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CONVEXITY AND INTEGRAL OPERATORS

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REZUMAT. - Convexitate şi operatori integrali. În prima parte a lucrării îmbunătăţim un rezultat al lui V. Zanelli şi dăm o demonstraţie uşoară a sa. Apoi considerăm câţiva operatori integrali şi studiem proprietăţile lor relative la conservarea convexităţii de ordin superior. Obţinem astfel o generalizare a rezultatului din [3].

1. A result of V. Zanelli. In [3] it is proved the following property:

LEMMA 0. *Let* $f: [a,\infty) \to R$ *(with* $a > 0$*) be a positive, decreasing, convex function and*

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

For $a \le y$, $k > 0$, $y + k \le x$, *we have the following inequality:*

$$F(y+k) - F(y) - F(x+k) + F(x) \le k\,[f(y) - f(x)]. \tag{2}$$

The proof is based on a rather complicated geometrical method. We want to eliminate some superfluous hypotheses from the enounce and to give a simple proof of it.

LEMMA 1. *Let* $f: [a,b] \to R$ *be a convex function and* F *be defined by*

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(1). *For $a \le y < x < x + k \le b$ we have the inequality* (2).

Proof. Let us consider the auxiliary function:

$$g(t) = t[f(y) - f(x)] - F(y+t) + F(y) + F(x+t) - F(x),\ t \in [0,k].$$

We have

$$g'(t) = [f(x+t) - f(x)] - [f(y+t) - f(y)] \ge 0$$

because, by the convexity of f, the conditions $x > y$ and $x + t > y + t$ give:

$$\frac{f(x+t) - f(x)}{t} \ge \frac{f(x+t) - f(y)}{x+t-y} \ge \frac{f(y+t) - f(y)}{t}.$$

Obviously $g(0) = 0$ so that $g'(t) \ge 0$ gives $g(k) \ge 0$, that is (2).

It can be remarked that we have renounced at the following hypotheses from Lemma 0: $a > 0$, f is positive and decreasing and $y + k \le x$.

2. Convex functions of higher order. We must remind some definitions.

Let f: $[a,b] \to R$ be an arbitrary function. For arbitrary distinct points x_1, x_2, ..., $x_{n+1} \in [a,b]$ the divided differences of the function f are defined by recurrence:

$$[x_1;f] = f(x_1),\ [x_1, \ldots, x_{n+1};f] =$$

$$= ([x_1, \ldots, x_{n-1}, x_{n+1};f] - [x_1, \ldots, x_n;f])/(x_{n+1} - x_n) \tag{3}$$

The function f is called convex of order n (or shortly n-convex) if:

$$[x_1, \ldots, x_{n+2}; f] \geq 0, \ \forall\, x_1, \ldots, x_{n+2} \in [a, b] \tag{4}$$

where the points are supposed, as in (3), distinct.

For $n = 1$ we get convexity and for $n = 0$ increasing monotony. It is known (see [2]) that a n-convex function, with $n \geq 1$, is continuous on (a,b), so it is integrable on any subinterval from $[a,b]$.

The main result that we will use is the following:

LEMMA 2. *If the function f is n-convex then:*

$$[x_1, \ldots, x_{n+1}; f] \leq [y_1, \ldots, y_{n+1}; f], \ \textit{if } x_i \leq y_i, \ \forall i. \tag{5}$$

Proof. From (3) and (4) we deduce that:

$[x_1, \ldots, x_{n-1}, x_{n+1}; f] \geq [x_1, \ldots, x_n; f]$ if $x_{n+1} > x_n$. This gives (5), step by step, because the divided differences are symmetric with respect to the points.

3. Arithmetic integral means. To generalize the result from [3] we consider, for a fixed $k > 0$, some operators.

Let $C[a,b]$ be the set of continuous functions on $[a,b]$. For $f \in C[a,b]$ we denote by $F_k(f)$ the function defined by:

$$F_k(f)(x) = \int_x^{x+k} f(t)\, dt, \ \forall\, x \leq b - k.$$

Then we define:

$$A_k(f)(x) = \frac{1}{k} F_k(f)(x)$$

a sort of arithmetic integral mean and:

$$E_k(f)(x) = A_k(f)(x) - f(x)$$

an "excess" function. We get so the operators F_k , A_k and E_k defined on $C[a,b]$ and with values in $C[a, b-k]$. To study some of their properties, we give simple representation formulas for them.

As:

$$F_k(f)(x) = \int_0^k f(x+t)\,dt$$

making the substitution $t = ks$, we have:

$$A_k(f)(x) = \int_0^1 f(x+ks)\,ds$$

and so

$$E_k(f)(x) = \int_0^1 [f(x+ks) - f(x)]\,ds.$$

Thus $E_k(f) \geq 0$ if f is increasing and Lemma 1 asserts in fact that $E_k(f)$ is increasing if f is convex. We generalize this result as follows.

THEOREM 1. *If the function f is n-convex, then $F_k(f)$ and $A_k(f)$ are also n-convex but $E_k(f)$ is $(n-1)$-convex.*

Proof. If $x_1, \ldots, x_{n+2}$ are distinct points from $[a, b-k]$ we have

$$[x_1, \ldots, x_{n+2}; A_k(f)] = \int_0^1 [x_1 + ks, \ldots, x_{n+2} + ks; f]\,ds \geq 0$$

and

$$[x_1, \dots, x_{n+1}; E_k(f)] = \int_0^1 ([x_1 + ks, \dots, x_{n+1} + ks; f] - [x_1, \dots, x_{n+1}; f])\, ds \geq 0$$

by Lemma 2. So the affirmation follows for $A_k(f)$ and $E_k(f)$. As $F_k(f) = kA_k(f)$, it is true also for $F_k(f)$.

We remark that the operator E_k can be defined similarly by:

$$E_k(f)(x) = f(x+k) - A_k(f)(x)$$

having the same properties.

Let us define also the operators F, A, E: $C[a,b] \to C[a,b]$ as follows. For f in $C[a,b]$ we put:

$$F(f)(x) = \int_a^x f(t)\, dt$$

$$A(f)(x) = F(f)(x)/(x-a)$$

and

$$E(f)(x) = f(x) - A(f)(x).$$

Using the substitution $t = a + s(x-a)$, we have:

$$A(f)(x) = \int_0^1 f(a + s(x-a))\, ds$$

and

$$E(f)(x) = \int_0^1 [f(x) - f(sx + (1-s)a)]\, ds.$$

Thus, as above, we can prove

THEOREM 2. *If the function f is n-convex then so is also* $A(f)$, *but* $E(f)$ *is* $(n-1)$*-convex.*

The first result is well known (see [1]) as it is also known that under the same hypotheses, $F(t)$ is $(n+1)$-convex.

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