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MONOTONE TECHNIQUE TO THE INITIAL VALUES PROBLEM FOR A DELAY INTEGRAL EQUATION FROM BIOMATHEMATICS

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REZUMAT. - Metoda iterațiilor monotone pentru problema cu valori inițiale relativă la o ecuație integrală din biomatematică. În lucrare este prezentată o matodă constructivă de rezolvare a problemei (1) - (2) în ipotezele (i) - (iv) presupunând că funcția $f(t,x)$ este monotonă în raport cu x. Un aspect nou conținut în acest articol îl constituie adaptarea metodei iterațiilor monotone la cazul operatorilor anti-izotoni, în particular, la cazul când $f(t,x)$ este o funcție necrescătoare în x.

1. Introduction. The following delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds \tag{1}$$

is a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $x(t)$ is the proportion of infectives in the population at time t, τ is the length of time an individual remains infectious and $f(t, x(t))$ is the proportion of new infectives per unit time.

In [1], [2], [4], [5], [6] sufficient conditions were given for the existence of nontrivial periodic nonnegative and continuous solutions to equation (1) in

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case of a periodic contact rate: $f(t+\omega,x) = f(t,x)$, $f(t,0) = 0$. The tools were Banach fixed point theorem [5], topological fixed point theorems [1], [2], [4], [6], fixed point index theory (the additivity property) [2] and monotone technique [2], [4].

In [3] we dealt with positive and continuous solutions $x(t)$ for equation (1), on a given interval of time $-\tau \le t \le T$, when it is known the proportion $\phi(t)$ of infectives in the population for $-\tau \le t \le 0$, i.e.

$$x(t) = \phi(t), \text{ for } -\tau \le t \le 0. \tag{2}$$

Clearly, we had to assume that ϕ satisfies the following condition:

$$b = \phi(0) = \int_{-\tau}^{0} f(s,\phi(s))\, ds. \tag{3}$$

Under this condition problem (1)-(2) is equivalent with the initial values problem:

$$x'(t) = f(t,x(t)) - f(t-\tau,x(t-\tau)),\ 0 \le t \le T \tag{4}$$

$$x(t) = \phi(t),\ -\tau \le t \le 0.$$

The existence of at least one solution to problem (4) was established in [3] under the following assumptions:

(i) $f(t,x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $x \ge 0$;

(ii) $\phi(t)$ is continuous, $0 < a \le \phi(t)$ for $-\tau \le t \le 0$ and satisfies condition

(3);

(iii) there exists an integrable function $g(t)$ such that

$$f(t,x) \geq g(t) \text{ for } -\tau \leq t \leq T \text{ and } x \geq a \tag{5}$$

and

$$\int_{t-\tau}^{t} g(s)\, ds \geq a \text{ for } 0 \leq t \leq T; \tag{6}$$

(iv) there exists a positive function $h(x)$ such that $1/h(x)$ is locally integrable on $[a, +\infty)$,

$$f(t,x) \leq h(x) \text{ for } 0 \leq t \leq T \text{ and } x \geq a \tag{7}$$

and

$$T < \int_{b}^{\infty} (1/h(x))\, dx. \tag{8}$$

THEOREM 1 [3]. *Suppose that assumptions* (i)-(iv) *are satisfied. Then equation* (1) *has at least one continuous solution* $x(t)$, $x(t) \geq a$, *for* $-\tau \leq t \leq T$, *which satisfies condition* (2).

Moreover, as follows from the proof, each continuous solution $x(t)$ to (1)-(2) satisfying $x(t) \geq a$ for $-\tau \leq t \leq T$, also satisfies

$$x(t) \leq R \text{ for } 0 \leq t \leq T, \tag{9}$$

where R is so that

$$T = \int_{b}^{R} (1/h(x))\, dx. \tag{10}$$

The proof of Theorem 1 was given by using the topological transversality theorem of Granas and can also be done by using Leray-Schauder continuation theorem. A constructive scheme to solve (1)-(2), namely the successive approximations method, was described in [3] only for the particular case where condition (iv) is replased by the more restrictive Lipschitz condition

(iv") there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

for all $t \in [-\tau, T]$ and $x,y \in [a, +\infty)$.

The aim of this paper is to give a constructive scheme to solve (1)-(2) under assumptions (i)-(iv) provided that $f(t,x)$ is .nonotone with respect to x. Uniqueness will be also discussed. In case $f(t,x)$ is nondecreasing in x, our results are somewhat similar with those in [2] referring to periodic solutions of (1).

2. Main results. Let E be the Banach space of all continuous functions $x(t)$, $0 \leq t \leq T$ with norm

$$\|x\| = \max_{0 \leq t \leq T} |x(t)|.$$

Consider the closed subset of E:

$$X = \{x \in E;\ x(0) = b \text{ and } x(t) \geq a \text{ for } 0 \leq t \leq T\}$$

and the delay integral operator

$$A\colon E \to X,\ Ax(t) = \int_{t-\tau}^{t} f(s, \tilde{x}(s))\, ds$$

where $\tilde{x}(s) = x(s)$ for $0 < s \leq T$ and $\tilde{x}(s) = \phi(s)$ for $-\tau \leq s \leq 0$. A is completely continuous as an operator from X into X.

THEOREM 2. *Let* (i)-(iv) *be satisfied. Suppose that* $f(t,x)$ *is nondecreasing in* x *for* $a \leq x \leq R$. *Denote*

$$U_0(t) = a \text{ for } 0 \leq t \leq T$$

$$U_n(t) = AU_{n-1}(t) \text{ for } 0 \leq t \leq T \ (n = 1, 2, \ldots).$$

Then, $U_n(t) \to x_*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$, $x_*(t)$ *is the minimal solution to* (1)-(2) *in* X *and*

$$a \leq U_1(t) \leq \ldots \leq U_n(t) \leq \ldots \leq x_*(t) \leq R \text{ for } 0 \leq t \leq T.$$

Proof. By Theorem 1 there exists at least one solution in X to (1)-(2). Let $x_1(t)$ be any solution to (1)-(2). We have

$$a = U_0(t) \leq x_1(t) \leq R \text{ for } 0 \leq t \leq T.$$

Consequently, since A is nondecreasing on interval $[a,R]$ of E

$$U_1(t) = AU_0(t) \leq Ax_1(t) = x_1(t).$$

On the other hand, by (iii), we have $a = U_0(t) \leq U_1(t)$. Hence

$$U_0(t) \le U_1(t) \le x_1(t) \text{ for } 0 \le t \le T.$$

Now we inductively find that

$$a \le U_1(t) \le U_2(t) \le \ldots \le U_n(t) \le \ldots \le x_1(t) \text{ for } 0 \le t \le T.$$

A being completely continuous on X, the sequence $(AU_n)_{n\ge1}$ must contain a subsequence, say $(AU_{n_k})_{k\ge1}$, convergent to some $x_* \in X$. But $AU_{n_k}(t) = U_{n_k+1}(t)$ and taking into account the monotonicity of $(U_n(t))_{n\ge1}$, we obtain that $U_n(t) \to x_*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$ and

$$U_n(t) \le x_*(t) \le x_1(t) \text{ for } 0 \le t \le T \ (n = 0, 1, \ldots).$$

Letting $n \to \infty$ in $AU_n(t) = U_{n+1}(t)$ we get $Ax_*(t) = x_*(t)$, i.e. $x_*(t)$ is a solution to (1)-(2). Finally, by $x_*(t) \le x_1(t)$ where $x_1(t)$ was any solution to (1)-(2), we see that $x_*(t)$ is the minimal solution to (1)-(2) in X.

The following result is concerning with the existence and approximation of the maximal solution in X to (1)-(2).

THEOREM 3. *Let* (i)-(iv) *be satisfied. Suppose that there exists* $R_0 \ge R$ *such that*

$$f(t, R_0) \le R_0/\tau \text{ for } -\tau \le t \le T \tag{11}$$

(i.e. $f(t, \phi(t)) \le R_0/\tau$ *for* $-\tau \le t \le 0$ *and* $f(t, R_0) \le R_0/\tau$ *for* $0 < t \le T$) *and* $f(t,x)$ *is nondecreasing in* x *for* $a \le x \le R_0$. *Denote* $V_0(t) = R_0$ *for* $0 \le t \le T$,

$$V_n(t) = AV_{n-1}(t) \text{ for } 0 \le t \le T \ (n = 1,2,\ldots).$$

Then, $V_n(t) \to x^*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$, $x^*(t)$ *is the maximal solution to* (1)-(2) *in* X *and*

$$x^*(t) \le \ldots \le V_n(t) \le \ldots \le V_2(t) \le V_1(t) \le R_0 \text{ for } 0 \le t \le T.$$

Proof. By (11) we have

$$V_1(t) \le V_0(t) = R_0 \text{ for } 0 \le t \le T.$$

Next, the proof is analog to that of Theorem 2.

THEOREM 4. *Let the conditions of Theorem* 2 *be satisfied. Suppose that there exists* $\alpha \in (0,1)$ *such that*

$$f(t,\gamma x) \ge \gamma^\alpha f(t,x) \text{ for all } \gamma \in (0,1),\ t \in [0,T],\ x \in [a,R]. \qquad (12)$$

Then, (1)-(2) *has a unique solution in* X.

Proof. Let $x_1(t)$ be any solution in X to (1)-(2). We will show that $x_1(t) = x_*(t)$. Let

$$\gamma_0 = \min_{0 \le t \le T} (x_*(t)/x_1(t)).$$

Since $a \le x_*(t) \le x_1(t) \le R$, we have $a/R \le \gamma_0 \le 1$. Now, we show $\gamma_0 = 1$. In fact, if $\gamma_0 < 1$, then (12) implies

$$x_*(t) = Ax_*(t) \ge A(\gamma_0 x_1)(t) = \int_{t-\tau}^{t} f(s, \gamma_0 x_1(s))\, ds$$

$$\geq \gamma_0^\alpha \int_{t-\tau}^{t} f(s, \tilde{x}_1(s))\, ds = \gamma_0^\alpha A x_1(t) = \gamma_0^\alpha x_1(t).$$

Thus $\gamma_0 \geq \gamma_0^\alpha$, which is impossible for $0 < \alpha < 1$. Therefore, $\gamma_0 = 1$ and $x_*(t) = x_1(t)$.

THEOREM 5. *Let the conditions of Theorem* 3 *and Theorem* 4 *be satisfied. Then,* (1)-(2) *has a unique solution* $x_*(t)$ *in* X *and for any* $x_0(t)$ *in* E *satisfying* $a \leq x_0(t) \leq R_0$ *for all* $t \in [0,T]$, *we have* $_n(t) \to x_*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$, *where*

$$x_n(t) = A x_{n-1}(t) \quad (n = 1, 2, \ldots).$$

Proof. We find from

$$a = U_0(t) \leq x_0(t) \leq V_0(t) = R_0$$

that

$$U_n(t) \leq x_n(t) \leq V_n(t) \quad (n = 1, 2, \ldots).$$

On the other hand, by Theorem 2 and Theorem 3, we have that

$$U_n(t) \to x_*(t) \text{ and } V_n(t) \to x_*(t)$$

uniformly in $t \in [0,T]$ as $n \to \infty$. Therefore, $x_n(t) \to x_*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$.

The following result refers to functions $f(t, x)$ which are nonincreasing in x.

THEOREM 6. *Let* (i)-(iv) *be satisfied. Denote* $R_0 = \max(R, \|U_1\|)$ *and suppose* $f(t,x)$ *is nonincreasing in* x *for* $a \leq x \leq R_0$. *Also suppose that there exists* $\alpha \in (-1,0)$ *such that*

$$f(t,\gamma x) \leq \gamma^{\alpha} f(t,x) \text{ for } \gamma \in (0,1),\ t \in [0,T],\ x \in [a,R_0]. \tag{13}$$

Then, (1)-(2) *has a unique solution* $x_*(t)$ *in* X,

$$a = U_0(t) \leq V_1(t) \leq \ldots \leq U_{2n}(t) \leq V_{2n+1}(t) \leq \ldots \leq x_*(t) \leq$$

$$\ldots \leq U_{2n+1}(t) \leq V_{2n}(t) \leq \ldots \leq U_1(t) \leq V_0(t) = R_0 \text{ for } 0 \leq t \leq T,$$

and $U_n(t) \to x_*(t)$, $V_n(t) \to x_*(t)$ *uniformly in* $t \in [0,T]$ *as* $n \to \infty$.

Proof. By Theorem 1 there exists as least one solution $x_1(t)$ to (1) - (2) and $a \leq x_1(t) \leq R$ for $0 \leq t \leq T$. We have

$$a = U_0(t) \leq x_1(t) \leq V_0(t) = R_0$$

whence

$$V_1(t) \leq x_1(t) \leq U_1(t).$$

But, by (iii), $a \leq V_1(t)$. Also $U_1(t) \leq \|U_1\| \leq R_0$. Hence

$$U_0(t) \leq V_1(t) \leq x_1(t) \leq U_1(t) \leq V_0(t).$$

It follows

$$U_0(t) \leq V_1(t) \leq U_2(t) \leq x_1(t) \leq V_2(t) \leq U_1(t) \leq V_0(t).$$

Finally

$$a = U_0(t) \le V_1(t) \le \dots \le U_{2n}(t) \le V_{2n+1}(t) \le \dots$$

$$\dots \le x_1(t) \le \dots \le U_{2n+1}(t) \le V_{2n}(t) \le \dots \le U_1(t) \le V_0(t) = R_0. \quad (14)$$

A being completely continuous on X, the sequence $(AU_{2n-1}(t))_{n\ge 1}$ contains a subsequence convergent to some $y_*(t)$ in X and similarly, $(AV_{2n-1}(t))_{n\ge 1}$ contains a subsequence converging to some $y^*(t)$ in X. Now, from (14) we see that

$$U_{2n}(t) \to y_*(t),\ V_{2n+1}(t) \to y_*(t)$$

$$U_{2n+1}(t) \to y^*(t),\ V_{2n}(t) \to y^*(t) \quad (15)$$

uniformly in $t \in [0,T]$ as $n \to \infty$ and

$$y_*(t) \le x_1(t) \le y^*(t). \quad (16)$$

By (15), it follows that

$$y^*(t) = Ay_*(t) \text{ and } y_*(t) = Ay^*(t).$$

Now, we prove that under assumption (13), we have indeed $y_*(t) = y^*(t)$. To do this, let

$$\gamma_0 = \min_{0\le t\le T} (y_*(t) / y^*(t)).$$

Obviously, $0 < a/R_0 \le \gamma_0 \le 1$. We will show that $\gamma_0 = 1$. In fact, if $\gamma_0 < 1$, then (13) implies

$$y^* = Ay_* \le A(\gamma_0 y^*) = \int_{t-\tau}^{t} f(s, \widetilde{\gamma_0 y^*}(s))\, ds \le$$

$$\leq \gamma_0^{\alpha} \int_{t-\tau}^{t} f(s, \tilde{y}^*(s))\, ds = \gamma_0^{\alpha} A y^* = \gamma_0^{\alpha} y_* .$$

Therefore, $\gamma_0^{-\alpha} \leq \gamma_0$ or, equivalently, $\alpha \leq -1$, a contradiction. Thus, $\gamma_0 = 1$ as claimed. Consequently, $y_* = y^*$. The proof is complete.

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