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SECOND-ORDER DIFFERENTIAL SUBORDINATIONS IN THE HALF-PLANE

Dorina RADUCANU*

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REZUMAT. - Subordonări diferenţiale de ordinul al doilea în semiplan. În lucrare, folosind subordonările diferenţiale, se obţin proprietăţi ale funcţiilor olomorfe în semiplanul complex care satisfac condiţia de normalizare $f(z)-z\to 0$ pentru $z\to\infty$.

Let Δ denote the upper half - plane

$$\Delta = \{z \in \mathbf{C} / \operatorname{Im} z > 0\}$$

and let $A(\Delta)$ denote the class of functions f which are holomorphic in Δ and have the normalization

$$\lim_{\Delta \ni z \to \infty} [f(z) - z] = 0$$

In this paper, using differential subordinations in the half - plane [2], we obtain some properties concerning functions of the class $A(\Delta)$.

DEFINITION 1 [2]. Let $f,g : \Delta \to \mathbf{C}$ be holomorphic functions in Δ. The function f is *subordinate* to the function g in Δ ($f \prec g$) if there is an holomorphic function φ: $\Delta \to \Delta$ such that $\lim_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0$ and

* *"Transilvania" University, Department of Mathematics, 2200 Braşov, Romania*

$f(z) = g(\varphi(z))$, for all $z \in \Delta$.

THEOREM 1. *Let $f \in A(\Delta)$, $g \in A(\Delta)$ and g is univalent in Δ. Then the function f is subordinate to the function g in Δ if and only if $f(\Delta) \subset g(\Delta)$.*

Proof. If $f \prec g$ then using Definition 1 and Schwarz's Lemma for the upper half-plane [3], [4], it results $f(\Delta) \subset g(\Delta)$.

If $f(\Delta) \subset g(\Delta)$ then, using the univalence of the function g, we obtain that $g^{-1}: g(\Delta) \to \Delta$ is an holomorphic function in Δ and we can define the function $\varphi: \Delta \to \Delta$, $\varphi(z) = g^{-1}(f(z))$, $z \in \Delta$. We have

$$|\varphi(z) - z| = |g^{-1}(f(z)) - z| \le |g^{-1}(f(z)) - f(z)| + |f(z) - z|, \ z \in \Delta$$

and since $\lim\limits_{\Delta \ni z \to \infty} [f(z) - z] = \lim\limits_{\Delta \ni z \to \infty} [g(z) - z] = 0$ it follows that $\lim\limits_{\Delta \ni z \to \infty} [\varphi(z) - z] = 0$.

DEFINITION 2 [2]. We denote by $Q(\Delta)$ the set of functions $q \in A(\Delta)$ which are holomorphic and injective on $\overline{\Delta} - E(q)$, where $E(q) = \{\zeta \in \partial\Delta /$ $\lim\limits_{z \in \zeta} q(z) = \infty\}$, and also $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \backslash E(q)$.

DEFINITION 3 [2]. Let Ω be a set in $\mathbf{C}$ and let $q \in Q(\Delta)$. We define the class of *admissible* functions $\psi_\Delta[\Omega, q]$ to be those functions $\psi: \mathbf{C}^3 \times \Delta \to \mathbf{C}$ that satisfy the following admissibility condition:

$$\begin{cases} \psi(r, s, t; z) \notin \Omega, \text{ when } r = q(\zeta),\ s = m \cdot q'(\zeta) \\ \operatorname{Im} \dfrac{t}{s} \ge m \cdot \operatorname{Im} \dfrac{q''(\zeta)}{q'(\zeta)} \text{ and } z \in \Delta \text{ for } \zeta \in \partial\Delta \backslash E(q),\ m \in \mathbf{R}_+ \end{cases} \quad (1)$$

We shall need the following theorem to prove our results:

THEOREM 2[2]. *Let* $\psi \in \psi_{\Delta}[\Omega, q]$ *and* $p: \Delta \to \mathbf{C}$ *be an holomorphic function in* Δ *such that there exists* $a \geq 0$ *with* $p(\Delta_a) \subset q(\Delta)$, *where* $\Delta_a = \{z \in \mathbf{C} / \mathrm{Im}\, z > a\}$. *If*

$$\psi(p(z), p'(z), p''(z); z) \in \Omega, \text{ for all } z \in \Delta \tag{2}$$

then $p \prec q$.

Remark 1. If $\lim\limits_{\Delta \ni z \to \infty} [p(z) - z] = \lim\limits_{\Delta \ni z \to \infty} [q(z) - z] = 0$ then we obtain that there exists $a \geq 0$ such that $p(\Delta_a) \subset q(\Delta)$. Thus, the condition "$p: \Delta \to \mathbf{C}$ be an holomorphic function in Δ such that there exists $a \geq 0$ with $p(\Delta_a) \subset q(\Delta)$" from Theorem 2 can be replaced by $p \in A(\Delta)$.

Let Ω be a set in Δ and let $q(z) = z$, $z \in \Delta$. We will obtain some applications of the Theorem 2 corresponding to this particular Ω and q.

THEOREM 3. *Let* $p \in A(\Delta)$ *and let* $\gamma \in \mathbf{R}$, $\gamma \leq 0$. *If*

$$\mathrm{Im}\left[p(z) + \gamma \cdot \frac{p''(z)}{p'(z)}\right] > 0, \; z \in \Delta \tag{3}$$

then $\mathrm{Im}\, p(z) > 0$.

Proof. If we let $\psi(r, s, t, z) = r + \gamma \cdot t/s$ then the conclusion will follow from Theorem 2 we show that $\psi \in \psi_{\Delta}[\Omega, q]$, where $\Omega = \Delta$ and $q(z) = z$. This follows from Definition 3 since

$$\operatorname{Im}\psi(r,s,t;z) = \operatorname{Im}(\zeta+\gamma\cdot t/s) = \operatorname{Im}\zeta+\gamma\cdot\operatorname{Im} t/s \le 0 \text{ for } r=\zeta\in\partial\Delta,$$

$\operatorname{Im} t/s \ge 0$ and $\gamma \le 0$. Hence $\psi \in \psi_\Delta[\Omega,q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

THEOREM 4. *Let $p \in A(\Delta)$ and let $\alpha,\beta \in \mathbf{R}$. If*

$$\operatorname{Im}\left[\alpha p(z) + \beta\frac{p'(z)}{p(z)}\right] > 0,\ z \in \Delta$$

then $\operatorname{Im} p(z) > 0$.

Proof. If we let $\psi(r,s,t;z) = \alpha r + \beta\cdot s/r$ then we have $\operatorname{Im}\psi(r,s,t;z) =$ $= \alpha\operatorname{Im}\zeta + \beta\cdot m\operatorname{Im} 1/\zeta = 0$ for $r=\zeta\in\partial\Delta$, $s=m\in\mathbf{R}_+$ and $\alpha,\beta\in\mathbf{R}$. Hence $\psi \in \psi_\Delta[\Omega,q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

COROLLARY. *Let $f\colon \Delta \to \mathbf{C}$ be an holomorphic function in Δ such that $-\dfrac{f'}{f}$ satisfies the conditions of Theorem 4 and $\alpha \in \mathbf{R}$. If*

$$\operatorname{Im}\left[(1-\alpha)\frac{f'(z)}{f(z)} + \alpha\frac{f''(z)}{f'(z)}\right] > 0,\ z \in \Delta \tag{5}$$

then $\operatorname{Im}\dfrac{f'(z)}{f(z)} < 0,\ z \in \Delta$.

Remark 2. A function $f \in A(\Delta)$, $f(z) \neq 0$, $z \in \Delta$ is starlike in the half-plane Δ if and only if

$$\operatorname{Im}\frac{f'(z)}{f(z)} < 0,\ z \in \Delta.$$

Using the Corollary, we obtain that a function which satisfies the condition 5 is a starlike function in Δ.

THEOREM 5. *Let* $p \in A(\Delta)$ *and* $\alpha, \beta, \gamma \in \mathbf{R}$, $\gamma \leq 0$. *If*

$$\operatorname{Im}\left[\alpha p(z) + \beta \frac{p'(z)}{p(z)} + \gamma \frac{p''(z)}{p'(z)}\right] > 0,\ z \in \Delta \tag{6}$$

then $\operatorname{Im} p(z) > 0$.

Proof. If we let $\psi(r,s,t;z) = \alpha r + \beta \cdot s/r + \gamma \cdot t/s$ then we have

$\operatorname{Im} \psi(r,s,t;z) = \alpha \operatorname{Im} \zeta + \beta \cdot m \operatorname{Im} 1/\zeta + \gamma \cdot \operatorname{Im} t/s \leq 0$ for $r = \zeta \in \partial\Delta$, $s = m \in \mathbf{R}_+$,

$\operatorname{Im} t/s \geq 0$, $\alpha, \beta, \gamma \in \mathbf{R}$, $\gamma \leq 0$. Hence $\psi \in \psi_\Delta[\Omega, q]$, $p \prec q$ and $\operatorname{Im} p(z) > 0$.

Remark 3.

i) If $\gamma = 0$ then Theorem 5 reduces to Theorem 4.

ii) If $\alpha = 1$ and $\beta = 0$ then Theorem 5 reduces to Theorem 3.

REFERENCES

1. S.S. Miller, P.T. Mocanu, *The theory and applications of second-order differential subordinations*, Studia Univ. Babeş-Bolyai, Mathematica, **34**, No. 4, 1989.
2. D. Răducanu, N.N. Pascu, *Differential subordinations for holomorphic functions in the upper half-plane*, International Conference on Complex Analysis and 7th Romanian Finnish Seminar on Complex Analysis, Timişoara 1993 (to appear).
3. N.N. Pascu, *On Schwarz's Lemma in the upper half-plane* (to appear).
4. J. Stankiewicz, *Geometric properties of functions regular in a half-plane*, Current topics in Analytic Function Theory, World Scientific Singapore, New-Jersey, London, Hong Kong.