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ON SOME INEQUALITIES USEFUL IN THE THEORY OF CERTAIN HIGHER ORDER DIFFERENTIAL AND DIFFERENCE EQUATIONS

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REZUMAT. - Asupra unor inegalităţi din teoria ecuaţiilor diferenţiale şi cu diferenţe de ordin superior. În lucrare sunt stabilite noi inegalităţi integrale şi discrete care pot fi folosite în teoria ecuaţiilor diferenţiale de ordin superior şi a ecuaţiilor cu diferenţe.

Abstract. In this paper we establish some new integral and discrete inequalities which can be used as handy tools in the theory of certain new classes of higher order differential and difference equations.

1. Introduction. The fundamental role played by the integral and discrete inequalities in the development of the theory of differential and difference equations is well known. In the literature there are many papers written on integral and discrete inequalities and their applications in the theory of differential and difference equations, see [1-8, 10-12] and the references cited

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therein. Although stimulating research work related to integral and discrete inequalities used in the theory of differential and difference equations has been undertaken in the literature, it appears that there are certain classes of differential and difference equations for which the existing results on such inequalities do not apply directly. This amounts to finding some useful integral and discrete inequalities in order to achieve a diversity of desired goals. Our main objective here is to establish some new integral inequalities and their discrete analogues, which can be used as handy tools in the study of certain new classes of higher order differential and difference equations. We also present some immediate applications to convey the importance of our results to the literature.

2. Statement of results. In what follows we let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$ and $N_0 = \{0, 1, 2, ...\}$. For any function $z(m)$, $m \in N_0$, we define the operator Δ by $\Delta z(m) = z(m+1) - z(m)$ and all $m > n$, $m, n \in N_0$, we use the usual conventions $\sum_{s=m}^{n} z(s) = 0$ and $\prod_{s=m}^{n} z(s) = 1$. We use the following notations for simplification of details of presentation. For $t \in R_+$ and some functions $r_i(t)$, $i = 1, 2, ..., n$, we set

$$F[t, r_1, r_2, \ldots, r_{n-1}, r_n] = r_1(t)\int_0^t r_2(s_2) \int_0^{s_2} r_3(s_3) \int_0^{s_3} r_4(s_4) \cdots \int_0^{s_{n-2}} r_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} r_n(s_n)\, ds_n\, ds_{n-1} \ldots ds_4\, ds_3\, ds_2$$

$$+ r_2(t) \int_0^t r_3(s_3) \int_0^{s_3} r_4(s_4) \int_0^{s_4} r_5(s_5) \cdots \int_0^{s_{n-2}} r_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} r_n(s_n)\, ds_n\, ds_{n-1} \ldots ds_5\, ds_4\, ds_3 +$$

$$+ r_{n-1}(t) \int_0^t r_n(s_n)\, ds_n + r_n(t) .$$

For $m \in N_0$ and some functions $r_i(m)$, $i = 1, 2, \ldots, n$, we set

$$H[m, r_1, r_2, \ldots, r_{n-1}, r_n]$$

$$= r_1(m) \sum_{s_2=0}^{m-1} r_2(s_2) \sum_{s_3=0}^{s_2-1} r_3(s_3) \sum_{s_4=0}^{s_3-1} r_4(s_4) \ldots \times \sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} r_n(s_n)$$

$$+ r_2(m) \sum_{s_3=0}^{m-1} r_3(s_3) \sum_{s_4=0}^{s_3-1} r_4(s_4) \sum_{s_5=0}^{s_4-1} r_5(s_5) \ldots \times \sum_{s_{n-1}=0}^{s_{n-2}-1} r_{n-1}(s_{n-1}) \sum_{s_n=0}^{s_{n-1}-1} r_n(s_n) +$$

$$+ r_{n-1}(m) \sum_{s_n=0}^{m-1} r_n(s_n) + r_n(m).$$

Our main result is given in the following theorem.

THEOREM 1. *Let $y(t)$, $a(t)$, $b(t)$, $p_i(t)$, $i = 1, 2, \ldots, n$ be real-valued nonnegative continuous functions defined for $t \in R_+$.*

(i) *If*

$$y(t) \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n y]\, ds, \tag{2.1}$$

for all $t \in R_+$, then

$$y(t) \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$

$$\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\, d\tau\right) ds, \tag{2.2}$$

for all $t \in R_+$.

(ii) *Let $G(r)$ be a continuous strictly increasing, convex and submultiplicative function for $r \geq 0$, $G(0) = 0$, $\lim_{r\to\infty} G(r) = \infty$, $\alpha(t)$, $\beta(t)$ be positive continuous functions for $t \in R_+$ and $\alpha(t) + \beta(t) = 1$. If*

$$y(t) \leq a(t) + b(t)G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds\right), \tag{2.3}$$

for all $t \in R_+$, where G^{-1} is the inverse of G, then

$$y(t) \leq a(t) + b(t)G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n \alpha G(a\alpha^{-1})] \times\right.$$

$$\left.\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n \beta G(b\beta^{-1})]\, d\tau\right) ds\right), \tag{2.4}$$

for all $t \in R_+$.

(iii) *Let $W(y)$ be a real-valued continuous, nondecreasing, subadditive and submultiplicative function defined on interval $I = [y_0, \infty)$ and $W(y) > 0$ on (y_0, ∞), $y_0 \geq 0$ is a real constant, $W(y_0) = 0$. If*

$$y(t) \leq a(t) + b(t) \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(y)]\, ds, \tag{2.5}$$

for all $t \in R_+$, then for $0 \leq t \leq t_1$,

$$y(t) \leq a(t) + b(t)\Omega^{-1}[\Omega(c(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds], \tag{2.6}$$

where

$$c(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(a)]\, ds, \tag{2.7}$$

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{W(s)},\ u \geq u_0 \text{ with } u_0 > y_0, \tag{2.8}$$

Ω^{-1} is the inverse of Ω and $t_1 \in R_+$ be chosen so that

$$\Omega(c(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\, ds \in \mathrm{Dom}\,(\Omega^{-1}),$$

for all $t \in R_+$ lying in the interval $0 \leq t \leq t_1$.

We next establish a more general version of Theorem 1 which may be convenient in some applications.

THEOREM 2. *Let $y(t)$, $a(t)$, $b(t)$, $p_i(t)$, $i = 1, 2, \ldots, n$ be real-valued nonnegative continuous functions defined for $t \in R_+$. Let $f\colon R_+^2 \to R_+$ be a continuous function which satisfies the condition*

$$\text{(A)} \qquad 0 \le f(t,u_1) - f(t,u_2) \le k(t,u_2)(u_1 - u_2),$$

for $t \in R_+$ *and* $u_1 \ge u_2 \ge 0$, *where* $k : R_+^2 \to R_+$ *is a continuous function.*

(iv) *If*

$$y(t) \le a(t) + b(t)\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(fy)]\,ds, \tag{2.9}$$

for all $t \in R_+$, *where* $(fy)(t) = f(t,y(t))$, *then*

$$y(t) \le a(t) + b(t)\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(fa)] \times$$

$$\times \exp\left(\int_s^t F[\tau,p_1,p_2,\dots,p_{n-1},p_n(ka)b]\,d\tau\right) ds, \tag{2.10}$$

for all $t \in R_+$, *where* $(ka)(t) = k(t, a(t))$.

(v) *Let* G, G^{-1}, α, β *be as in* (ii). *If*

$$y(t) \le a(t) + b(t)G^{-1}\left(\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(f(G(y)))]\,ds\right), \tag{2.11}$$

for all $t \in R_+$, *where* $(f(G(y)))(t) = f(t,G(y(t)))$, *then*

$$y(t) \le a(t) + b(t)\,G^{-1}\left(\int_0^t F[s,p_1,p_2,\dots,p_{n-1},p_n(f(\alpha G(a\alpha^{-1})))] \times\right.$$

$$\left.\times \exp\left(\int_s^t F[\tau,p_1,p_2,\dots,p_{n-1},p_n(k(\alpha G(a\alpha^{-1})))(\beta G(b\beta^{-1}))]\,d\tau\right) ds\right), \tag{2.12}$$

for all $t \in R_+$, *where* $(f(\alpha G(a\alpha^{-1})))(t) = f(t,\alpha(t)\,G(a(t)\alpha^{-1}(t)))$,

$(k(\alpha G(a\alpha^{-1})))(t) = k(t,\alpha(t)G(a(t)\alpha^{-1}(t)))$, $(\beta G(b\beta^{-1}))(t) = \beta(t)\,G(b(t)\beta^{-1}(t))$.

(vi) *Let W, Ω, Ω^{-1} be as in* (iii). *If*

$$y(t) \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(f(W(y)))]\, ds, \tag{2.13}$$

for $t \in R_+$, where $(f(W(y)))(t) = f(t, W(y(t)))$, then for $0 \leq t \leq t_2$,

$$y(t) \leq a(t) + b(t)\Omega^{-1}[\Omega(\bar{c}(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(k(W(a)))\, W(b)]\, ds], \tag{2.14}$$

where $(k(W(a)))(t) = k(t, W(a(t)))$, $W(b)(t) = W(b(t))$,

$$\bar{c}(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(f(W(a)))]\, ds, \tag{2.15}$$

and $t_2 \in R_+$ is chosen so that

$$\Omega(\bar{c}(t)) + \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n(k(W(a)))\, W(b)]\, ds \in \mathrm{Dom}\,(\Omega^{-1}),$$

for all $t \in R_+$ lying in the interval $0 \leq t \leq t_2$.

The discrete analogues of Theorems 1 and 2 are given in the following theorems.

THEOREM 3. *Let $y(m)$, $a(m)$, $b(m)$, $p_i(m)$, $i = 1, 2, \ldots, n$ be real-valued nonnegative functions defined for $m \in N_0$.*

(vii) *If*

$$y(m) \leq a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n y], \tag{2.16}$$

for all $m \in N_0$, then

$$y(m) \leq a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$

$$\times \sum_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n b]], \tag{2.17}$$

for all $m \in N_0$.

(viii) *Let* G, G^{-1} *be as in* (ii) *and* $\alpha(m)$, $\beta(m)$ *be positive functions defined for* $m \in N_0$ *and* $\alpha(m) + \beta(m) = 1$. *If*

$$y(m) \leq a(m) + b(m) G^{-1} \Big(\sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n G(y)] \Big), \tag{2.18}$$

for all $n \in N_0$, *then*

$$y(m) \leq a(m) + b(m) G^{-1} \left(\sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n \alpha G(a\alpha^{-1})] \times \right.$$
$$\left. \times \sum_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n \beta G(b\beta^{-1})]] \right), \tag{2.19}$$

for all $m \in N_0$.

(ix) *Let* W, Ω, Ω^{-1} *be as in* (iii). *If*

$$y(m) \leq a(m) + b(m) \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(y)], \tag{2.20}$$

for all $m \in N_0$, *then for* $0 \leq m \leq m_1$, m, $m_1 \in N_0$,

$$y(m) \leq a(m) + b(m) \Omega^{-1} [\Omega(d(m) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)]], \tag{2.21}$$

where

$$d(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(a)], \tag{2.22}$$

for $m \in N_0$ *and* $m_1 \in N_0$ *is chosen so that*

$$\Omega(d(m)) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)] \in \mathrm{Dom}\,(\Omega^{-1}),$$

for $m \in N_0$ *and* $0 \leq m \leq m_1$.

THEOREM 4. *Let* $y(m)$, $a(m)$, $b(m)$, $p_i(m)$, $i = 1,2,\ldots,n$ *be real-valued nonnegative functions defined for* $m \in N_0$. *Let* $h : N_0 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$\text{(B)} \qquad 0 \le h(m,u_1) - h(m,u_2) \le q(m,u_2)(u_1 - u_2),$$

for $m \in N_0$ *and* $u_1 \ge u_2 \ge 0$, *where* $q(m,r)$ *is a real-valued function defined for* $m \in N_0$, $r \in R_+$.

(x) *If*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s,p_1,p_2,\ldots,p_{n-1},p_n(hy)], \tag{2.23}$$

for $m \in N_0$, *where* $(hy)(m) = h(m,y(m))$, *then*

$$y(m) \le a(m) + b(m)\sum_{s=0}^{m-1} H[s,p_1,p_2,\ldots,p_{n-1},p_n(ha)] \times$$
$$\times \sum_{\tau=s+1}^{m-1} [1 + H[\tau,p_1,p_2,\ldots,p_{n-1},p_n(qa)b]], \tag{2.24}$$

for $m \in N_0$, *where* $(qa)(m) = q(m,a(m))$.

(xi) *Let* G, G^{-1}, α, β *be as in* (viii). *If*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s,p_1,p_2,\ldots,p_{n-1},p_n(h(G(y)))]\right), \tag{2.25}$$

for $m \in N_0$, *where* $(h(G(y)))(m) = h(m,G(y(m)))$, *then*

$$y(m) \le a(m) + b(m)G^{-1}\left(\sum_{s=0}^{m-1} H[s,p_1,p_2,\ldots,p_{n-1},p_n(h(\alpha G(a\alpha^{-1})))] \times\right.$$

$$\times \prod_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \ldots, p_{n-1}, p_n(q(\alpha G(a\alpha^{-1}))) \times (\beta(G(b\beta^{-1})))]]\Bigg), \tag{2.26}$$

for $m \in N_0$, *where* $(h(\alpha G(a\alpha^{-1})))(m) = h(m, \alpha(m)G(a(m)\alpha^{-1}(m)))$,

$(q(\alpha G(a\alpha^{-1})))(m) =$

$= q(m, \alpha(m)G(a(m)\alpha^{-1}(m))), (\beta(G(b\beta^{-1})))(m) = \beta(m)G(b(m)\beta^{-1}(m))$.

(xii) *Let* W, Ω, Ω^{-1} *be as in* (iii). *If*

$$y(m) \leq a(m) + b(m)\sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n(h(W(y)))], \tag{2.27}$$

for $m \in N_0$, *where* $(h(W(y)))(m) = h(m, W(y(m)))$, *then for* $0 \leq m \leq m_2$,

$$y(m) \leq a(m) + b(m)\Omega^{-1}[\Omega(\bar{d}(m))$$

$$+ \sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n(qW(a)))\, W(b))]], \tag{2.28}$$

where $(q(W(a)))(m) = q(m, W(a(m)))$, $(W(b))(m) = W(b(m))$,

$$\bar{d}(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n(h(W(a)))],$$

and $m_2 \in N_0$ *be chosen so that*

$$\Omega(\bar{d}(m)) + \sum_{s=0}^{m-1} H[s, p_1, p_2, \ldots, p_{n-1}, p_n(q(W(a)))(W(b))] \in \mathrm{Dom}(\Omega^{-1}),$$

for $m \in N_0$ *and* $0 \leq m \leq m_2$.

3. Proofs of Theorems 1 and 2

(i) Define a function $u(t)$ by

$$u(t) = \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n y]\, ds. \tag{3.1}$$

From (3.1) and using $y(t) \leq a(t) + b(t)\,u(t)$ and the fact that $u(t)$ is monotone nondecreasing for $t \in R_+$, we observe that

$$u'(t) \leq F[t, p_1, p_2, \ldots, p_{n-1}, p_n a] + F[t, p_1, p_2, \ldots, p_{n-1}, p_n b]\,u(t). \quad (3.2)$$

From (3.2) we obtain

$$u(t) \leq \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$
$$\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\, d\tau\right) ds. \quad (3.3)$$

Using (3.3) in $y(t) \leq a(t) + b(t)u(t)$ we get the required inequality in (2.2).

(ii) Rewrite (2.3) as

$$y(t) \leq \alpha(t)\, a(t)\, \alpha^{-1}(t) + \beta(t)\, b(t)\, \beta^{-1}(t) \times$$
$$\times\, G^{-1}\left(\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds\right). \quad (3.4)$$

Since G is convex, submultiplicative and strictly increasing, from (3.4) we have

$$G(y(t)) \leq \alpha(t) G(a(t)\alpha^{-1}(t)) + \beta(t) G(b(t)\beta^{-1}(t)) \times$$
$$\times \int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n G(y)]\, ds. \quad (3.5)$$

The estimate given in (2.4) follows by first applying the inequality proved in (i) with $a(t) = \alpha(t)G(a(t)\alpha^{-1}(t))$, $b(t) = \beta(t)G(b(t)\beta^{-1}(t))$ and $y(t) = G(y(t))$ and then applying G^{-1} to both sides of the resulting inequality.

(iii) Define a function $u(t)$ by

$$u(t) = \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n W(Y)]\, ds. \tag{3.6}$$

Using $y(t) \le a(t) + b(t)\, u(t)$ on the right side of (3.6) we observe that

$$u(t) \le \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n W(a + bu)]\, ds$$

$$\le c(t) + \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)\, W(u)]\, ds. \tag{3.7}$$

For an arbitrary $T \in R_+$, it follows from (3.7) that

$$u(t) \le c(T) + \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)\, W(u)]\, ds,\ 0 \le t \le T. \tag{3.8}$$

Define a function $v(t)$ by

$$v(t) = \varepsilon + c(T) + \int_0^t F[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)\, W(u)]\, ds,\ 0 \le t \le T, \tag{3.9}$$

where $\varepsilon > 0$ is an arbitrary small constant. From (3.9) and using the facts that $u(t) \le v(t)$ and $v(t)$ is monotone nondecreasing for $0 \le t \le T$, we observe that

$$v'(t) \le F[t, p_1, p_2, \dots, p_{n-1}, p_n W(b)]\, W(v(t)),\ 0 \le t \le T. \tag{3.10}$$

From (2.8) and (3.10) we have

$$\frac{d}{dt}\Omega(v(t)) \le F[t, p_1, p_2, \dots, p_{n-1}, p_n W(b)],\ 0 \le t \le T. \tag{3.11}$$

Now integrating both sides of (3.11) from 0 to T we have

$$\Omega(v(T)) \le \Omega(\varepsilon + c(T)) + \int_0^T F[s, p_1, p_2, \dots, p_{n-1}, p_n W(b)]\, ds. \tag{3.12}$$

Since T is arbitrary, the inequality (3.12) holds for $t = T$, for all $t \in R_+$ and hence from (3.12) we have

$$v(t) \le \Omega^{-1}\Big[\,\Omega(\varepsilon + c(t)) + \int_0^t F\,[s, p_1, p_2, \ldots, p_{n-1}, p_n W(b)]\,ds\,\Big]. \quad (3.13)$$

Using (3.13) in $u(t) \le v(t)$ and the fact that $y(t) \le a(t) + b(t)\,u(t)$ and letting $\varepsilon \to 0$ we get the desired inequality in (2.6). The subdomain of R_+ for t is obvious. This completes the proof of Theorem 1.

(iv) Define a function $u(t)$ by

$$u(t) = \int_0^t F\,[s, p_1, p_2, \ldots, p_{n-1}, p_n(fy)]\,ds. \quad (3.14)$$

From (3.14) and using the condition (A) and the facts that $y(t) \le a(t) + b(t)\,u(t)$ and $u(t)$ is monotone nondecreasing for $t \in R_+$, we observe that

$$\begin{aligned} u'(t) &\le F\,[t, p_1, p_2, \ldots, p_{n-1}, p_n(f(a + bu))] \\ &= F\,[t, p_1, p_2, \ldots, p_{n-1}, p_n\{(f(a + bu)) - (fa) + (fa)\}] \\ &\le F\,[t, p_1, p_2, \ldots, p_{n-1}, p_n(fa)] \\ &\quad + F\,[t, p_1, p_2, \ldots, p_{n-1}, p_n(ka)b]\,u(t). \end{aligned} \quad (3.15)$$

From (3.15) we obtain

$$\begin{aligned} u(t) \le \int_0^t F\,[s, p_1, p_2, \ldots, p_{n-1}, p_n(fa)] \times \\ \times \exp\Big(\int_s^t F\,[\tau, p_1, p_2, \ldots, p_{n-1}, p_n(ka)b]\,d\tau\Big)\,ds. \end{aligned} \quad (3.16)$$

Using (3.16) in $y(t) \le a(t) + b(t)\,u(t)$ we get the desired inequality in (2.10).

The proofs of (v) and (vi) can be completed by following the same arguments as in the proofs of (ii), (iii) and (iv) given above with suitable modifications. Here we omit the details.

4. Proofs of Theorems 3 and 4. Since the proofs resemble one another, we give the details for (vii) only, the proofs of (viii)-(xii) can be completed by following the proofs of similar results given in [7, ', 11] and closely looking at the proofs of (i)-(iv) and (vii).

(vii) Define a function $z(m)$ by

$$z(m) = \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n y]. \tag{4.1}$$

From (4.1) and using $y(m) \leq a(m) + b(m)z(m)$ and the fact that $z(m)$ is monotone nondecreasing for $m \in N_0$, we observe that

$$z(m+1) - z(m) = H[m, p_1, p_2, \dots, p_{n-1}, p_n y]$$

$$\leq H[m, p_1, p_2, \dots, p_{n-1}, p_n a] + H[m, p_1, p_2, \dots, p_{n-1}, p_n b]\, z(m). \tag{4.2}$$

The inequality (4.2) implies the estimate (see [7])

$$z(m) \leq \sum_{s=0}^{m-1} H[s, p_1, p_2, \dots, p_{n-1}, p_n a] \times$$
$$\times \prod_{\tau=s+1}^{m-1} [1 + H[\tau, p_1, p_2, \dots, p_{n-1}, p_n b]]. \tag{4.3}$$

The required inequality in (2.17) now follows by using (4.3) in $y(m) \leq a(m) + b(m)z(m)$.

5. Some applications. In this section we present some applications of our results to obtain bounds on the solutions of certain higher order differential and difference equations for which the inequalities available in the existing literature do not apply directly.

Let $p_i(t)$, $0 \le i \le n$ be positive continuous functions defined for $t \in R_+$. We define the differential operators L_i, $0 \le i \le n$ by

$$L_0 x(t) = \frac{x(t)}{p_0(t)}, \; L_i x(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1} x(t), \; 1 \le i \le n.$$

Consider the nonlinear differential equation of the form

$$L_n x(t) = g(t, L_0 x(t), L_1 x(t), \dots, L_{n-2} x(t), L_{n-1} x(t)), \tag{5.1}$$

with the initial conditions

$$L_{i-1} x(0) = 0, \; i = 1, 2, \dots, n, \tag{5.2}$$

where $g : R_+ \times R^n \to R$ is a continuous function. For the study of (5.1)-(5.2), see [9] and the references cited therein.

We first convert the problem (5.1)-(5.2) into an equivalent integral equation. Let $y(t) = L_n x(t)$, then we have

$$L_{n-1} x(t) = \int_0^t p_n(s_n)\, y(s_n)\, ds_n, \tag{5.3}$$

$$L_{n-2} x(t) = \int_0^t p_{n-1}(s_{n-1}) \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1}, \tag{5.4}$$

$$L_1 x(t) = \int_0^t p_2(s_2) \int_0^{s_2} p_3(s_3) \int_0^{s_3} p_4(s_4) \dots \int_0^{s_{n-2}} p_{n-1}(s_{n-1})$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \dots ds_4\, ds_3\, ds_2 , \tag{5.5}$$

$$L_0 x(t) = \int_0^t p_1(s_1) \int_0^{s_1} p_2(s_2) \int_0^{s_2} p_3(s_3) \dots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \dots ds_3\, ds_2\, ds_1 . \tag{5.6}$$

Consequently the problem (5.1)-(5.2) is equivalent to the following integral equation

$$y(t) = g\Bigg(t, \int_0^t p_1(s_1) \int_0^{s_1} p_2(s_2) \int_0^{s_2} p_3(s_3) \dots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \dots ds_3\, ds_2\, ds_1 ,$$

$$\int_0^t p_2(s_2) \int_0^{s_2} p_3(s_3) \int_0^{s_3} p_4(s_4) \dots \int_0^{s_{n-2}} p_{n-1}(s_{n-1}) \times$$

$$\times \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} \dots ds_4\, ds_3\, ds_2 ,$$

$$\int_0^t p_{n-1}(s_{n-1}) \int_0^{s_{n-1}} p_n(s_n)\, y(s_n)\, ds_n\, ds_{n-1} ,$$

$$\int_0^t p_n(s_n)\, y(s_n)\, ds_n\Bigg) . \tag{5.7}$$

Suppose that the function g in (5.1) satisfies

$$|g(t, w_0, w_1, \ldots, w_{n-2}, w_{n-1})|$$
$$\leq a(t) + b(t)[\,|w_0| + |w_1| + \ldots + |w_{n-2}| + |w_{n-1}|\,], \tag{5.8}$$

where $a(t)$ and $b(t)$ are real-valued nonnegative continuous functions defined for $t \in R_+$. From (5.7) and (5.8) we observe that

$$|y(t)| \leq a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n|y|]\,ds. \tag{5.9}$$

Now an application of inequality proved in (i) Theorem 1 yields

$$|y(t)| \leq Q(t), \tag{5.10}$$

where

$$Q(t) = a(t) + b(t)\int_0^t F[s, p_1, p_2, \ldots, p_{n-1}, p_n a] \times$$
$$\times \exp\left(\int_s^t F[\tau, p_1, p_2, \ldots, p_{n-1}, p_n b]\,d\tau\right) ds.$$

Now using (5.10) in (5.3)-(5.6) we get the bounds on $|L_{n-1}x(t)|$, $|L_{n-2}x(t)|$, $\ldots$, $|L_1x(t)|$, $|L_0x(t)|$ in terms of the known quantities. Thus by using the definition of $L_0x(t)$, we get the bound on the solution $x(t)$ of (5.1)-(5.2) in terms of known quantities.

Further, it is be noted that the inequality given in (vii) can be used to obtain upper bound on the solution of the nonlinear difference equation of the form

$$E_n x(m) = g(m, E_0 x(m), E_1 x(m), \ldots, E_{n-2} x(m), E_{n-1} x(m)), \qquad (5.11)$$

with the initial conditions

$$E_{i-1} x(0) = 0, \; i = 1, 2, \ldots, n, \qquad (5.12)$$

where g is a real-valued function defined on $N_0 \times R^n$, the operators E_j, $0 \leq j \leq n$ are defined by

$$E_0 x(m) = \frac{x(m)}{p_0(m)}, \; E_j x(m) = \frac{1}{p_j(m)} \Delta E_{j-1} \cdot (m), \; j = 1, 2, \ldots, n.$$

$p_j(m)$, $0 \leq j \leq n$ are positive functions defined for $m \in N_0$. By letting $z(m) = E_n x(m)$ and converting the problem (5.11)-(5.12) into an equivalent form of sum-difference equation and following the same arguments as explained above for the problem (5.1)-(5.2) we get the bound on the solution $x(m)$ of (5.11)-(5.12).

We also note that the inequalities established in (iv) and (x) can be used to obtain bounds on the solutions of the following more general nonlinear higher order differential and difference equations of the forms:

$$L_n x(t) = g(t, f(t, L_0 x(t)), f(t, L_1 x(t)), \ldots, f(t, L_{n-2} x(t)), f(t, L_{n-1} x(t))), \qquad (5.13)$$

$$L_{i-1} x(0) = 0, \; i = 1, 2, \ldots, n, \qquad (5.14)$$

and

$$E_n x(m) = g(m, h(m, E_0 x(m)), h(m, E_1 x(m)),$$

$$\ldots, h(m, E_{n-2}x(m)), h(m, E_{n-1}x(m))), \tag{5.15}$$

$$E_{i-1}x(0) = 0, \; i = 1, 2, \ldots, n, \tag{5.16}$$

respectively, under some suitable conditions on the functions involved in (5.13) and (5.15). Since the details of these results are very close to that of given above with suitable modifications, and hence we do not discuss it here.

In concluding this paper we note that there are many possible applications of the inequalities established in this paper to certain classes of higher order differential and difference equations, but those presented here are sufficient to convey the importance of our results to the literature. Various other applications of these inequalities will appear elsewhere.

REFERENCES

1. E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, New York, 1961.
2. P.R. Beesack, *Gronwall Inequalities*, Carleton University Mathematical Lecture Notes No. 11, 1975.
3. I. Bihari, *A generalization of a Lemma of Bellman and its applications to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar. 7(1956), 81-94.
4. S.S. Dragomir, *The Gronwall type Lemmas and Applications*, Monografii Matematice, University Timişoara No. 29, 1987.
5. H.E. Gollwitzer, *A note on functional inequality*, Proc. Amer. Math. Soc. **43**(1969), 642-647.
6. D.S. Mitrinović and J.L Pečarić, *Differential and Integral Inequalities*, Naucna Knjiga Belgrade, 1988.
7. B.G. Pachpatte, *On the discrete generalizations of Gronwall's inequality*, J. Indian Math. Soc. **37**(1973), 147-156.

8. B.G. Pachpatte, *On some nonlinear discrete inequalities of Gronwall type*, Bull. Inst. Math. Academia Sinica **5**(1977), 305-315.
9. B.G. Pachpatte, *On a class of nonlinear higher order differential equations*, Indian J. Pure Appl. Math. **20**(1989), 121-128.
10. B.G. Pachpatte, *On a certain inequality arising in the theory of differential equations*, J. Math. Anal. Appal. **182**(1994), 143-157.
11. B.G. Pachpatte, *Some new finite difference inequalities*, Computers Math. Applic. **28**(1994), 227-241.
12. W. Walter, *Differential and Integral Inequalities*, Springer-Verlag, Berlin, New York, 1970.