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New midpoint and trapezoidal-type inequalities for prequasiinvex functions via generalized fractional integrals

Seth Kermausuor and Eze R. Nwaeze

Abstract. In this work, we establish some new midpoint and trapezoidal type inequalities for prequasiinvex functions via the Katugampola fractional integrals. Some of the results obtained in this paper are generalizations of some earlier results in the literature.

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1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$ (see [26, 28]). The following result which holds for convex functions is known in the literature as the Hermite-Hadamard inequality.

Theorem 1.1 ([10])**.** *If $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$ with $a < b$, then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Many authors have studied and generalized the Hermite-Hadamard inequality in several ways via different classes of convex functions. For some recent results related to the Hermite-Hadamard inequality, we refer the interested reader to the papers [1, 22, 23, 13, 20, 21, 4, 9, 3, 2, 18, 19].

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The concept of quasi-convexity which generalizes the concept of convexity is defined as follows.

Definition 1.2 (See [26, 28]**).** A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [12], Ion introduced the following Hermite-Hadamard type inequalities also known as trapezoidal-type inequalities for quasi-convex functions as follows.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^{\frac{p}{p-1}}$, $p > 1$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left(\max\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}.$$

For more results related to quasi-convex functions, we refer the interested reader to the papers [9, 3, 1, 2]. The concept of preinvexity was introduced in [5, 11, 32] as a generalization of convexity as follows.

Definition 1.5. Let $I \subseteq \mathbb{R}$ and $\eta : I \times I \to \mathbb{R}$ be a bifunction. I is said to be an invex set with respect to η, if

$$x + t\eta(y, x) \in I \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

If $I \subseteq \mathbb{R}$ is an invex set with respect to the bifunction η, then a function $f : I \to \mathbb{R}$ is said to be a preinvex function with respect to η, if

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y) \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

Remark 1.6. If $\eta(y, x) = y - x$ in Definition 1.5, then we have that f is a convex function. Thus, every convex function is a preinvex function with respect to the bifunction $\eta(y, x) = y - x$. However, not every preinvex function is a convex function (see [32] for more details).

In a similar way, the concept of quasi-convexity has been generalized in the following definition.

Definition 1.7 ([24]**).** If $I \subseteq \mathbb{R}$ is an invex set with respect to the bifunction η, then a function $f : I \to \mathbb{R}$ is said to be prequasiinvex with respect to η, if

$$f(x + t\eta(y, x)) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

Remark 1.8. Every quasi-convex function is a prequasiinvex function with respect to the bifunction $\eta(y, x) = y - x$. However, not every prequasinvex function is a quasi-convex function (see [33] for more details).

Barani et al. [4] established the following trapezoidal-type inequalities for prequasiinvex functions which are generalizations of Theorem 1.3 and Theorem 1.4.

Theorem 1.9. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{4}\max\{|f'(a)|,|f'(b)|\}.$$

Theorem 1.10. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{|\eta(b,a)|}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.$$

For more information and results related to prequasiinvex functions, we refer the interested reader to the papers [24, 33, 13, 20, 21]. In [13], the author generalized Theorem 1.9 and Theorem 1.10 using the Riemann-Liouville fractional integrals.

Our goal in this paper is to provide some midpoint and trapizoidal type inequalities for functions whose derivative in absolute value to some exponents are prequasiinvex via the Katugampola fractional integrals. Some of our results generalize the results in [13]. We end this section with the definitions of the Riemann-Liouville, Hadamard and Katugampola fractional integrals and some preliminary results.

Definition 1.11 ([25])**.** The left- and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ of f are defined by

$$J^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt$$

and

$$J^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) := \int_0^{\infty} t^{x-1}e^{-t}dt,\ Re(x) > 0$$

with the property that $\Gamma(x+1) = x\Gamma(x)$.

Definition 1.12 ([29])**.** The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f are defined by

$$H^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}dt$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

Definition 1.13. $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm $\|\cdot\|_{X_c^p}$ is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \quad (1 \le p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = \operatorname{ess} \sup_{a \le t \le b} |t^c f(t)|.$$

In 2011, Katugampola [14] introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows:

Definition 1.14. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$${}^{\rho}I_{a+}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 1.15. It is shown in [14] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.

Theorem 1.16 ([14]). *Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$*

1. $\lim_{\rho \to 1} {}^{\rho}I_{a+}^{\alpha} f(x) = J_{a+}^{\alpha} f(x)$,
2. $\lim_{\rho \to 0^+} {}^{\rho}I_{a+}^{\alpha} f(x) = H_{a+}^{\alpha} f(x)$.

Similar results also hold for the right-sided operators.

For more information about the Katugampola fractional integrals and related results, we refer the interested reader to the papers [6, 14, 15, 16, 17].

Lemma 1.17 (See [27, 31]**).** *For any $\alpha \in [0,1]$ and $x, y \in [0,1]$, we have*

$$|x^\alpha - y^\alpha| \le |x - y|^\alpha.$$

2. Main results

2.1. Midpoint-type inequalities

The following lemma is a generalization of [7, Lemma 16] via the Katugampola fractional integrals.

Lemma 2.1. *Let $\alpha, \rho > 0$, $I \subseteq \mathbb{R}$ be an open invex set with respect to the bifunction $\eta : I \times I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I. If $a, b > 0$ with $a < b$ such that $a^\rho, b^\rho \in I$, $\eta(b^\rho, a^\rho) > 0$ and $f' \in L_1\Big([a^\rho, a^\rho + \eta(b^\rho, a^\rho)]\Big)$, then the following equality via the fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \\ & + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho)\Big] \\ & = \frac{\eta(b^\rho, a^\rho)\rho}{2}\Big(I_1 + I_2 + I_3 + I_4\Big), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} I_1 &= \int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt, \\ I_2 &= -\int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt, \\ I_3 &= \int_{\sqrt[\rho]{1/2}}^1 (t^{\alpha\rho} - 1)t^{\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \end{aligned}$$

and

$$I_4 = \int_{\sqrt[\rho]{1/2}}^1 (1 - t^{\alpha\rho})t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt.$$

Proof. By integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\ &= \frac{t^{\alpha\rho}}{(b^\rho - a^\rho)\rho} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))\Big|_0^{\sqrt[\rho]{1/2}} \\ &\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\ &= \frac{2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) \\ &\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt. \end{aligned} \tag{2.2}$$

Similarly, we have

$$I_2 = \frac{2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt, \tag{2.3}$$

$$\begin{aligned}
I_3 &= \int_{\sqrt[\rho]{1/2}}^1 (t^{\alpha\rho} - 1)t^{\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\
&= \frac{t^{\alpha\rho} - 1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))\Big|_{\sqrt[\rho]{1/2}}^1 \\
&\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\
&= \frac{1 - 2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) \\
&\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt
\end{aligned} \tag{2.4}$$

and

$$I_4 = \frac{1 - 2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt. \tag{2.5}$$

Now, by using (2.2), (2.3), (2.4) and (2.5), we have

$$\begin{aligned}
\frac{2}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\alpha}{\eta(b^\rho, a^\rho)}\bigg[\int_0^1 t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\
+ \int_0^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt\bigg] \\
= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.6}$$

By using change of variables and Definition 1.14, we have

$$\int_0^1 t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt = \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(b^\rho, a^\rho)^\alpha}\, {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)^-} f(a^\rho) \tag{2.7}$$

and

$$\int_0^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt = \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(b^\rho, a^\rho)^\alpha}\, {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)). \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6), we obtain

$$I_1+I_2+I_3+I_4=\frac{2}{\eta(b^\rho,a^\rho)\rho}f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho,a^\rho)^{\alpha+1}}\times\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))+{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]. \tag{2.9}$$

The desired identity in (2.1) follows from (2.9). Hence, the proof is complete. □

Remark 2.2. If we choose $\rho=1$ in Lemma 2.1, then we obtain [7, Lemma 16]. Also, if $\rho\neq 1$ and $\eta(x,y)=x-y$ in Lemma 2.1, then we obtain [8, Lemma 2.1] with a minor mistake in the identities obtained in [8] where $\Gamma(\alpha+1)$ should have been $\Gamma(\alpha)$ instead.

Theorem 2.3. *Under the conditions of Lemma* 2.1*, if* $|f'|^q, q\geq 1$ *is prequasiinvex on* I*, then the following inequality holds:*

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))+{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right|$$
$$\leq\eta(b^\rho,a^\rho)\Big(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^\alpha(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}.$$

Proof. By using Lemma 2.1 and the properties of the absolute value, we have

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta_1(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))+{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right|$$
$$\leq\frac{\eta(b^\rho,a^\rho)\rho}{2}\Big(|I_1|+|I_2|+|I_3|+|I_4|\Big). \tag{2.10}$$

By using the power mean inequality, we have

$$|I_1|\leq\left(\int_0^{\sqrt[\rho]{1/2}}t^{(\alpha+1)\rho-1}dt\right)^{1-1/q}\left(\int_0^{\sqrt[\rho]{1/2}}t^{(\alpha+1)\rho-1}|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^qdt\right)^{1/q}. \tag{2.11}$$

Using the prequasiinvexity of $|f'|^q$, we have

$$|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^q\leq\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}. \tag{2.12}$$

Substituting (2.12) in (2.11), we obtain

$$|I_1|\leq\frac{1}{2^{\alpha+1}(\alpha+1)\rho}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.13}$$

Using similar arguments, we deduce that

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)\rho}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}, \tag{2.14}$$

$$\begin{aligned}
|I_3| &\leq \int_{\sqrt[\rho]{1/2}}^{1} |t^{\alpha\rho} - 1| t^{\rho-1} dt \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q} \\
&= \frac{1}{\rho}\int_{1/2}^{1} (1-u^\alpha)du \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right)\right\}\right)^{1/q} \\
&= \frac{1}{\rho}\Big(\frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{2^{\alpha+1}(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}
\end{aligned} \tag{2.15}$$

and

$$|I_4| \leq \frac{1}{\rho}\Big(\frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{2^{\alpha+1}(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right)\right\}\right)^{1/q}. \tag{2.16}$$

The desired inequality follows from (2.10) by using (2.11)-(2.12). □

Corollary 2.4. *If in Theorem* 2.3 *we take* $\eta(x,y) = x-y$ *for all* $x, y \in I$, *i.e,* $|f'|^q, q \geq 1$, *is quasiconvex, then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{a^\rho + b^\rho}{2}\right) - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\leq (b^\rho - a^\rho)\Big(\frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{2^\alpha(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.
\end{aligned}$$

Remark 2.5. It is worth noting that in [8, Theorem 2.8] the authors established another estimate for the left hand side of the inequality in Corollary 2.4 under the condition that $|f'|$ is convex. On the other hand, since every convex function is quasiconvex it follows that the inequality in Corollary 2.4 holds if $|f'|^q, q \geq 1$ is convex.

Theorem 2.6. *Under the conditions of Lemma* 2.1, *if* $|f'|^q, q > 1$ *is prequasiinvex on* I, *then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \right. \\
&\qquad\qquad \left. + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho)\Big]\right| \\
&\leq \frac{\eta(b^\rho, a^\rho)}{2}\left[\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r} + \left(2\int_{1/2}^{1} |u^\alpha - 1|^r du\right)^{1/r}\right] \\
&\qquad \times \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q},
\end{aligned} \tag{2.17}$$

where $\frac{1}{r}+\frac{1}{q}=1$*. In addition, if* $\alpha\in(0,1]$*, then we have the inequality*

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho)) + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right| \le \eta(b^\rho,a^\rho)\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \quad (2.18)$$

Proof. By using Lemma 2.1 and the properties of the absolute value, we have

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho)) + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right| \le \frac{\eta(b^\rho,a^\rho)\rho}{2}\Big(|I_1|+|I_2|+|I_3|+|I_4|\Big). \quad (2.19)$$

By using the Hölder's inequality, we have

$$|I_1|\le\left(\int_0^{\sqrt[\rho]{1/2}}t^{\alpha\rho r}t^{\rho-1}dt\right)^{1/r}\left(\int_0^{\sqrt[\rho]{1/2}}t^{\rho-1}|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^q dt\right)^{1/q}. \quad (2.20)$$

Using the prequasiinvexity of $|f'|^q$, we have

$$|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^q\le\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}. \quad (2.21)$$

Substituting (2.21) in (2.20), we obtain

$$\begin{aligned}|I_1|&\le\left(\frac{1}{2^{\alpha r+1}(\alpha r+1)\rho}\right)^{1/r}\left(\frac{1}{2\rho}\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}\\&=\frac{1}{2\rho}\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \quad (2.22)\end{aligned}$$

Using similar arguments, we deduce that

$$|I_2|\le\frac{1}{2\rho}\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}, \quad (2.23)$$

$$\begin{aligned}|I_3|&\le\left(\int_{\sqrt[\rho]{1/2}}^1|t^{\alpha\rho}-1|^r t^{\rho-1}dt\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\int_{\sqrt[\rho]{1/2}}^1 t^{\rho-1}dt\right)^{1/q}\\&=\left(\frac{1}{\rho}\int_{1/2}^1|u^\alpha-1|^r du\right)^{1/r}\left(\frac{1}{2\rho}\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}\\&=\frac{1}{2\rho}\left(2\int_{1/2}^1|u^\alpha-1|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q} \quad (2.24)\end{aligned}$$

and

$$|I_4| \leq \frac{1}{2\rho}\left(2\int_{1/2}^{1}|u^{\alpha}-1|^{r}du\right)^{1/r}\left(\max\left\{|f'(a^{\rho})|^{q},|f'(b^{\rho})|^{q}\right\}\right)^{1/q}. \tag{2.25}$$

The inequality in (2.17) follows from (2.19) by using (2.20)-(2.21). Now, if $\alpha \in (0,1]$, then it follows from Lemma 1.17 that

$$\int_{1/2}^{1}|u^{\alpha}-1|^{r}du \leq \int_{1/2}^{1}(1-u)^{\alpha r}du = \frac{1}{2^{\alpha r+1}(\alpha r+1)}. \tag{2.26}$$

The inequality in (2.18) follows from (2.17) by using (2.26). Hence, the proof is complete. □

Corollary 2.7. *If in Theorem 2.6 we take $\eta(x,y)=x-y$ for all $x,y\in I$, i.e, $|f'|^{q}, q>1$, is quasiconvex, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{a+}f(b^{\rho})+{}^{\rho}I^{\alpha}_{b-}f(a^{\rho})\right]\right|\\&\leq \frac{b^{\rho}-a^{\rho}}{2}\left[\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}+\left(2\int_{1/2}^{1}|u^{\alpha}-1|^{r}du\right)^{1/r}\right]\\&\quad\times\left(\max\left\{|f'(a^{\rho})|^{q},|f'(b^{\rho})|^{q}\right\}\right)^{1/q},\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=1$. In addition, if $\alpha\in(0,1]$, then we have the inequality

$$\begin{aligned}&\left|f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{a+}f(b^{\rho})+{}^{\rho}I^{\alpha}_{b-}f(a^{\rho})\right]\right|\\&\leq (b^{\rho}-a^{\rho})\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^{\rho})|^{q},|f'(b^{\rho})|^{q}\right\}\right)^{1/q}.\end{aligned}$$

2.2. Trapezoidal-type inequalities

The following lemma is a generalization of Lemma 2.4 in [6] for the invex case.

Lemma 2.8. *Let $\alpha,\rho>0$, $I\subseteq\mathbb{R}$ be an open invex set with respect to the bifunction $\eta: I\times I\to\mathbb{R}$ and $f: I\to\mathbb{R}$ be a differentiable mapping on I. If $a,b>0$ with $a<b$ such that $a^{\rho},b^{\rho}\in I$, $\eta(b^{\rho},a^{\rho})>0$ and $f'\in L_1\Big([a^{\rho},a^{\rho}+\eta(b^{\rho},a^{\rho})]\Big)$, then the following equality via the fractional integrals holds:*

$$\begin{aligned}&\frac{f(a^{\rho})+f(a^{\rho}+\eta(b^{\rho},a^{\rho}))}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\eta(b^{\rho},a^{\rho})^{\alpha}}\Bigg[{}^{\rho}I^{\alpha}_{\left(\sqrt[\rho]{a^{\rho}+\eta(b^{\rho},a^{\rho})}\right)-}f(a^{\rho})\\&\qquad\qquad+{}^{\rho}I^{\alpha}_{a+}f(a^{\rho}+\eta(b^{\rho},a^{\rho}))\Bigg]\\&=\frac{\eta(b^{\rho},a^{\rho})\rho}{2}\int_{0}^{1}[(1-t^{\rho})^{\alpha}-t^{\rho\alpha}]t^{\rho-1}f'(a^{\rho}+(1-t^{\rho})\eta(b^{\rho},a^{\rho}))dt.\end{aligned} \tag{2.27}$$

Proof. We observe that

$$\int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}]t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt = I_1 - I_2,$$

where

$$I_1 = \int_0^1 (1-t^\rho)^\alpha t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt$$

and

$$I_2 = \int_0^1 t^{\alpha\rho} t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt.$$

By integrating by parts and change of variables, we have

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\rho)^\alpha t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\ &= -\frac{(1-t^\rho)^\alpha}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))\Big|_0^1 \\ &\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^1 (1-t^\rho)^{\alpha-1} t^{\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\ &= \frac{1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + \eta(b^\rho, a^\rho)) \\ &\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^1 (1-t^\rho)^{\alpha-1} t^{\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\ &= \frac{1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + \eta(b^\rho, a^\rho)) \\ &\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)^{\alpha+1}} \int_a^{\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}} (u^\rho - a^\rho)^{\alpha-1} u^{\rho-1} f(u^\rho)du. \end{aligned} \tag{2.28}$$

By using Definition 1.14 and (2.28), we have

$$I_1 = \frac{f(a^\rho + \eta(b^\rho, a^\rho))}{\eta(b^\rho, a^\rho)\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho). \tag{2.29}$$

By a similar argument, we have

$$I_2 = -\frac{f(a^\rho)}{\eta(b^\rho, a^\rho)\rho} + \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)). \tag{2.30}$$

By using (2.29) and (2.30), we have

$$\begin{aligned} I_1 - I_2 = \frac{f(a^\rho) + f(a^\rho + \eta(b^\rho, a^\rho))}{\eta(b^\rho, a^\rho)\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} \Big[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho) \\ + {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \Big]. \end{aligned} \tag{2.31}$$

The desired identity in (2.27) follows from (2.31). $\square$

Remark 2.9. If $\eta(x,y) = x - y$ in Lemma 2.8, then we obtain [6, Lemma 2.4] with minor mistakes in the identity obtained in [6] where $\Gamma(\alpha+1)$ should have been $\Gamma(\alpha)$ and $\frac{b^\rho - a^\rho}{2}$ should have been $\frac{(b^\rho - a^\rho)\rho}{2}$ instead.

Theorem 2.10. *Under the conditions of Lemma* 2.8*, if* $|f'|^q, q \geq 1$ *is prequasiinvex on* I*, then the following inequality holds:*

$$\left| \frac{f(a^\rho) + f(a^\rho + \eta(b^\rho, a^\rho))}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha} \left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \right] \right|$$
$$\leq \frac{\eta(b^\rho, a^\rho)}{\alpha+1} \left(1 - \frac{1}{2^\alpha}\right) \left(\max\left\{ |f'(a^\rho)|^q, |f'(b^\rho)|^q \right\} \right)^{1/q}. \tag{2.32}$$

Proof. Using Lemma 2.8, the power mean inequality and the prequasiinvexity of $|f'|^q$, we have

$$\left| \frac{f(a^\rho) + f(a^\rho + \eta(b^\rho, a^\rho))}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha} \left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \right] \right|$$
$$\leq \frac{\eta(b^\rho, a^\rho)\rho}{2} \left(\int_0^1 \left| (1-t^\rho)^\alpha - t^{\rho\alpha} \right| t^{\rho-1} dt \right)^{1-1/q}$$
$$\times \left(\int_0^1 \left| (1-t^\rho)^\alpha - t^{\rho\alpha} \right| t^{\rho-1} \left| f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho)) \right|^q dt \right)^{1/q}$$
$$\leq \frac{\eta(b^\rho, a^\rho)\rho}{2} \left(\int_0^1 \left| (1-t^\rho)^\alpha - t^{\rho\alpha} \right| t^{\rho-1} dt \right) \left(\max\left\{ |f'(a^\rho)|^q, |f'(b^\rho)|^q \right\} \right)^{1/q}$$
$$= \frac{\eta(b^\rho, a^\rho)\rho}{2} \left(\frac{1}{\rho} \int_0^1 \left| (1-u)^\alpha - u^\alpha \right| du \right) \left(\max\left\{ |f'(a^\rho)|^q, |f'(b^\rho)|^q \right\} \right)^{1/q}. \tag{2.33}$$

Now, we observe that

$$\int_0^1 \left| (1-u)^\alpha - u^\alpha \right| du = \int_0^{1/2} \left((1-u)^\alpha - u^\alpha \right) du + \int_{1/2}^1 \left(u^\alpha - (1-u)^\alpha \right) du$$
$$= \frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)} + \frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)}$$
$$= \frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha}\right). \tag{2.34}$$

The inequality in (2.32) follows from (2.33) and (2.34). □

Remark 2.11. If $\eta(x,y) = x - y$ in Theorem 2.10, then we recover the result in [30, Theorem 2.4]. Also, if $\rho = 1$ in Theorem 2.10, then we obtain the result in [13, Theorem 2.3].

Theorem 2.12. *Under the conditions of Lemma* 2.8*, if* $|f'|^q, q > 1$ *is prequasiinvex on* I*, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)}{2}\left(\int_0^1 \left|(1-u)^\alpha - u^\alpha\right|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}, \quad (2.35)$$

where $\frac{1}{r}+\frac{1}{q}=1$*. In addition, if* $\alpha \in (0,1]$*, then we have the inequality*

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)}{2}\left(\frac{1}{\alpha r+1}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}. \quad (2.36)$$

Proof. Using Lemma 2.8, the Hölder's inequality and the prequasiinvexity of $|f'|^q$, we have

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\int_0^1 \left|(1-t^\rho)^\alpha - t^{\rho\alpha}\right|^r t^{\rho-1}dt\right)^{1/r}$$
$$\times\left(\int_0^1 t^{\rho-1}\left|f'(a^\rho+(1-t^\rho)\eta(b^\rho,a^\rho))\right|^q dt\right)^{1/q}$$
$$\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\frac{1}{\rho}\int_0^1 \left|(1-u)^\alpha - u^\alpha\right|^r du\right)^{1/r}\left(\frac{1}{\rho}\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}$$
$$= \frac{\eta(b^\rho,a^\rho)}{2}\left(\int_0^1 \left|(1-u)^\alpha - u^\alpha\right|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.$$

This proves the inequality in (2.35). By using Lemma 1.17 with $\alpha \in (0,1]$, we deduce that

$$\begin{aligned}\int_0^1 \left|(1-u)^\alpha - u^\alpha\right|^r du &\leq \int_0^1 \left|1-2u\right|^{\alpha r} du \\ &= \int_0^{1/2} (1-2u)^{\alpha r} du + \int_{1/2}^1 (2u-1)^{\alpha r} du \\ &= \frac{1}{2(\alpha r+1)} + \frac{1}{2(\alpha r+1)} \\ &= \frac{1}{\alpha r+1}. \end{aligned} \tag{2.37}$$

The inequality in (2.36) follows from (2.35) and (2.37). □

Remark 2.13. If $\rho = 1$ in the inequality (2.36) in Theorem 2.12, then we obtain the result in [13, Theorem 2.4].

Corollary 2.14. *If in Theorem 2.12 we take $\eta(x,y) = x - y$ for all $x, y \in I$, i.e, $|f'|^q, q > 1$, is quasiconvex, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I_{b-}^\alpha f(a^\rho) + {}^\rho I_{a+}^\alpha f(b^\rho)\right]\right| \\ &\leq \frac{b^\rho-a^\rho}{2}\left(\int_0^1 \left|(1-u)^\alpha - u^\alpha\right|^r du\right)^{1/r} \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q},\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$. In addition, if $\alpha \in (0,1]$, then we have the inequality

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I_{b-}^\alpha f(a^\rho) + {}^\rho I_{a+}^\alpha f(a^\rho + \eta(b^\rho, a^\rho))\right]\right| \\ &\leq \frac{b^\rho-a^\rho}{2}\left(\frac{1}{\alpha r+1}\right)^{1/r} \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.\end{aligned}$$

3. Conclusion

We established two midpoint-type inequalities and two trapezoidal-type inequalities for functions whose derivatives in absolute value to some powers are prequasiinvex with respect to a bifunction η via the Katugampola fractional integral operators. By considering the bifunction $\eta(x,y) = x - y$, the results for quasiconvex functions has been obtained from our main results. Several other results can be obtained from our results by considering different bifunctions and/or different values of the parameters involved. In particular, if we take $\rho = 1$, then our results are in terms of the Riemann-Liouville fractional integrals. Also, we hope that under certain conditions on f and η, similar results via the Hadamard fractional integrals could be derived from our results by taking the limit as $\rho \to 0^+$. The details are left for the interested reader.

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Seth Kermausuor
Department of Mathematics and Computer Science,
Alabama State University,
Montgomery, AL 36101, USA
e-mail: `skermausour@alasu.edu`

Eze R. Nwaeze
Department of Mathematics and Computer Science,
Alabama State University, Montgomery, AL 36101, USA
e-mail: `enwaeze@alasu.edu`