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ON FACTORIZATION OF GROUP

I. VIRÁG*

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REZUMAT. - Asupra factorizării unui grup. În această lucrare sunt stabilite câteva rezultate privind posibilitatea descompunerii unui grup finit în produs de două subgrupuri.

1. Introduction. All groups in this paper are finite. Let G be a group and let M be a subgroup of G (in symbols $M \leq G$). G is factorizable over M if there are $H \leq G$ and $K \leq G$ such that

$$G = HK, \; H \cap K = M.$$

In this case H and K furnish a factorization of G over M and we call K a complement in G of H over M.

Assume that $M \leq H \leq G$. In this paper we present three theorems which give criteria for the existence of a complement in G of H over M. Some special cases are also presented.

We shall begin by reviewing the folloving notions:

* *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 3400 Cluj-Napoca, Romania*

Suppose that G is a group and that S is a (left) G-set (i.e. S is a set on which G acts from the left as a group of permutations). For each element $\alpha \in S$ its G-orbit is the subset $G(\alpha) = \{x\alpha \mid x \in G\}$ of S and its stabilizer in G is the subgroup $G_\alpha = \{x \in G \mid x\alpha = \alpha\}$ of G. It is known that $|G(\alpha)| = |G: G_\alpha|$.

2. Results.

THEOREM 1. *Suppose that $H \leq G$ and that S is a G-set. For an element $\alpha \in S$, the following statements are equivalent:*

(i) *The subgroup H and the stabilizer G_α of α in G furnish a factorization of G over the stabilizer H_α of α in H.*

(ii) *The G-orbit $G(\alpha)$ of α coincides with the H-orbit $H(\alpha)$ of α.*

(iii) $|G : G_\alpha| = |H : H_\alpha|$.

Proof. Assume that $G = HG_\alpha$, $H \cap G_\alpha = H_\alpha$. Then for $x \in G$ we have $x = yz$, $y \in H$, $z \in G_\alpha$. It follows, that $G(\alpha) = \{x\alpha \mid x \in G\} = \{(yz)\alpha \mid y \in H, z \in G_\alpha\} = \{y(z\alpha) \mid y \in H, z \in G_\alpha\} = \{y\alpha \mid y \in H\} = H(\alpha)$. Hence (i) implies (ii).

Suppose that $G(\alpha) = H(\alpha)$. Then it is immediate that $|G : G_\alpha| = |H : H_\alpha|$. Thus (ii) implies (iii).

Suppose that $|G : G_\alpha| = |H : H_\alpha|$. Then $|G| = \frac{|H||G_\alpha|}{|H_\alpha|}$. Since $H \cap G_\alpha = H_\alpha$, it follows, that $|G| = \frac{|H||G_\alpha|}{|H \cap G_\alpha|} = |HG_\alpha|$. Therefore $G = H G_\alpha$, $H \cap G_\alpha = H_\alpha$. Hence (iii) implies (i).

THEOREM 2. *Assume that $M \leq K \leq G$. Then for a subgroup H of G the following statements are equivalent:*

(i) $G = H K$, $H \cap K = M$.

(ii) *There exists a G-set S and $\alpha \in S$ such that the G-orbit G_α of α coincide with the H-orbit H_α of α and $G_\alpha = K$, $H_\alpha = M$.*

Proof. It is easily that the set of left cosets of K in G is a G-set with the operation $(x, yK) \rightarrow xyK$. Assume that $\alpha = K$.

If $G = HK$, $H \cap K = M$, then for every $x \in G$, $x = yz$, $y \in H$, $z \in K$. It follows, that $xK = yzK = yK$. Therefore the G-orbit $G(K)$ of K coincides with the H-orbit $H(K)$ of K. If $x \in G$, then $xK = K$ iff $x \in K$. Hence the stabilizer G_K of K coincides with K and $K \cap H = M$ is the stabilizer of K in H. Therefore (i) implies (ii).

The implication (ii) implies (i) is an immediate consequence of Theorem 1.

We note the following particular case of Theorem 1.

THEOREM 3. *Let H be a subgroup of the group G. Then for a subset T of G the following statements are equivalent:*

(i) *The subgroup H and the normalizer $N_G(T)$ of T in G furnish a factorization of G over the normalizer $N_H(T)$ of T in H.*

(ii) *If a subset R of G is conjugate to T in G, then R is conjugate to T with an element of H.*

(iii) $|G : N_G(T)| = |H : N_H(T)|$.

Proof. The set $P(G)$ of the subsets of G is a G-set with the operation $(x,Z) \to x^{-1}Zx$, $x \in G$, $Z \in P(G)$. If $\alpha = T \in P(G)$, then $G_\alpha = N_G(T)$, $H_\alpha = N_H(T)$, $G(\alpha) = \{x^{-1}Tx \mid x \in G\}$, $H(\alpha) = \{y^{-1}Ty \mid y \in H\}$. Hence the statements of Theorem 3 are easily from Theorem 1.

3. Applications. We note that if G is a group, then the condition (ii) of Theorem 3 is satisfied in the following particular cases:

a. H is a normal subgroup of G and T is a Sylow subgroup of H.

b. H is a normal subgroup of G and T is a nilpotent Hall subgroup of H ([1], Th. 5.8., p.285).

c. H is a normal solvable subgroup of G and T is a Hall subgroup of H

([1], Th. 1.8., p.662).

Hence the following applications of Theorem 3 are immediate:

COROLLARY 1 (The Frattini argument). *If H is a normal subgroup of G and T is a Sylow subgroup of H, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

COROLLARY 2. *If H is a normal subgroup of G and T is a nilpotent Hall subgroup of H, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

COROLLARY 3. *If H is a normal solvable subgroup of G and T is a Hall subgroup of G, then*

$$G = H\,N_G(T),\ H \cap N_G(T) = N_H(T).$$

REFERENCES

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