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JACK'S, MILLER'S AND MOCANU'S LEMMA FOR HOLOMORPHIC MAPPINGS DEFINED ON DOMAINS WITH DIFFERENTIABLE BOUNDARY OF CLASS C^2

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REZUMAT. - Lema lui Jack-Miller-Mocanu pentru aplicaţii olomorfe pe domenii cu frontieră de clasă C^2. În acest articol vom prezenta varianta n-dimensională a lemei Jack-Miller-Mocanu pentru aplicaţii olomorfe definite pe domenii din $\mathbf{C}^n$ ce au frontieră de clasă C^2. De asemenea vom prezenta şi interpretări geometrice ale rezultatului.

1. Introduction. In several papers [4,5] S.S. Miller and P.T. Mocanu gave the following generalization of the one dimensional Jack's lemma [2] and used it as a basic tool in developing the theory of admissible functions.

LEMMA (Jack-Miller-Mocanu). *Let $f: D \to \mathbf{C}$ be a holomorphic function with $f(0) = 0$ and $f \neq 0$. If $|f(z_0)| = \max_{|z| \le |z_0|} |f(z)|$, $z_0 \in D = \{z \in \mathbf{C} |\ |z| < 1\}$ then there exists a real number $m \geq 1$ such that:*

(i) $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$ *and*

(ii) $\operatorname{Re} \dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m.$

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In a previous paper we extend this result to the case of holomorphic mappings defined on the unit ball of $\mathbf{C}^n$. Since in several complex variables the Riemann mapping theorem fails to be true the purpose of this paper is to study an analogous problem to that studied in [2] for holomorphic mappings defined on arbitrary domains. Also we shall give geometric interpretations of result.

We let $\mathbf{C}^n$ denote the space of n-complex va iables $z = (z_1, \ldots, z_n)'$, with the euclidian inner product $<z, w> = \sum_{j=1}^{n} z_j \bar{w}_j$ and the norm $\|z\| = (<z, z>)^{1/2}$.

Vector and matrices marked with the symbol ' and * denote the t nsposed and the transposed conjugate vector or matrix, respectively.

We denote by $\mathcal{L}(\mathbf{C}^n)$ the space of continuou linear operators from $\mathbf{C}^n$ into $\mathbf{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm:

$$\|A\| = \sup \{\|Az\| : \|z\| \leq 1\}, A \in \mathcal{L}(\mathbf{C}^n).$$

The class of holomorphic mappings $f(z) = (f_1(z), \ldots, f_n(z))'$ from D ($D \subseteq \mathbf{C}^n$ domain) into $\mathbf{C}^n$ is denoted by $\mathcal{H}(D)$.

We denote by $Df(z)$ and $D^2f(z)$ the first and the second Fréchet derivatives of f at z.

We say that $f \in \mathcal{H}(D)$ is locally biholomorphic (locally univalent) at $z \in D$ if f has a local holomorphic inverse at z, or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j, k \le n} \quad \text{is nonsingular.}$$

The open set $D \subseteq \mathbf{C}^n$ is said to have differentiable boundary bD of class C^2, at the point $\dot{z} \in bD$ if there are an open neighborhood U of $\dot{z}$ and a real valued function $\varphi \in C^2(D)$ with the following properties:

$$U \cap D = \{z \in U: \varphi(z) < 0\} \tag{1}$$

$$\frac{\partial \varphi}{\partial z}(z) = \left(\frac{\partial \varphi}{\partial z_1}(z), \ldots, \frac{\partial \varphi}{\partial z_n}(z)\right)' \neq 0 \text{ for } z \in U. \tag{2}$$

bD is of class C^2 if it is of class C^2 at every $z \in bD$.

Notice that (1) and (2) imply

$$U \cap bD = \{z \in U: \varphi(z) = 0\} \text{ and } U - \bar{D} = \{z \in U: \varphi(z) > 0\}. \tag{3}$$

Any function $\varphi \in C^2(U)$ which satisfies (1) and (2) is called a (local) defining function for bD at $\dot{z}$.

For a real valued function $\varphi \in C^2(U)$ ($U \in \mathbf{C}^n$) we define:

$$\frac{\partial^2 \varphi}{\partial z^2}(z) = \left(\frac{\partial^2 \varphi(z)}{\partial z_j \partial z_k}\right)_{1 \le j, k \le n} \tag{4}$$

and

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}}(z) = \left(\frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k}\right)_{1 \le j, k \le n} \tag{5}$$

2. Main result.

THEOREM. *Let D be a bounded domain in $\mathbf{C}^n$ with $0 \in D$. Suppose $f \in C(\bar{D}) \cap \mathcal{H}(D \cup \{\dot{z}\})$, $f(0)=0, f \neq 0$ and $Df(\dot{z})$ is nonsingular where $\dot{z} \in bD$ is defined by:*

$$\|f(\dot{z})\| = \max_{z \in \bar{D}} \|f(z)\|.$$

If D has differentiable boundary bD of class C^2 at the point $\dot{z} \in bD$ with the locally defining function φ then there exists a real positive number m such that:

(i) $$((Df(\dot{z}))^*)^{-1}\left(\overline{\frac{\partial \varphi}{\partial z}}(\dot{z})\right) = m f(\dot{z}) \tag{6}$$

and

(ii) $$\frac{w^* \dfrac{\partial^2 \varphi}{\partial z \partial \bar{z}} w + \operatorname{Re}\left(w' \dfrac{\partial^2 \varphi}{\partial z^2} w - \left(\dfrac{\partial \varphi}{\partial z}(\dot{z})\right)' (D^2 f(\dot{z}))^{-1} D^2 f(\dot{z})(w,w)\right)}{\|Df(\dot{z}) w\|^2} \geq m \tag{7}$$

for all $w \in \mathbf{C}^n \setminus \{0\}$ which satisfy $\operatorname{Re} < w, \overline{\dfrac{\partial \varphi}{\partial z}}(\dot{z}) > = 0.$

Proof. For $z = (z_1, \dots, z_n)' \in \mathbf{C}^n$, each coordinate z_j can be written as $z_j = a_j + i a_{j+n}$, with $a_j, a_{n+j} \in \mathbf{R}$

The mapping $z \to (a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})' \in \mathbf{R}^{2n}$ establishes an $\mathbf{R}$ linear isomorphism between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$, i.e. we obtain the natural identification between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$.

By using the weak maximum modulus theorem [1] we obtain that $\mathring{z} \in bD$ and $\mathring{z} \in \bar{D}$ is a point of local conditional maximum of the function $\|f(z)\|$ under the condition $z \in bD$.

Since $Df(\mathring{z})$ is nonsingular and D has a differentiable boundary at $\mathring{z} \in bD$ it follows that there exists an open neighborhood U of $\mathring{z}$ and φ a real function such that (1), (2) and (3) hold and also f is injective on U.

Next, we shall use method of Lagrange's multipliers.

Let F: $(\mathring{a}_1 - \varepsilon, \mathring{a}_1 + \varepsilon) \times \ldots \times (\mathring{a}_{2n} - \varepsilon, \mathring{a}_{2n} + \varepsilon) \to \mathbf{R}$

$$F(a_1, \ldots, a_{2n}) = \sum_{i=1}^{2n} |f_i(a_1, \ldots, a_{2n})|^2 - \lambda\varphi(a_1, \ldots, a_{2n}) \tag{8}$$

where $\lambda \in \mathbf{R}$ and $\varepsilon > 0$ is sufficiently small so that

$$(\mathring{a}_1 - \varepsilon, \mathring{a}_1 + \varepsilon) \times \ldots \times (\mathring{a}_{2n} - \varepsilon, \mathring{a}_{2n} + \varepsilon) \subset U.$$

Since $(\mathring{a}_1, \ldots, \mathring{a}_{2n})$ is a point of local maximum for the function $\|f(a_1, \ldots, a_{2n})\|^2$ under the condition $\varphi(a_1, \ldots, a_{2n}) = 0$ we obtain:

$$\frac{\partial F}{\partial a_i}(\mathring{a}_1, \ldots, \mathring{a}_{2n}) = 0, \; i \in \{1, \ldots, 2n\} \tag{9}$$

and

$$d^2F(\mathring{a}_1, \ldots, \mathring{a}_{2n})(t, t) \leq 0 \text{ for all } t \in \mathbf{R}^{2n} \setminus \{0\} \tag{10}$$

which satisfy $\sum_{i=1}^{2n} t_i \frac{\partial \varphi}{\partial a_i}(\mathring{a}_1, \ldots, \mathring{a}_{2n}) = 0.$

A simple calculation yields:

$$\frac{\partial F}{\partial a_j}(\mathring{a}_1,\ldots,\mathring{a}_{2n})=\sum_{l=1}^{n}\left(\frac{\partial f_l}{\partial z_j}(\dot{z})\bar{f}_l(\dot{z})+f_l(\dot{z})\frac{\partial \bar{f}_l}{\partial \bar{z}_j}(\dot{z})\right)-\lambda\frac{\partial\varphi}{\partial a_j}(\dot{z})=0, \tag{11}$$

for $j \in \{1, \ldots, n\}$ and

$$\frac{\partial F}{\partial a_{j+n}}(\mathring{a}_1,\ldots,\mathring{a}_{2n})=i\sum_{l=1}^{n}\left(\frac{\partial f_l}{\partial z_j}(\dot{z})\bar{f}_l(\dot{z})-f_l(\dot{z})\frac{\partial \bar{f}_l}{\partial \bar{z}_j}(\dot{z})\right)-\lambda\frac{\partial\varphi}{\partial a_{j+n}}(\dot{z})=0 \tag{12}$$

for $j \in \{1, \ldots, n\}$.

Since

$$\frac{\partial F}{\partial a_j}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) - i\frac{\partial F}{\partial a_{j+n}}(\mathring{a}_1,\ldots,\mathring{a}_{2n}) = 0 \text{ for } j\in\{1,\ldots,n\},$$

we easily obtain:

$$\sum_{l=1}^{n}\frac{\partial f_l}{\partial z_j}(\dot{z})\bar{f}_l(\dot{z}) = \lambda\frac{\partial\varphi}{\partial z_j}(\dot{z}) \text{ for all } j\in\{1,\ldots,n\}. \tag{13}$$

From the relations (13) we get:

$$((Df(\dot{z}))^{*})f(\dot{z}) = \lambda\overline{\frac{\partial\varphi}{\partial z}}(\dot{z}). \tag{14}$$

By using the fact that $Df(\dot{z})$ is nonsingular we obtain:

$$((Df(\dot{z}))^{*})^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}}(\dot{z})\right) = mf(\dot{z}) \tag{15}$$

where by m we denote the real number $\frac{1}{\lambda}$ (Indeed, if $\lambda = 0$ we obtain $f(\dot{z}) = 0$ which contradicts the assumption $f \neq 0$).

In order to prove (i) of the Theorem it remains to show that m is positive.

We now let ψ: $f(U) \to \mathbf{R}$ defined by

$$\psi(w) = \varphi((f\,|\,U)^{-1}(w))$$

If t is a small enough real positive number we have that $(1 + t)f(\dot{z}) \in f(U)$ and $(1 + t)f(\dot{z}) \notin f(D)$. Hence $\psi((1 + t)f(\dot{z})) > 0$.

A simple calculations yields:

$$0 \leq \lim_{\substack{t\to 0\\ t>0}} \frac{\psi((1+t)f(\dot{z})) - \psi(f(\dot{z}))}{t} = \sum_{i=1}^{n} \frac{\partial\psi}{\partial w_i}(f(\dot{z}))f_i(\dot{z}) =$$

$$= \sum_{i,k=1}^{n} f_i(\dot{z}) \frac{\partial (f^{-1})k}{\partial w_i}(f(\dot{z})) \frac{\partial\varphi(\dot{z})}{\partial z_k} = \sum_{i=1}^{n} f_i(\dot{z})\frac{1}{\lambda}\overline{f_i(\dot{z})} = \frac{1}{\lambda}\|f(\dot{z})\|^2.$$

Hence $\lambda > 0$ and in consequence $m > 0$ too.

The second differential of F at the point $\dot{z}$ is negative semidefinite.

Straightforward calculations given us:

$$d^2F(\dot{z})(t,t) = \sum_{i,j,k=1}^{n} \frac{\partial^2 f_i(\dot{z})}{\partial z_j \partial z_k}\overline{f_i(\dot{z})}(t_j + it_{j+n})(t_k + it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial^2 \overline{f}_i(\dot{z})}{\partial \overline{z}_j \partial \overline{z}_k} f_i(\dot{z})(t_j - it_{j+n})(t_k - it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial f_i}{\partial z_j}\frac{\partial \overline{f}_i}{\partial \overline{z}_k}(t_j + it_{j+n})(t_k - it_{k+n}) +$$

$$+ \sum_{i,j,k=1}^{n} \frac{\partial f_i}{\partial z_k}\frac{\partial \overline{f}_i}{\partial \overline{z}_j}(t_k + it_{k+n})(t_j - it_{j+n}) -$$

$$- \frac{1}{m}\sum_{i,j=1}^{2n} \frac{\partial\varphi}{\partial a_j \partial a_k} t_j t_k$$

for $t \in \mathbb{R}^{2n}\setminus\{0\}$ with $\sum_{i=1}^{2n} t_i \frac{\partial\varphi}{\partial a_i}(\dot{z}) = 0$.

If we note $w_j = t_j + it_{j+n}$, $j \in \{1,\dots,n\}$, $w = (w_1,\dots,w_n)$, and use (4) and (5) then the above inequality becomes:

$$2\operatorname{Re}\left((f(\dot z))^* D^2 f(\dot z)(w,w)\right) + 2\,\|Df(\dot z)w\|^2 - \\ - \frac{2}{m} w^* \frac{\partial^2\varphi}{\partial z\,\partial \bar z} w - \frac{2}{m}\operatorname{Re} w' \frac{\partial^2\varphi}{\partial z^2} w \le 0. \tag{16}$$

From (15) we get that $(f(\dot z))^* = \frac{1}{m}\left(\frac{\partial\varphi}{\partial z}(\dot z)\right)'(Df(\dot z))^{-1}$ and substituting into 16) we obtain:

$$\frac{1}{m}\operatorname{Re}\left(\left(\frac{\partial\varphi}{\partial z}(\dot z)\right)'(Df(\dot z))^{-1}D^2 f(\dot z)(w,w)\right) + \|Df(\dot z)w\|^2 - \\ - \frac{1}{m} w^* \frac{\partial^2\varphi}{\partial z\,\partial \bar z} w - \frac{1}{m}\operatorname{Re} w' \frac{\partial^2\varphi}{\partial z^2} w \le 0$$

which is equivalent with (7).

The condition for $t \in \mathbf{R}^{2n}\setminus\{0\}$ gives the following condition for $w \in \mathbf{C}^n\setminus\{0\}$ (obtained by the natural identification between $\mathbf{R}^{2n}$ and $\mathbf{C}^n$ mentioned above) $\operatorname{Re} < w, \frac{\partial\varphi}{\partial z}(\dot z) > = = \sum_{j=1}^{2n} t_j \frac{\partial\varphi}{\partial a_j}(\dot z) = 0$ and this completes the proof.

3. Geometric interpretation of the main result. In the following remarks we shall give some geometric consequences of Theorem.

First we note that if $Df(\dot z)$ is nonsingular there exists a neighborhood U of $\dot z$ so that f is injective on U and since f is holomorphic we obtain that f is a biholomorphic mapping between U and $f(U)$. So, if we note by M the intersection between U and bD we obtain that $f(M)$ is a real hypersurface.

Since $((Df(\dot{z}))^*)^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}}(\dot{z})\right)$ is an outer normal vector to $f(M)$ at the point $f(\dot{z})$ the part (i) of the Theorem has the following geometric interpretation.

Remark 1. If f is a function which satisfies the requirements of the Theorem and M is the set defined above then the outher normal vector to $f(M)$ at the point $f(\dot{z})$ and the position vector $f(\dot{z})$ are in the same direction.

Let $v = (v_1, \ldots, v_n)'$ be a real tangent vector to $f(M)$ at $f(\dot{z})$.

It follows that $\mathrm{Re} < ((Df(\dot{z}))^*)^{-1}\left(\overline{\frac{\partial\varphi}{\partial z}}(\dot{z})\right), v > = 0.$

We define on $F(M)$ an orientation such as the second fundamental form of the real hypersurface $f(M)$ at $f(\dot{z})$ is

$b(u,u) = \sum_{i,j=1}^{2n} \frac{\partial^2 |f^{-1}(f(\dot{z}))|^2}{\partial b_i \partial b_j} u_i u_j$ where $u \in \mathbf{R}^{2n}\setminus\{0\}$ is a real tangent vector at $f(M)$ in the point $f(\dot{z})$.

It is easy to check that the second fundamental form of the real hypersurface $f(M)$ at $f(\dot{z})$ can be written as:

$$b(v,v) = \frac{v^* \frac{\partial^2\psi(f(\dot{z}))}{\partial w \partial \overline{w}} v + \mathrm{Re}\, v' \frac{\partial^2\psi(f(\dot{z}))}{\partial w^2} v}{\left\| ((Df(\dot{z}))^*)^{-1} \overline{\left(\frac{\partial\varphi(\dot{z})}{\partial z}\right)} \right\|}. \tag{17}$$

We can compute as follows:

$$\frac{\partial^2\psi(f(\mathring{z}))}{\partial w_j \partial w_k} = \sum_i \frac{\partial\varphi}{\partial z_i}(\mathring{z}) \frac{\partial^2(f^{-1})_i(f(\mathring{z}))}{\partial w_j \partial w_k} +$$

$$+ \sum_{i,k} \frac{\partial^2\varphi(\mathring{z})}{\partial z_i \partial z_k} \frac{\partial(f^{-1})_i(f(\mathring{z}))}{\partial w_k} \frac{\partial(f^{-1})_k(f(\mathring{z}))}{\partial w_j} \tag{18}$$

$$\frac{\partial^2\psi(f(\mathring{z}))}{\partial w_j \partial \bar{w}_k} = \sum_i \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} \frac{\partial(f^{-1})_i(f(\mathring{z}))}{\partial w_j} \overline{\frac{\partial(f^{-1})_i(f(\mathring{z}))}{\partial w_k}} \tag{19}$$

Next, by using the following connection between the second derivative of a biholomorphic function f and the second derivative of the inverse function f^{-1}:

$D^2f^{-1}(f(z))(a,b) = -(Df(z))^{-1}D^2f(z)((Df(z))^{-1}a, (Df(z))^{-1}b)$, $a, b \in \mathbf{C}^n$ and substituting (18) and (19) into (17), we obtain:

$$b(v,v) = \frac{u^* \dfrac{\partial\varphi(\mathring{z})}{\partial z \partial \bar{z}} u + \operatorname{Re}\left(u' \dfrac{\partial^2\varphi}{\partial z^2}(\mathring{z}) u - \left(\dfrac{\partial\varphi}{\partial z}(\mathring{z})\right)' (Df(\mathring{z}))^{-1} D^2f(\mathring{z})(u,u)\right)}{\left\| ((Df(\mathring{z}))^*)^{-1} \overline{\left(\dfrac{\partial\varphi}{\partial z}(\mathring{z})\right)} \right\|} \tag{20}$$

where u is defined by $u = Df^{-1}(f(\mathring{z}))(v) = (Df(\mathring{z}))^{-1}(v)$.

Since v is a real tangent vector to $f(M)$ at $f(\mathring{z})$ we have:

$$0 = \operatorname{Re} < v, ((Df(\mathring{z}))^*)^{-1} \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) > = \operatorname{Re} < (Df(\mathring{z}))^{-1} v, \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) > =$$

$$= \operatorname{Re} < u, \overline{\frac{\partial\varphi}{\partial z}}(\mathring{z}) >$$

and by using (ii) of Theorem we obtain:

$$\frac{b(v,v)}{\|v\|^2} \geq \frac{m\|Df(\dot{z})u\|^2}{\|v\|^2\left\|((Df(\dot{z}))^*)^{-1}\left(\frac{\partial\varphi}{\partial z}(\dot{z})\right)\right\|} = \frac{m\|v\|^2}{\|v\|^2\left\|((Df(\dot{z}))^*)^{-1}\left(\frac{\partial\varphi}{\partial z}(\dot{z})\right)\right\|}$$

According to part (i) of the Theorem we get

$$\frac{b(v,v)}{\|v\|^2} \geq \frac{1}{\|f(\dot{z})\|}$$

Since a principal curvature of a real hypersurface $f(M)$ at $f(\dot{z})$ can be writen as $\frac{b(v,v)}{\|v\|^2}$ where v is a principal direction (so v is a real tangent vector) we get the following geometric interpretation of the (ii) of Theorem.

Remark 2. If f is a function which satisfies the requirements of the Theorem and M is the set defined above then all the principal curvatures k_j $(j \in \{1, \dots, 2n-1\})$ of $f(M)$ at the point $f(\dot{z})$ satisfy

$$k_j \geq \frac{1}{\|f(\dot{z})\|},\ j \in \{1, \dots, 2n-1\}$$

Also the mean curvature of $f(M)$ at $f(\dot{z})$ and the Gaussian curvature of $f(M)$ at $f(\dot{z})$ satisfy the same inequality.

REFERENCES

1. B. Chabat, *Introduction à l'analyse complexe.* Tome 2. Fonctions de plusieurs variables, Mir Moscou, 1990.
2. P. Curt, Cs. Varga, *Jack's, Miller's and Mocanu's lemma for holomorphic mappings*

in $\mathbf{C}^n$, to appear.
3. I.S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc., 3(1971), 469-474.
4. S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. of Math. Anal. and Appl., **69**(1978), 289-305.
5. S.S. Miller, P.T. Mocanu, *The theory and applications of second-order differential subordinations*, Studia Univ. Babeş-Bolyai, **34**(1989), 3-34.