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BERNSTEIN POLYNOMIALS OVER SIMPLICES

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REZUMAT. - Polinoame Bernstein pe simplexuri. În această lucrare autorii se ocupă de studiul unor proprietăţi ale polinoamelor Bernstein definite pe un simplex arbitrar din $\mathbf{R}^s$. Se pun în evidenţă anumite relaţii care au loc între funcţiile convexe în T şi şirurile polinoamelor Bernstein corespunzătoare.

Abstract. In this paper the authors are concerned with a study of the multivariate Bernstein polynomials over an arbitrary simplex in $\mathbf{R}^s$. Some relations between convex functions in T and the sequences of the corresponding Bernstein polynomials are shown.[**]

Let $T_0, T_1, \ldots, T_s$ be $(s + 1)$ affinely independent points of $\mathbf{R}^s$, $s \geq 1$. The s-dimensional simplex T is defined by

$$T = \operatorname{span}\{T_0, \ldots, T_s\}.$$

Each point $P \in T$ can be uniquely expressed by

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$$P = \sum_{i=0}^{s} u_i T^i$$

such that $u_i \geq 0$, $i = 0, \ldots, s$, $\sum_{i=0}^{s} u_i = 1$; the $(s + 1)$ tuple $u = (u_0, \ldots, u_s)$ is called the barycentric coordinates of P with respect to the simplex T.

Let us define the basic functions

$$B_\alpha^n(P) = \frac{|\alpha|!}{\alpha!} u^\alpha, \tag{1}$$

$\alpha = (\alpha_0, \ldots, \alpha_s) \in Z_+^{s+1}$, $|\alpha| = \sum_{i=0}^{s}$, $\alpha! = \alpha_0! \ldots \alpha_s$, $|\alpha| = n$, $u^\alpha = u_0^{\alpha_0} \ldots u_s^{\alpha_s}$, $\sum_{i=0}^{s} B_\alpha^n(P) = 1$.

The points $x_\alpha = \dfrac{\alpha}{n}$, $\alpha \in Z_+^{s+1}$ are called nodes of the simplex T, it means that their barycentric coordinates are $\left(\dfrac{\alpha_0}{n}, \ldots, \dfrac{\alpha_s}{n}\right)$.

For any function $f(P)$ continuous on T the multivariate Bernstein polynomials defined by

$$B_n(f; P) = \sum_{\substack{|\alpha| = n \\ \alpha \in Z_+^{s+1}}} B_\alpha^n(P) f\left(\frac{\alpha}{n}\right) \tag{2}$$

converge to $f(P)$ uniformly on T as $n \to \infty$. Properties of the multivariate Bernstein polynomials have been also studied in [2], [3], [4], [10], [11], [12], [13], [14].

Now some properties of multivariate Bernstein polynomials are stated.

For a given interior point $P \in T$, $P = (u_0, \ldots, u_s)$ and a number δ, $u_i > \delta$

> 0, $i = 0, \ldots, s$ we define

$$T_{P,\delta} = \{Q = (v_0, \ldots, v_s) \mid v_i \geq u_i - \delta,\ i = 0, \ldots, s\}.$$

This is a closed simplex contained by the simplex T and containing P as its focal point. Each edge of $T_{P,\delta}$ is parallel to the corresponding edge of T.

LEMMA 1. *Let* $P = (u_0, \ldots, u_s)$, $P \in \operatorname{int} T$ *and* $0 < \delta < u_i$, $i = 0, \ldots, s$. *Then*

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \leq \sum_{i=0}^{s} e^{-\frac{n\delta^2}{4(1-u_i)}}. \tag{3}$$

Proof. By the definition of $T_{P,\delta}$ it is clear that $\frac{\alpha}{n} \notin T_{P,\delta}$ if there exists $k \in \{0, \ldots, s\}$ such that $\frac{\alpha_k}{n} < u_k - \delta$. Then

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \leq \sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} < u_i - \delta}} P_\alpha^n(P)$$

Let us define functions $G_i(x)$, $i = 0, \ldots, s$, as follows

$$G_i(x) = \sum_{|\alpha| = n} B_\alpha^n(P)\, e^{x(\alpha_i - u_i n)},\ x \in \mathbf{R}. \tag{4}$$

It is easy to show (using the fact $\sum_{i=0}^{s} u_i = 1$) that

$$G_i(x) = (e^{-xu_i}(1 - u_i) + u_i e^{x(1-u_i)})^n.$$

Let us denote

$$\varphi_i(x) = e^{-xu_i}(1 - u_i) + u_i e^{x(1-u_i)}.$$

And now in the same way as in [9], [6] it can be shown that

$$\varphi_i(x) \le 1 + u_i x^2(1 - u_i) \le e^{u_i x^2(1 - u_i)}$$

under the assumption $|x| \le 3/2$.

From it follows

$$G_i(x) \le e^{n u_i x^2(1 - u_i)}. \tag{5}$$

Let t be an arbitrary positive real number. Then

$$G_i(x) = \sum_{|\alpha| = n} B_\alpha^n(P) e^{x(\alpha_i - u_i n)} \ge \sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i(x)}} B_\alpha^n(P) e^{x(\alpha_i - n u_i)} >$$

$$> \sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i(x)}} B_\alpha^n(P) e^t G_i(x).$$

This gives the following estimate

$$\sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t G_i x)}} B_\alpha^n(P) < e^{-t} \tag{6}$$

Now, using (5) we obtain

$$\sum_{\substack{|\alpha| = n \\ e^{x(\alpha_i - n u_i)} > e^t e^{n x^2(1 - u_i)}}} B_\alpha^n(P) < e^{-t}. \tag{7}$$

Let $t = \dfrac{n\delta^2}{4(1 - u_i)}$, $x = -\dfrac{\delta}{2(1 - u_i)}$ then $|x| \le 3/2$ and (7) gives

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha_i}{n} < u_i - \delta}} B_\alpha^n(P) < e^{-\frac{n\delta^2}{4(1 - u_i)}}.$$

And this estimate concludes our proof

$$\sum_{\substack{|\alpha| = n \\ \frac{\alpha}{n} \notin T_{P,\delta}}} P_\alpha^n(P) \le \sum_{i=0}^{s} e^{-\frac{n\delta^2}{4(1 - u_i)}}. \qquad \blacksquare$$

LEMMA 2. *Let* $P = (u_0, \dots, u_s)$ *be an interior point of* T, $0 < \delta < \frac{u_i}{4s}$, $i = 0, \dots, s$. *Then for* $\frac{\alpha}{n} \in T_{P,\delta}$ *the following inequality*

$$B_\alpha^n(P) \geq K \frac{1}{n^{s/2}} e^{-\frac{3n\delta^2}{4} \sum_{i=0}^{s} \frac{1}{u_i}} \tag{8}$$

holds, where K *is an positive constant independent only on* s.

Proof. Let us remind Stirling's formula

$$n! = \sqrt{2\pi n}\, n^n e^{-n} H_n,\ H_n = e^{\frac{\theta}{12n}},\ 0 \leq \theta < 1$$

i.e.

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Then

$$B_\alpha^n(P) = \frac{n!}{\alpha!} u^\alpha > \frac{\sqrt{2\pi n}\, n^n e^{-n} \prod_{i=0}^{s} u_i^{\alpha_i}}{\prod_{i=0}^{s} \sqrt{2\pi\alpha_i}\, \alpha_i^{\alpha_i} e^{-\alpha_i} e^{\frac{1}{12\alpha_i}}} =$$

$$= \frac{1}{\prod_{i=0}^{s} e^{\frac{1}{12\alpha_i}}} \frac{1}{(\sqrt{2\pi})^s} \frac{\sqrt{|\alpha|}}{\sqrt{\alpha_0 \dots \alpha_s}} \prod_{i=0}^{s} \left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} \prod_{i=0}^{s} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)}. \tag{9}$$

Denote

$$L_{\alpha_i} = \left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)},\ i = 0, \dots, s.$$

As it was proved in [5]

$$\left(\frac{|\alpha|}{\alpha_i} u_i\right)^{\alpha_i} e^{|\alpha|\left(\frac{\alpha_i}{|\alpha|} - u_i\right)} \geq e^{-\frac{3|\alpha|}{4u_i}\left(\frac{\alpha_i}{|\alpha|} - u_i\right)^2} \tag{10}$$

provided that

$$\left|\frac{\alpha_i}{|\alpha|} - u_i\right| < \frac{u_i}{4},\ i = 0,\dots,s. \tag{11}$$

It is easy to see that these assumptions are satisfied. From $\frac{\alpha_i}{n} \ge u_i - \delta$ it follows immediately $s\delta \ge \delta \ge u_i - \frac{\alpha_i}{n}$. On the other hand the equalities $\sum_{i=0}^{s} \frac{\alpha_i}{n} = 1$ and $\sum_{i=0}^{s} u_i = 1$ give $s\delta \ge \frac{\alpha_i}{n} - u_i$. Together with the assumptions of lemma we have $\frac{u_i}{4} > s\delta \ge \left|\frac{\alpha_i}{|\alpha|} - u_i\right|$.

Therefore if inequalities (11) are satisfied the

$$\prod_{i=0}^{s} L\alpha_i \ge \prod_{i=0}^{s} e^{-\frac{3|\alpha|}{4u_i}\delta^2} = e^{-\frac{3|\alpha|}{4u_i}\delta^2 \sum_{i=0}^{s}\frac{1}{u_i}} \tag{12}$$

Further

$$\prod_{i=0}^{s} e^{\frac{1}{12\alpha_i}} = e^{\sum_{i=0}^{s}\frac{1}{12\alpha_i}} < e^{s+1} = C \tag{13}$$

and

$$\left(\frac{|\alpha|}{\prod_{i=0}^{s}\alpha_i}\right)^{\frac{1}{2}} \ge \frac{M}{|\alpha|^{\frac{s}{2}}} \tag{14}$$

for

$$\frac{|\alpha|}{\alpha_i} \ge \frac{1}{\varepsilon + u_i},\ i = 0,\dots,s$$

where the constants C and M are independent on α.

Summarizing (9), (10), (13) and (14) we obtain

$$B_\alpha^n(P) \ge K\frac{1}{n^{\frac{s}{2}}} e^{-\frac{3n\delta^2}{4}\sum_{i=0}^{s}\frac{1}{u_i}}. \qquad \blacksquare$$

LEMMA 3. *Let $\Omega \subset T$ be a simplex with edges parallel to those of the given simplex T. Let N_Ω be a number of nodes belonging to Ω. Then there exists a positive number n_0 such that*

$$N_\Omega > \gamma n^s \tag{15}$$

if $n \geq n_0$, where $\gamma > 0$ is a constant.

The proof is simple.

The following theorem can be proved

THEOREM 1. *Let $f \in C(T)$ be convex on T. Then*

$$B_n(f;P) \geq f(P), \quad B_n(f;P) \geq B_{n+1}(f;P)$$

for all $n \geq 1$ and all $P \in T$.

See [3] for the proof.

It is well-known that for univariate Bernstein polynomials so-called converse theorems hold ([5], [7], [8], [15]):

(i) $B_n(f;x) \geq f(x)$, $x \in [0,1]$, $n \geq 1 \Rightarrow f$ is convex in [0,1].

(ii) $B_n(f;x) \geq B_{n+1}(f;x)$, $x \in [0,1]$, $n \geq 1 \Rightarrow f$ is convex in [0,1].

But it is impossible to extend directly these converse theorems to the Bernstein polynomials over simplices.

As concerns Bernstein polynomials over triangles this problem was solved in [1]. In [6] there was given a different approach to this problem. Now we are going to prove the following theorem:

THEOREM 2. *Let* $f \in C(T)$ *and* $B_n(f; P) \geq f(P)$ *for all* $P \in T$ *and all natural numbers n. Then the function f does not attain its strict local maximum inside T.*

Proof. Let us suppose that f attains a strict local maximum at the interior point $Q = (u_0, \ldots, u_s)$. Without lost of generality it is possible to put $f(Q) = 0$. Then there exists a subsimplex T_{Q,δ_1}, $0 < \delta_1 < u_i$, $i = 0, \ldots, s$ containing Q as an interior point such $f(P) \leq 0$ for all $P \in T_{Q,\delta_1}$ and let $t = \min\{u_0, \ldots, u_t\}$.

Let us choose δ_2 in such a way that

$$0 < \delta_2 < \min\left(\frac{\delta_1}{4s}, \frac{\delta_1}{\sqrt{3\varepsilon(1-t)}}\right), \quad \varepsilon = \sum_{i=0}^{s} \frac{1}{u_i}.$$

Then $T_{Q,\delta_1} \supset T_{Q,\delta_2}$. Further let $\Omega \subset T_{Q,\delta_2}$ be a subsimplex with edges parallel to the corresponding edges of T_{Q,δ_2} and $f(P) < 0$ for all $P \in \Omega$. Let $(-h)$ be a maximum of f over the subsimplex Ω and let $M = \max_{P \in T} |f(P)|$.

Now let us evaluate $B_n(f; Q)$. It is

$$B_n(f;Q) = \sum_{\frac{\alpha}{n} \notin T_{Q,\delta_1}} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) + \sum_{\frac{\alpha}{n} \in T_{Q,\delta_1} - \Omega} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) + \sum_{\frac{\alpha}{n} \in \Omega} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) \quad (16)$$

Using lemma 1 we obtain for the first sum

$$\left| \sum_{\frac{\alpha}{n} \notin T_{Q,\delta_1}} f\left(\frac{\alpha}{n}\right) B_\alpha^n(Q) \right| \le M \sum_{\frac{\alpha}{n} \notin T_{Q,\delta_1}} B_\alpha^n(Q) \le M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)}}. \qquad (17)$$

Further as far as the second sum is concerned one can state it is nonpositive.

And now the sum will be estimated: $\Omega \subset T_{\Omega,\delta_2}$ and due to this reason it is

$$\sum_{\frac{\alpha}{n} \in \Omega} B_\alpha^n(Q) \ge K \sum_{\frac{\alpha}{n} \in \Omega} \frac{1}{|\alpha|^{\frac{\delta}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon}$$

Now the use of lemma 3 gives

$$\sum_{\frac{\alpha}{n} \in \Omega} B_\alpha^n(Q) \le -hL \frac{n^s}{n^{\frac{s}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon}$$

Then

$$B_n(f;Q) \le M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)}} - hL \frac{n^s}{n^{\frac{s}{2}}}\, e^{-\frac{3n\delta_2^2}{4}\varepsilon} =$$

$$= e^{-\frac{3n\delta_2^2}{4}\varepsilon} \left(M(s+1)\, e^{-\frac{n\delta_1^2}{4(1-t)} + \frac{3n\delta_2^2}{4}\varepsilon} - hLn^{\frac{s}{2}} \right)$$

Under given assumptions from here it follows that $B_n(f;Q) < 0$ and this contradiction concludes our proof. ■

The following theorem can be proved as the consequence of the theorem 2.

THEOREM 3. *If $f \in C(T)$ and the inequality*

$$B_n(f;P) \ge B_{n+1}(f;P)$$

holds for all natural numbers n and all points on T, then the function f does not attain a strict local maximum inside T.

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