Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 2, 255–268
DOI: 10.24193/subbmath.2020.2.07

$\mathcal{A}-$Summation process in the space of locally integrable functions

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Abstract. In this paper, using the concept of summation process, we give a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_{p,q}(loc)$, the space of locally integrable functions, into itself. We also study rate of convergence of these operators.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: Summation process, positive linear operators, locally integrable functions, Korovkin type theorem, modulus of continuity, rate of convergence.

1. Introduction

Approximation theory has many connections with theory of polynomial approximation, functional analysis, numerical solutions of differential and integral equations, summability theory, measure theory and probability theory ([1], [14], [7]).

A Korovkin type theorem for positive linear operators acting from $L_p(a,b)$ to $L_p(a,b)$ was studied in [5], [8], [11] and [20]. Note that all the results just mentioned are devoted to the case of a finite interval (a,b). Roughly speaking a Korovkin type approximation theorem provides conditions for whether a given sequence of positive linear operators converges strongly to the identity operator [1], [12] and [14]. These theorems exhibit a variety of test functions which guarantee that convergence property holds on the whole space provided it holds on them ([1], [14]). If the sequence of positive linear operators does not converge, then it might be useful to use matrix summability methods. The main aim of using summability methods has always been to make a non-convergent sequence to converge. This was the motivation behinde Fejer's famous theorem showing that Cesàro method being effective in making the Fourier series of a continuous periodic function to converge ([22]). Summability methods are also considered in physics ([6]) to make a nonconvergent sequence to converge.

In this paper, using matrix summability methods which includes both convergence and almost convergence, we obtain a Korovkin type approximation theorem of a function f in $L_{p,q}(loc)$. We also give rate of convergence in $L_{p,q}(loc)$ approximation by means of the modulus of continuity. We recall that some results concerning the approximation in $L_{p,q}(loc)$ may be found in [9], [10], [18], [19], [21]. Also $L_{p,q}$ approximation via Abel convergence has been studied in [4]. We remark that matrix summability methods are quite effective, in summing sequences of nonlinear integral operators ([2]).

First of all, we recall some notation and basic definitions used in this paper.

Let $q(x) = 1 + x^2$; $-\infty < x < \infty$. For $h > 0$, by $L_{p,q}(loc)$ we will denote the space of measurable functions f satisfying the inequality,

$$\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p} \leq M_f \; q(x), -\infty < x < \infty \tag{1.1}$$

where $p \geq 1$ and M_f is a positive constant which depends on the function f.

It is known [13] that $L_{p,q}(loc)$ is a linear normed space with norm,

$$\|f\|_{p,q} = \sup_{-\infty < x < \infty} \frac{\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p}}{q(x)}. \tag{1.2}$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. To simplify the notation, we need the following. For any real numbers a and b put

$$\|f; L_p(a,b)\|_{p,q} := \left(\frac{1}{b-a} \int_a^b |f(t)|^p \, dt \right)^{1/p},$$

$$\|f; L_{p,q}(a,b)\|_{p,q} = \sup_{a<x<b} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)},$$

$$\|f; L_{p,q}(|x| \geq a)\|_{p,q} = \sup_{|x| \geq a} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)}.$$

With this notation the norm in $L_{p,q}(loc)$ may be written in the form

$$\|f\|_{p,q} = \sup_{x \in \mathbb{R}} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}.$$

It is known [13] that $L_{p,q}^k(loc)$ is the subspace of all functions $f \in L_{p,q}(loc)$ for which there exists a constant k_f such that

$$\lim_{|x| \to \infty} \frac{\|f - k_f q; L_p(x-h, x+h)\|}{q(x)} = 0.$$

As usual, if T is a positive linear operator from $L_{p,q}(loc)$ into $L_{p,q}(loc)$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_{p,q}}{\|f\|_{p,q}}$.

Let $\mathcal{A} : = \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. A sequence $\{T_j\}$ of positive linear operators from $L_{p,q}(loc)$ into itself is called a *strong $\mathcal{A}-$summation process* in $L_{p,q}(loc)$ if $\{T_j f\}$ is strongly $\mathcal{A}-$*summable* to f for every $f \in L_{p,q}(loc)$, i.e.,

$$\lim_k \sum_j a_{kj}^n \|T_j f - f\|_{p,q} = 0, \quad \text{uniformly in } n.$$

Some results concerning strong summation processes in $L_{p,q}(loc)$ may be found in [3].

2. $\mathcal{A}-$summation process in $L_{p,q}(loc)$

The main aim of the present work is to study a Korovkin type approximation theorem for a sequence of positive linear operators acting on the space $L_{p,q}(loc)$ by using matrix summability method which includes both convergence and almost convergence. We also present an example of positive linear operators which verifies our Theorem 2.6 but does not verify the classical one (see Theorem 2.2 below).

Let $\mathcal{A} := \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. A sequence $\{T_j\}$ of positive linear operators from $L_{p,q}(loc)$ into itself is called an $\mathcal{A}-$*summation process* in $L_{p,q}(loc)$ if $\{T_j f\}$ is $\mathcal{A}-$*summable* to f for every f in $L_{p,q}(loc)$, i.e.,

$$\lim_k \left\| \sum_j a_{kj}^n T_j f - f \right\|_{p,q} = 0, \quad \text{uniformly in } n, \tag{2.1}$$

where it is assumed that the series converges for each k, n and f. Some results concerning summation processes on some other spaces may be found in [16], [17] and [20].

The next result establishes a relationship between strong summation process and summation process in $L_{p,q}(loc)$.

Proposition 2.1. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries and assume that*

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} = 1.$$

Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself If $\{T_j\}$ is a strong $\mathcal{A}-$summation process in $L_{p,q}(loc)$ then $\{T_j\}$ is an $\mathcal{A}-$summation process in $L_{p,q}(loc)$.

Proof. The proof may be obtained by using the idea given in [16].

Throughout the paper let

$$B_k^{(n)}(f) = B_k^{(n)}(f;x) := \sum_j a_{kj}^n T_j(f;x)$$

where we assume that the series on the right is convergent for each $k, n \in \mathbb{N}$ and $f \in L_{p,q}(loc)$.

We recall the following result of [13] that we need in the sequel.

Theorem 2.2. *Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself and satisfy the conditions*

i) The sequence (T_j) is uniformly bounded, that is, $\|T_j\| \leq C < \infty$, where C is a constant independent of j,

ii) For $f_i(y) = y^i$, $i = 0, 1, 2$;

$$\lim_j \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0.$$

Then

$$\lim_j \|T_j f - f\|_{p,q} = 0$$

for each function $f \in L^k_{p,q}(loc)$, (see [13]).

We show that the Korovkin type theorem holds in the subspace $L^k_{p,q}(loc)$. First we give the following

Lemma 2.3. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying the condition*

$$\lim_k \sup_n \left\|B^{(n)}_k(f_i; x) - f_i(x)\right\|_{p,q} = 0.$$

Then, for any continuous and bounded function f on the real axis, we have

$$\lim_k \sup_n \left\|B^{(n)}_k(f; x) - f(x); L_{p,q}(a,b)\right\| = 0$$

where a and b are any real numbers.

Proof. By the uniform continuity of f on the interval $[a,b]$ and by the positivity and linearity of T_j, we may write that

$$\begin{aligned}\left\|B^{(n)}_k(f(t); x) - f(x); L_{p,q}(a,b)\right\| &\leq \left\|B^{(n)}_k(f(t) + f(x) - f(x); x) - f(x)\right\|_{p,q} \\ &\leq \left\|B^{(n)}_k(|f(t) - f(x)|; x)\right\|_{p,q} \\ &+ |f(x)| \left\|B^{(n)}_k(1; x) - 1\right\|_{p,q} \\ &< \varepsilon + \frac{2M}{\delta^2}\left\|B^{(n)}_k(t^2; x) - x^2\right\|_{p,q} \\ &+ \frac{4Mc}{\delta^2}\left\|B^{(n)}_k(t; x) - x\right\|_{p,q} \\ &+ \left(\frac{2Mc^2}{\delta^2} + \varepsilon + M\right)\left\|B^{(n)}_k(1; x) - 1\right\|_{p,q}.\end{aligned}$$

Hence the proof is completed.

Theorem 2.4. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from*

$L_{p,q}(loc)$ *into itself. Assume that*

$$H := \sup_{n,k} \sum_j a_{k,j}^n \|T_j\| < \infty. \tag{2.2}$$

Then $\{T_j\}$ *is an* $\mathcal{A}-$ *summation process in* $L_{p,q}^k(loc)$ *, i.e.,for any function* $f \in L_{p,q}^k(loc)$

$$\lim_k \sup_n \left\| B_k^{(n)}(f;x) - f(x) \right\|_{p,q} = 0$$

if and only if

$$\lim_k \sup_n \left\| B_k^{(n)}(f_i;x) - f_i(x) \right\|_{p,q} = 0$$

where $f_i(y) = y^i$ *for* $i = 0, 1, 2$.

Proof. We follow [13] up to a certain stage. If $f \in L_{p,q}^k(loc)$ then $f - k_f.q \in L_{p,q}^0(loc)$. So it is sufficient to prove the theorem for the function $f \in L_{p,q}^0(loc)$. For $\varepsilon > 0$, there exists a point x_0 such that the inequality

$$\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p} < \varepsilon q(x) \tag{2.3}$$

holds for all x, $|x| \geq x_0$. By the well known Lusin theorem, there exists a continuous function φ on the finite interval $[-x_0 - h, x_0 + h]$ such that the inequality

$$\|f - \varphi; L_p(-x_0, x_0)\| < \varepsilon \tag{2.4}$$

is fulfilled. Setting

$$\delta < \min\left\{ \frac{2h\varepsilon^p}{M^p(x_0)}, h \right\}, \tag{2.5}$$

where $M(x_0) = \max\left\{ \max_{|x| \leq x_0 + h} |\varphi(x)|, 1 \right\}$, we define a continuous function g by

$$g(x) = \begin{cases} \varphi(x), & \text{if } |x| \leq x_0 + h \\ 0, & \text{if } |x| \geq x_0 + h + \delta \\ \text{linear}, & \text{otherwise.} \end{cases}$$

Then by (2.3), (2.4), (2.5) and the Minkowski inequality, we obtain

$$\|f - g\|_{p,q} < \varepsilon \tag{2.6}$$

for any $\varepsilon > 0$ (see [13]).
Now we can find a point $x_1 > x_0$ such that

$$q(x_1) > \frac{M(x_0)}{\varepsilon} \text{ and } g(x) = 0 \text{ for } |x| > x_1, \tag{2.7}$$

where $M(x_0)$ is defined above. Then by (2.4), (2.5), (2.6) and by Lemma 2.3 we get

$$\begin{aligned}\left\|B_k^{(n)}(f;x)-f(x)\right\|_{p,q} &\le \left\|B_k^{(n)}(f-g)\right\|_{p,q}+\left\|B_k^{(n)}g-g\right\|_{p,q}+\|f-g\|_{p,q}\\ &\le \varepsilon\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+\varepsilon+\left\|B_k^{(n)}g-g\right\|_{p,q}\\ &\le \varepsilon\left(\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+1\right)+\left\|B_k^{(n)}g-g;L_{p,q}(-x_1,x_1)\right\|\\ &+\left\|B_k^{(n)}g-g;L_{p,q}(|x|\ge x_1)\right\|\\ &\le \varepsilon\left(\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+2\right)+\left\|B_k^{(n)}g;L_{p,q}(|x|\ge x_1)\right\|.\end{aligned} \tag{2.8}$$

Since $|g(x)|\le M(x_0)$ for all $x\in\mathbb{R}$, we can write

$$\begin{aligned}\left\|B_k^{(n)}g;L_{p,q}(|x|\ge x_1)\right\|_{p,q} &\le M(x_o)\left\|B_k^{(n)}1;L_{p,q}(|x|\ge x_1)\right\|\\ &\le M(x_o)\left\|B_k^{(n)}1-1;L_{p,q}(|x|\ge x_1)\right\|\\ &+M(x_o)\|1;L_{p,q}(|x|\ge x_1)\|\\ &\le M(x_o)\left\|B_k^{(n)}1-1\right\|_{p,q}+\frac{M(x_o)}{q(x_1)}.\end{aligned}$$

Considering hypothesis and (2.7) we get by (2.8) that

$$\lim_k\sup_n\left\|B_k^{(n)}f-f\right\|_{p,q}=0.$$

The next result shows that Korovkin type theorem does not hold in the whole space $L_{p,q}(loc)$.

Theorem 2.5. *Let $\mathcal{A}:=\{A^{(n)}\}=\left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying*

$$\lim_k\sup_n\left\|\sum_j a_{kj}^{(n)}T_j(f_i;x)-f_i(x)\right\|_{p,q}=0.$$

Then there exists a function f^* in $L_{p,q}(loc)$ for which

$$\lim_k\sup_n\left\|\sum_j a_{kj}^{(n)}T_jf^*-f^*\right\|_{p,q}\ge 2^{1-\frac{1}{p}}. \tag{2.9}$$

Proof. We consider the sequence of operators T_j given in [13] $(for\ j = 1, 2, \ldots)$:

$$T_j\left(f; x\right) = \begin{cases} \dfrac{x^2}{\left(x+h\right)^2} f\left(x+h\right), & x \in \left[\left(2j-1\right)h, \left(2j+1\right)h\right) \\ f\left(x\right), & \text{otherwise.} \end{cases}$$

As observed in [13] that $T_j : L_{p,q}\left(loc\right) \to L_{p,q}\left(loc\right)$. Assume now that

$$\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$$

is a sequence of infinite matrices defined by

$$a_{kj}^{(n)} = \begin{cases} \dfrac{1}{k+1}, & n \le j \le n+k \\ 0, & \text{otherwise.} \end{cases}$$

It is shown in [13] that

$$\left\|T_j f_i - f_i\right\|_{p,q} \to 0, \ \left(as\ j \to \infty\right).$$

Now it is easy to verify that, for each $i = 0, 1, 2$

$$\begin{aligned} \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f_i - f_i \right\|_{p,q} &= \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f_i - \frac{1}{k+1} \sum_{j=n}^{k+n} f_i \right\|_{p,q} \\ &= \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} \left(T_j f_i - f_i\right) \right\|_{p,q} \\ &\le \frac{1}{k+1} \sum_{j=n}^{k+n} \left\|T_j f_i - f_i\right\|_{p,q} \\ &\to 0 \ (k \to \infty, \text{ uniformly in } n). \end{aligned}$$

Consider the following function f^* given in [13] :

$$f^*\left(x\right) = \begin{cases} x^2, & \text{if } \ x \in \bigcup\limits_{k=1}^{\infty} [(2k-1)h, 2kh) \\ -x^2, & \text{if } \ x \in \bigcup\limits_{k=0}^{\infty} \left[2kh, \left(2k+1\right)h\right) \\ 0, & \text{if } \ x < 0. \end{cases}$$

Then $f^{*}\in L_{p,q}\left(loc\right)$ and we get

$$\begin{aligned}\left\Vert \frac{1}{k+1}\sum_{j=n}^{k+n}T_{j}f^{*}-f^{*}\right\Vert _{p,q} &\geq \sup_{x\in[(2n-1)h,2(n+k)h]}\frac{\left(\frac{1}{2h}\int\limits_{x-h}^{x+h}\left|\frac{1}{k+1}\sum\limits_{j=n}^{k+n}T_{j}f^{*}-f^{*}\right|^{p}dt\right)^{\frac{1}{p}}}{q(x)}\\ &\geq \frac{\left(\frac{1}{2h}\int\limits_{2nh-h}^{2nh+h}\left|\frac{1}{k+1}\sum\limits_{j=n}^{k+n}\frac{\xi^{2}}{(\xi+h)^{2}}f^{*}\left(\xi+h\right)-f^{*}\left(\xi\right)\right|^{p}d\xi\right)^{\frac{1}{p}}}{q\left(2nh\right)}\\ &> \frac{\left(\frac{1}{2h}2^{p}\left(\left(2n-1\right)h\right)^{2p}h\right)^{\frac{1}{p}}}{1+4n^{2}h^{2}}\\ &= \frac{2^{1-\frac{1}{p}}\left(2n-1\right)^{2}h^{2}}{1+4n^{2}h^{2}}.\end{aligned}$$

On applying the operator $\lim\limits_{k}\sup\limits_{n}$ on both sides one can see that

$$\lim_{k}\sup_{n}\left\Vert \frac{1}{k+1}\sum_{j=n}^{k+n}T_{j}f^{*}-f^{*}\right\Vert _{p,q}\geq 2^{1-1/p}$$

Therefore the theorem is proved.
In the whole space $L_{p,q}\left(loc\right)$ we have the following

Theorem 2.6. *Let* $\mathcal{A}:=\left\{A^{(n)}\right\}=\left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entries for which (2.2) holds. Let* $\{T_{j}\}$ *be a sequence of positive linear operators from* $L_{p,q}\left(loc\right)$ *into itself. Assume that*

$$\lim_{k}\sup_{n}\left\Vert B_{k}^{(n)}(f_{i};x)-f_{i}\left(x\right)\right\Vert _{p,q}=0$$

where $f_{i}\left(y\right)=y^{i}$ *for* $i=0,1,2$. *Then for any functions* $f\in L_{p,q}\left(loc\right)$ *we have*

$$\lim_{k}\sup_{n}\left(\sup_{x\in\mathbb{R}}\frac{\left\Vert B_{k}^{(n)}f-f;L_{p}\left(x-h,x+h\right)\right\Vert }{q^{*}\left(x\right)}\right)=0$$

where q^{*} *is a weight function such that* $\lim\limits_{|x|\to\infty}\frac{1+x^{2}}{q^{*}(x)}=0.$

Proof. By hypothesis, given $\varepsilon>0$, there exists x_{0} such that for all x with $|x|\geq x_{0}$ we have

$$\frac{1+x^{2}}{q^{*}\left(x\right)}<\varepsilon. \tag{2.10}$$

Let $f \in L_{p,q}(loc)$. Then, for all n,k we get

$$\begin{aligned}\gamma_k^{(n)} &:= \left\| B_k^{(n)} f - f; L_p(|x| > x_0) \right\| \\ &\le \left\| B_k^{(n)} f - f \right\|_{p,q} \\ &\le \sum_j a_{k,j}^{(n)} \|T_j f\|_{p,q} + \|f\|_{p,q} \\ &\le \|f\|_{p,q} \left(\sum_j a_{k,j}^{(n)} \|T_j\|_{p,q} + 1 \right) < N, \text{ say.}\end{aligned}$$

Hence we have $\sup_{n,k} \gamma_k^{(n)} < \infty$. By Lusin's theorem we can find a continuous function φ on $[-x_0 - h, x_0 + h]$ such that

$$\|f - \varphi; L_p(-x_0 - h, x_0 + h)\| < \varepsilon. \tag{2.11}$$

Now we consider the function G defined in [13] given by

$$G(x) := \begin{cases} \varphi(-x_0 - h), & x \le -x_0 - h \\ \varphi(x_0), & |x| < x_0 + h \\ \varphi(x_0 + h), & x \ge x_0 + h. \end{cases}$$

We see that G is continuous and bounded on the whole real axis.
Now let $f \in L_{p,q}(loc)$. Then we get for all n,k that

$$\begin{aligned}\beta_k^{(n)} &:= \left\| \sum_j a_{k,j}^{(n)} T_j f - f; L_{p,q}(-x_0, x_0) \right\| \\ &\le \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{k,j}^{(n)} \|T_j\|_{p,q} + 1 \right) \\ &+ \left\| \sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\|.\end{aligned}$$

Hence by the hypothesis and Lemma 2.3 we have

$$\lim_k \sup_n \beta_k^{(n)} = 0. \tag{2.12}$$

On the other hand, a simple calculation shows that

$$\begin{aligned}u_k^{(n)} &:= \left\| \sum_j a_{k,j}^{(n)} T_j f - f \right\|_{p,q^*} \\ &< \beta_k^{(n)} \sup_{|x| < x_0} \frac{q(x)}{q^*(x)} + \gamma_k^{(n)} \sup_{|x| \ge x_0} \frac{q(x)}{q^*(x)} \\ &< \beta_k^{(n)} M + \varepsilon \gamma_k^{(n)}, \text{ (for some } M > 0).\end{aligned} \tag{2.13}$$

It follows from (2.10), (2.11), (2.12), (2.13) and Lemma 2.3 that

$$\begin{aligned} u_k^{(n)} &< q(x_0)\left\|f-G; L_{p,q}(-x_0-h, x_0+h)\right\| \left(\sum_j a_{k,j}^{(n)} \left\|T_j\right\|_{p,q}+1\right) \\ &+ q(x_0)\left\|\sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0)\right\| + \varepsilon N \\ &= K\varepsilon + q(x_0)\left\|\sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0)\right\| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. By Lemma 2.3 we get

$$\lim_k \sup_n \left(\sup_{x\in\mathbb{R}} \frac{\left\|B_k^{(n)} f - f; L_p(x-h, x+h)\right\|_{p,q}}{q^*(x)} \right) = 0.$$

Remark 2.7. We now present an example of a sequence of positive linear operators which satisfies Theorem 2.6 but does not satisfy Theorem 2.2. Assume now that $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ is a sequence of infinite matrices defined by

$$a_{k,j}^{(n)} = \begin{cases} \frac{1}{k+1}, & n \le j \le n+k \\ 0, & \text{otherwise.} \end{cases}$$

In this case $\mathcal{A}-$summability method reduces to almost convergence, ([15]).

Let $T_j : L_{p,q}(loc) \to L_{p,q}(loc)$ be given by

$$T_j(f;x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h), & x \in [(2j-1)h, (2j+1)h) \\ f(x), & \text{otherwise.} \end{cases}$$

The sequence $\{T_j\}$ satisfies Theorem 1 in [13]. It is also shown in [13] that for all $j \in \mathbb{N}$, $\|T_j f\|_{p,q} \le 4\|f\|_{p,q}$. Hence $\{T_j\}$ is an uniformly bounded sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Also

$$\lim_k \sup_n \left\|B_k^{(n)}(f_i;x) - f_i(x)\right\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Now we define $\{P_j\}$ by

$$P_j(f;x) = (1+u_j)\, T_j(f;x)$$

where

$$u_j = \begin{cases} 1, & j = 2^n,\ n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\{u_j\}$ is almost convergent to zero. Therefore the sequence of positive linear operators $\{P_j\}$ satisfies Theorem 2.6 but does not satisfy Theorem 2.2.

3. Rates of convergence for $\mathcal{A}-$summation process in $L_{p,q}(loc)$

In this section, using the modulus of continuity, we study rates of convergence of operators given in Theorem 2.6.

We consider the following modulus of continuity:

$$w(f,\delta)=\sup_{|x-y|\le\delta}|f(y)-f(x)|$$

where δ is a positive constant, $f\in L_{p,q}(loc)$. It is easy to see that, for any $c>0$ and all $f\in L_{p,q}(loc)$,

$$w(f,c\delta)\le(1+[c])\,w(f,\delta), \tag{3.1}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

We first need the following lemma

Lemma 3.1. *Let $\mathcal{A}:=\{A^{(n)}\}=\left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Then for each $j\in\mathbb{N}$ and $\delta>0$, and for every function f that is continuous and bounded on the whole real axis, we have*

$$\left\|B_k^{(n)}f-f;L_{p,q}(a,b)\right\|\le w(f;\delta)\left\|B_k^{(n)}f_0-f_0\right\|_{p,q}+2w(f;\delta)+C_1\left\|B_k^{(n)}f_0-f_0\right\|_{p,q}$$

where $f_0(t)=1,\varphi_x(t):=(t-x)^2,C_1=\sup_{a\le x\le b}|f(x)|$ and

$$\delta:=\alpha_k^{(n)}=\sqrt{\left\|B_k^{(n)}\varphi_x\right\|_{p,q}}.$$

Proof. Let f be any continuous and bounded function on the real axis, and let $x\in[a,b]$ be fixed. Using linearity and monotonicity of T_j and for any $\delta>0$, by (3.1), we get

$$\begin{aligned}\left|B_k^{(n)}(f;x)-f(x)\right|&\le B_k^{(n)}(|f(t)-f(x)|;x)+|f(x)|\left|B_k^{(n)}(f_0;x)-f_0(x)\right|\\&B_k^{(n)}\left(w\left(f,\frac{|t-x|}{\delta}\delta\right),x\right)\\&+|f(x)|\left|B_k^{(n)}(f_0;x)-f_0(x)\right|\\&\le w(f,\delta)\,B_k^{(n)}\left(1+\left[\frac{t-x}{\delta}\right],x\right)\\&+|f(x)|\left|B_k^{(n)}(f_0;x)-f_0(x)\right|\\&\le w(f,\delta)\left|B_k^{(n)}(f_0;x)-f_0(x)\right|+w(f,\delta)\\&+\frac{w(f,\delta)}{\delta^2}\left|B_k^{(n)}\varphi_x\right|+|f(x)|\left|B_k^{(n)}(f_0;x)-f_0(x)\right|.\end{aligned}$$

Now let $C_1 = \sup_{a<x<b} |f(x)|$ and $\delta := \alpha_k^{(n)} = \sqrt{\left\| B_k^{(n)} \varphi_x \right\|_{p,q}}$. Then we have

$$\left\| B_k^{(n)} f - f \right\|_{p,q} \leq w(f,\delta) \left\| B_k^{(n)}(f_0;x) - f_0(x) \right\|_{p,q} + 2w(f,\delta) + C_1 \left\| B_k^{(n)}(f_0;x) - f_0(x) \right\|_{p,q}.$$

Theorem 3.2. *Let* $\mathcal{A} : = \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entires. Let* $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself. Assume that for each continuous and bounded function* f *on the real line, the following conditions hold:*

(i) $\lim_k \sup_n \left\| B_k^{(n)}(f_0;x) - f_0(x) \right\|_{p,q} = 0,$

(ii) $\lim_k \sup_n w(f,\delta) = 0.$

Then we have

$$\lim_k \sup_n \left\| B_k^{(n)} f - f \right\|_{p,q} = 0.$$

Proof. Using Lemma 3.1 and considering (i) and (ii), we have

$$\lim_k \sup_n \left\| B_k^{(n)} f - f; L_{p,q}(a,b) \right\| = 0$$

for all continuous and bounded functions on the real axis.

Theorem 3.3. *Let* $\mathcal{A} : = \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entries for which* (2.2) *holds. Let* $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself. For a given* $f \in L_{p,q}(loc)$ *assume that*

$$\lim_k \sup_n \left\| B_k^{(n)}(f_i;x) - f_i(x) \right\|_{p,q} = 0$$

where $f_i(y) = y^i$ *for* $i = 0,1,2.$ *If*

(i) $\lim_k \sup_n \left\| B_k^{(n)}(f_0;x) - f_0(x) \right\|_{p,q} = 0,$

(ii) $\lim_k \sup_n w(G,\delta) = 0$

where G *is given as in the proof of Theorem* 2.6*. Then we have*

$$\lim_k \sup_n \left(\sup_{x\in\mathbb{R}} \frac{\left\| B_k^{(n)} f - f; L_p(x-h,x+h) \right\|}{q^*(x)} \right) = 0$$

where q^* *is a weight function such that*

$$\lim_{|x|\to\infty} \frac{1+x^2}{q^*(x)} = 0.$$

Proof. It is known from Theorem 2.6 that

$$u_k^{(n)} < q(x_0) \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + 1 \right)$$
$$+ q(x_0) \left\| B_k^{(n)} G - G; L_{p,q}(-x_0, x_0) \right\| + \varepsilon N$$
$$= K\varepsilon + q(x_0) \left\| B_k^{(n)} G - G; L_{p,q}(-x_0, x_0) \right\|$$

where $K := Mq(x_0) + N$ and $M := H + 1$. Then by Lemma 3.1 and Theorem 3.2 we get

$$u_k^{(n)} \leq K\varepsilon + q(x_0) w(G;\delta) \left\| B_k^{(n)}(f_0; x) - f_0(x) \right\|_{p,q} + 2q(x_0) w(G;\delta)$$
$$+ q(x_0) C_1' \left\| B_k^{(n)}(f_0; x) - f_0(x) \right\|_{p,q}$$

where $C_1' := \sup_{-x_0 < x < x_0} |G(x)|$ and the proof is completed.

Acknowledgment. The authors wish to thank the referee for several helpful suggestions that have improved the exposition of these results.

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