - \frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)} - CM_6^{-(\beta_2-1)} - 1\right)\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\left(c_0/c_1 - CM_1^{-(m-1)} - CM_2^{-(r-1)} - \frac{C}{4}M_3^{-1} - \frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)} - CM_6^{-(\beta_2-1)}\right) H(t), \end{aligned} \tag{3.29}$$

where $\overline{c} = k/\alpha - 1 - kE_1\zeta_2^{-2} = c_0/(c_1\alpha) - 1 - (c_0/c_1) E_1\zeta_2^{-2} > 0$ since $\zeta_2 > \zeta_1$.
At this point, and for large values of M_1, M_2, M_3, M_4, M_5 and M_6, we can find positive constants Λ_1 and Λ_2 such that (3.29) becomes

$$\begin{aligned} L^{'}(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H^{'}(t) \\ &+\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ &+\varepsilon\Lambda_1(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) + \varepsilon\Lambda_2 H(t). \end{aligned} \tag{3.30}$$

Once M_1, M_2, M_3, M_4, M_5 and M_6 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $((1-\sigma) - M\varepsilon) \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \int_\Omega [u_0.u_t + v_0.v_t]\, dx > 0.$$

From these and (3.30) becomes

$$\begin{aligned} L^{'}(t) &\geq \varepsilon\Gamma(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &+ \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha). \end{aligned} \tag{3.31}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \\ &\leq C\left(\|\nabla u\|_\alpha^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|\nabla v\|_\alpha^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right), \end{aligned} \tag{3.32}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We take $s = 2(1-\sigma)$, to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$.
By using (3.11) and (3.25) we get

$$\|\nabla u\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d(\|\nabla u\|_\alpha^\alpha + H(t)),$$

and

$$\|\nabla v\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d(\|\nabla v\|_\alpha^\alpha + H(t)),\ \forall t \geq 0.$$

Therefore, (3.32) becomes

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ &\leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + \|u_t\|_2^2 + \|v_t\|_2^2 \\ &\quad + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)), \forall t \geq 0. \end{aligned} \tag{3.33}$$

Also, since

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon \int_\Omega (u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ &\leq C\left(H(t) + \left|\int_\Omega (u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ &\leq C[H(t) + \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + \|u_t\|_2^2 + \|v_t\|_2^2 \\ &\quad + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2],\ \forall t \geq 0. \end{aligned} \tag{3.34}$$

Combining withe (3.34) and (3.31), we arrive at

$$L'(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t),\ \forall t \geq 0. \tag{3.35}$$

Finally, a simple integration of (3.35) gives the desired result.This completes the proof of Theorem (3.1) □

References

[1] Ball, J., *Remarks on blow up and nonexistence theorems for nonlinear evolutions equations,* Quart. J. Math., Oxford, **28**(1977), no. 2, 473-486.

[2] Benaissa, A., Ouchenane, D., Zennir, Kh., *Blow up of positive initial-energy solutions to systems of nonlinear wave equation withe degenerate damping and source terms,* Nonl. Stud., **19**(2012), no. 4, 523-535.

[3] Braik, A., Miloudi, Y., Zennir, Kh., *A finite-time blow-up result for a class of solutions with positive initial energy for coupled system of heat equations with memories,* Math. Meth. Appl. Sci., **41**(2018), 1674-1682.

[4] Georgiev, V., Todorova, G., *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Diff. Equ., **109**(1994), 295-308.

[5] Haraux, A., Zuazua, E., *Decay estimates for some semilinear damped hyperbolic problems,* Arch. Rational Mech. Anal., **150**(1988), 191-206.

[6] Kalantarov, V.K., Ladyzhenskaya, O.A., *The occurence of collapse for quasilinear equations of parabolic and hyperbolic type,* J. Soviet. Math., **10**(1978), 53-70.

[7] Kopackova, M., *Remarks on bounded solutions of a semilinear dissipative hyperbolic equation,* Comment. Math. Univ. Carolin, **30**(1989), no. 4, 713-719.

[8] Levine, H.A., *Instability and nonexistence of global solutions to nonlinear wave equations of the form,* Trans. Amer. Math. Soc., **192**(1974), 1-21.

[9] Levine, H.A., *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations,* SIAM J. Math. Anal., **5**(1974), 138-146.

[10] Levine, H.A., Park, S.R., *Global existence and global nonexistence of solutions of the Cauchy problem for a non-linearly damped wave equation,* J. Math. Anal. App., **228**(1998), no. 1, 181-205.

[11] Levine, H.A., Serrin, J., *Global nonexistence theorems for quasilinear evolution equations with dissipation,* Arch. Rational Mech. Anal., **137**(1997), no. 4, 341-361.

[12] Messaoudi, S., *Blow up in a nonlinearly damped wave equation,* Mathematische Nachrichten, **231**(2001), 1-7.

[13] Messaoudi, S., Said-Houari, B., *Global nonexistence of solutions of a class of wave equations with non-linear damping and source terms,* Math. Meth. Appl. Sci., **27**(2004), 1687-1696.

[14] Messaoudi, S., Said-Houari, B., *Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms,* J. Math. Anal. Appl., **365**(2010), no. 1, 277-287.

[15] Ouchenane, D., Zennir, Kh., Bayoud, M., *Global nonexistence of solutions for a system of nonlinear viscoelastic wave equations with degenerate damping and source terms,* Ukranian. Math. J., **65**(7)(2013), 723-739.

[16] Rahmoune, A., Ouchenane, D., *Global nonexistence of solution of a system wave equations with nonlinear damping and source terms,* J. Math. Anal. Appl., **6**(2018), no. 1, 12-24.

[17] Said-Houari, B., *Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equation with damping and source terms,* Diff. Integ. Equ., **23**(2010), 79-92.

[18] Vitillaro, E., *Global existence theorems for a class of evolution equations with dissipation,* Arch. Rational Mech. Anal., **149**(1999), 155-182.

[19] Yang, Z., *Existence and asymptotic behavior of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms,* Math. Meth. Appl. Sci., **25**(2002), 795-814.

[20] Zennir, Kh., *Growth of solutions with positive initial energy to system of degeneratly damped wave equations with memory,* Lobachevskii J. Math., **35**(2014), no. 2, 147-156.

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