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Applications of the deferred generalized de la Vallée Poussin means in approximation of continuous functions

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Abstract. In this paper we have proved a theorem which show the degree of approximation of periodic functions by some generalized means of their Fourier series. In addition, our result is extended to two-dimensional setting as well.

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1. Introduction

Let f be a 2π-periodic function, $f \in L[0, 2\pi]$, and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{1.1}$$

its Fourier series at the point x, where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx,\ (k = 0, 1, \dots);\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx,\ (k = 1, 2, \dots).$$

By

$$\|f\| = \sup_{0\leq x\leq 2\pi} |f(x)|$$

we denote the sup-norm of f over $[0, 2\pi]$, and by $C[0, 2\pi]$ the class of all 2π-periodic continuous functions defined in $[0, 2\pi]$.

In 1928, was G. Alexits [4] who studied the degree of approximation of function a $f \in \text{Lip}\alpha$ by Cesàro means (C, δ) of its Fourier series. This study may be considered as a starting point for other studies of this nature, and another type of similar studies

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can be found in [6]-[9]. Recent studies of other researchers can be found in [1], [5], and [7].

For our purpose, we are going to recall a result proved in [6]. To do this we need first to present the generalized Vallée Poussin mean given in [10].

Let $\sum_{n=1}^{\infty} w_n$ be a given infinite series and let s_n be its n-th partial sum. Let $\lambda := (\lambda_n)$ be a monotone non-decreasing sequence of integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$.

The mean

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{m=n-\lambda_n}^{n-1} s_m, \quad (n \geq 1), \tag{1.2}$$

is called the n-th generalized de la Vallée Poussin mean of the sequence (s_n) generated by sequence (λ_n).

For n-th partial sum

$$s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of the series (1.1), its n-th generalized de la Vallée Poussin mean is defined by

$$V_n(\lambda; f; x) = \frac{1}{\lambda_n} \sum_{m=n-\lambda_n}^{n-1} s_m(f;x), \quad (n \geq 1), \tag{1.3}$$

and the modulus of continuity of $f(x)$, for a given real number $\delta > 0$, is defined as follows

$$\omega(f;\delta) := \sup_{|x-y|\leq\delta} |f(x) - f(y)|,$$

where $x, y \in [0, 2\pi]$.

Throughout this paper we write $u = \mathcal{O}(v)$ if there exists a positive constant K, such that $u \leq Kv$. Now, we are ready to recall the result mentioned above.

Theorem 1.1 ([6]). *Let $f \in C[0, 2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying the following conditions as $t \to +0$:*

$$\int_t^{\frac{\pi}{2}} u^{-2} \omega(f;u) du = \mathcal{O}(F(t)), \tag{1.4}$$

where $F(t) \geq 0$, and

$$\int_0^t F(u) du = \mathcal{O}(tF(t)). \tag{1.5}$$

Then

$$\|f - V_n(\lambda; f)\| = \mathcal{O}\left(\frac{1}{\lambda_n} F\left(\frac{\pi}{2\lambda_n}\right)\right). \tag{1.6}$$

For our further investigation let $a := (a_n)$ and $b := (b_n)$ be sequences of non-negative integers with condition

$$1 \leq b_n - a_n + \lambda_n, \quad (n = 1, 2, \dots). \tag{1.7}$$

Whence, we are in able to generalize the mean $V_n(\lambda)$ defined in (1.2) as follows.

The mean

$$V_n(\lambda,a,b)=\frac{1}{b_n-a_n+\lambda_n}\sum_{m=a_n-\lambda_n}^{b_n-1}s_m,\quad (n\geq 1), \tag{1.8}$$

is called the n-th deferred generalized de la Vallée Poussin mean of the sequence (s_n) generated by sequences λ, a, and b.

It is the purpose of this paper to estimate the deviation $f-V_n(\lambda,a,b)$ in the sup-norm, which in fact generalize Theorem 1.1 (as well as we extend it in the two-dimensional setting, see subsec. 3.2). To do this we need some helpful lemmas given in next section.

2. Auxiliary lemma

Next lemma has been proved implicitly in [6].

Lemma 2.1. *Let* (1.4) *hold. Then,* $\omega(f;t)=\mathcal{O}(tF(t))$.

Now, we prove next helpful lemma.

Lemma 2.2. *Denote by*

$$\mathbb{K}_n^{a,b}(t):=\sum_{m=a_n-\lambda_n}^{b_n-1}D_m(t)=\sum_{m=a_n-\lambda_n}^{b_n-1}\frac{\sin(2m+1)t}{\sin t}$$

the deferred de la Vallée Poussin kernel, where $D_m(t):=\frac{\sin(2m+1)t}{\sin t}$.
Then,

(i) $\mathbb{K}_n^{a,b}(t)=\dfrac{\sin(b_n-a_n+\lambda_n)t\sin(b_n+a_n-\lambda_n)t}{\sin^2 t}$,

(ii) $|\mathbb{K}_n^{a,b}(t)|=\mathcal{O}\left(\dfrac{b_n-a_n+\lambda_n}{t}\right),\quad 0<t\leq\dfrac{\pi}{2(b_n-a_n+\lambda_n)}$,

(iii) $|\mathbb{K}_n^{a,b}(t)|=\mathcal{O}\left(\dfrac{1}{t^2}\right),\quad \dfrac{\pi}{2(b_n-a_n+\lambda_n)}<t\leq\dfrac{\pi}{2}$.

Proof. (i) We have

$$\begin{aligned}\mathbb{K}_n^{a,b}(t) &= \sum_{m=a_n-\lambda_n}^{b_n-1}\frac{\sin(2m+1)t}{\sin t}\\ &= \sum_{m=0}^{b_n-1}\frac{2\sin(2m+1)t\sin t}{2\sin^2 t}-\sum_{m=0}^{a_n-\lambda_n-1}\frac{2\sin(2m+1)t\sin t}{2\sin^2 t}\\ &= \frac{1-\cos(2b_nt)}{2\sin^2 t}-\frac{1-\cos(a_n-\lambda_n)t}{2\sin^2 t}\\ &= \frac{\sin^2(b_nt)-\sin^2(a_n-\lambda_n)t}{\sin^2 t}\\ &= \frac{\sin(b_n-a_n+\lambda_n)t\sin(b_n+a_n-\lambda_n)t}{\sin^2 t}.\end{aligned}$$

(ii) Using the inequalities $|\sin\beta|\le 1$, $|\sin\beta|\le\beta$, and $\sin\beta\ge\frac{2}{\pi}\beta$ for $0<\beta\le\frac{\pi}{2}$, we have:

$$|\mathbb{K}_n^{a,b}(t)|\le\frac{\pi^2(b_n-a_n+\lambda_n)t}{4t^2}=\mathcal{O}\left(\frac{b_n-a_n+\lambda_n}{t}\right).$$

(iii) Similarly, using the inequalities $|\sin\beta|\le 1$ and $\sin\beta\ge\frac{2}{\pi}\beta$ for $0<\beta\le\frac{\pi}{2}$, we also have:

$$|\mathbb{K}_n^{a,b}(t)|\le\frac{\pi^2}{4t^2}=\mathcal{O}\left(\frac{1}{t^2}\right).$$

The proof is completed. □

In the sequel we pass to the main result.

3. Main result

3.1. Approximation by deferred generalized de la Vallée Poussin mean of single Fourier series

Here, we prove the following.

Theorem 3.1. *Let $f\in C[0,2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t\to+0$, where $F(t)\ge 0$.*

Then

$$\|f-V_n(\lambda,a,b;f)\|=\mathcal{O}\left(\frac{1}{b_n-a_n+\lambda_n}F\left(\frac{\pi}{2(b_n-a_n+\lambda_n)}\right)\right). \tag{3.1}$$

Proof. After some calculation we have:

$$s_m(f;x)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}[f(x+2t)+f(x-2t)]D_m(t)dt,$$

where $D_m(t)=\frac{\sin(2m+1)t}{\sin t}$.

Denoting by $V_n(\lambda,a,b;f;x)$ the deferred generalized de la Vallée Poussin mean of $s_m(f;x)$, i.e.,

$$V_n(\lambda,a,b;f;x):=\frac{1}{b_n-a_n+\lambda_n}\sum_{m=a_n-\lambda_n}^{b_n-1}s_m(f;x),$$

we get:

$$V_n(\lambda,a,b;f;x)-f(x)=\frac{1}{(b_n-a_n+\lambda_n)\pi}\int_0^{\frac{\pi}{2}}\psi_x(t)\mathbb{K}_n^{a,b}(t)dt,$$

where

$$\psi_x(t):=f(x+2t)+f(x-2t)-f(x).$$

Whence,

$$\|V_n(\lambda,a,b;f)-f\| \le \frac{1}{(b_n-a_n+\lambda_n)\pi}\int_0^{\frac{\pi}{2}} |\psi_x(t)||\mathbb{K}_n^{a,b}(t)|dt$$
$$\le \frac{4}{(b_n-a_n+\lambda_n)\pi}\left(\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}} + \int_{\frac{\pi}{2(b_n-a_n+\lambda_n)}}^{\frac{\pi}{2}}\right)\omega(f;t)|\mathbb{K}_n^{a,b}(t)|dt$$
$$:= \mathbb{P}_1+\mathbb{P}_2. \tag{3.2}$$

Using Lemma 2.2, part (ii), we obtain:

$$|\mathbb{P}_1| = \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}} t^{-1}\omega(f;t)dt,$$

and applying Lemma 2.1, (1.4) and (1.5), we get:

$$\begin{aligned}|\mathbb{P}_1| &= \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}}\int_t^{\frac{\pi}{2}} u^{-2}\omega(f;u)dudt\\ &= \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}} F(t)dt\\ &= \mathcal{O}\left(\frac{1}{b_n-a_n+\lambda_n}F\left(\frac{\pi}{2(b_n-a_n+\lambda_n)}\right)\right).\end{aligned} \tag{3.3}$$

To estimate $\mathbb{P}_2$, we use Lemma 2.2, part (iii). Namely, based on (1.4), we have

$$\begin{aligned}|\mathbb{P}_2| &= \mathcal{O}\left(\frac{1}{(b_n-a_n+\lambda_n)\pi}\right)\int_{\frac{\pi}{2(b_n-a_n+\lambda_n)}}^{\frac{\pi}{2}} t^{-2}\omega(f;t)dt\\ &= \mathcal{O}\left(\frac{1}{b_n-a_n+\lambda_n}F\left(\frac{\pi}{2(b_n-a_n+\lambda_n)}\right)\right).\end{aligned} \tag{3.4}$$

Finally, inserting (3.2) and (3.3) into (3.4), we immediately obtain (3.1) as required. The proof is completed. □

Remark 3.2. Since, in general, $\lambda_n \le b_n-a_n+\lambda_n$, then we observe that the degree of approximation obtained in Theorem 3.1 is not worse than that appears in Theorem 1.1.

Remark 3.3. For $b_n=a_n=n$, we immediately obtain the result given in [6].

Further, let the sequences $a:=(a_n)$ and $b:=(b_n)$ be of non-negative integers with conditions

$$a_n < b_n, \quad n=1,2,\ldots, \tag{3.5}$$

and

$$\lim_{n\to\infty} b_n = +\infty. \tag{3.6}$$

If $\lambda_n = 1$ for all $n \ge 1$, then the deferred de la Vallée Poussin mean

$$V_n(1,a+2,b+1;f;x)$$

reduces to

$$D_a^b(f;x) := \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} s_m(f;x),$$

which is the deferred Cesàro mean of the sum $s_n(f;x)$ introduced in [2]. In the same paper, it was shown that (3.5) and (3.6) are conditions of regularity for D_a^b. Consequently, if conditions (3.5) and (3.6) are satisfied, then from Theorem 3.1 we deduce the following.

Corollary 3.4. *Let $f \in C[0,2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t \to +0$, where $F(t) \geq 0$.*
Then

$$\|f - D_a^b(f)\| = \mathcal{O}\left(\frac{1}{b_n - a_n} F\left(\frac{\pi}{2(b_n - a_n)}\right)\right).$$

Also, if we take $\lambda_n = n$, $a_n = n$, $b_n = n+1$, $\forall n \geq 1$, then the deferred generalized de la Vallée Poussin mean reduces to ordinary Cesàro mean of the sum $s_n(f;x)$,

$$\sigma_n(f;x) := \frac{1}{n+1} \sum_{m=0}^{n} s_m(f;x).$$

Therefore, Theorem 3.1 also implies:

Corollary 3.5. *Let $f \in C[0,2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t \to +0$, where $F(t) \geq 0$.*
Then

$$\|f - \sigma_n(f)\| = \mathcal{O}\left(\frac{1}{n+1} F\left(\frac{\pi}{2(n+1)}\right)\right).$$

Let us specify the function $F(t)$ as follows:

$$F(t) = \begin{cases} t^{\gamma-1}, & 0 < \gamma < 1; \\ \log\left(\frac{\pi}{t}\right), & \gamma = 1. \end{cases}$$

Using this function the following estimations from Theorem 3.1, Corollary 3.4, and Corollary 3.5 can be deduced (of course all other conditions are maintaining):

(a) From Theorem 3.1:

$$\|f - V_n(\lambda, a, b; f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(b_n - a_n + \lambda_n)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(b_n - a_n + \lambda_n))}{b_n - a_n + \lambda_n}, & \gamma = 1. \end{cases}$$

(b) From Corollary 3.4:

$$\|f - D_a^b(f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(b_n - a_n)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(b_n - a_n))}{b_n - a_n}, & \gamma = 1. \end{cases}$$

(c) From Corollary 3.5 (this is a particular case of a result given in [4]):

$$\|f - \sigma_n(f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(n+1)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(n+1))}{n+1}, & \gamma = 1. \end{cases}$$

3.2. Approximation by deferred generalized de la Vallée Poussin mean of double Fourier series

Let $C([-\pi,\pi]^2)$ be the class of real-valued functions of two variables that are continuous on $[-\pi,\pi]\times[-\pi,\pi] := [-\pi,\pi]^2$ and 2π periodic with respect to x and y. We recall that the double Fourier series of the function $f(x,y)\in C([-\pi,\pi]^2)$ is defined by

$$f(x,y)\sim\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big[a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny \\ +c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny\Big],$$

where

$$\lambda_{mn}=\begin{cases}1/4, & \text{if } m=n=0,\\ 1/2, & \text{if } m>0, n=0\vee m=0, n>0,\\ 1, & \text{if } m>0, n>0,\end{cases}$$

and

$$a_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\cos nv du dv,$$
$$b_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\cos nv du dv,$$
$$c_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\sin nv du dv,$$
$$d_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\sin nv du dv,$$

are the Fourier coefficients of the function $f(x,y)$.

The sequence $\{s_{m,n}(f;x,y)\}$ represents the sequence of partial sums of the double Fourier series which can be rewritten in integral form by

$$s_{m,n}(x,y):=s_{m,n}(f;x,y):=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)D_m(u)D_n(v)du dv.$$

To my best knowledge the double de la Vallée Poussin mean of $s_{m,n}(x,y)$ is defined by (see [3])

$$V_{m,n}^{(p,q)}(f;x,y):=\frac{1}{(p+1)(q+1)}\sum_{k=n}^{n+p}\sum_{\ell=m}^{m+q}s_{k,\ell}(x,y),\quad p\ge 0,\ q\ge 0. \tag{3.7}$$

The mean $V_{m,n}^{(p,q)}(f;x,y)$ is generalized in [11] as follows (for our purposes we modify it "a little bit"). Let $\lambda:=(\lambda_m)$ and $\mu:=(\mu_n)$ be two monotone non-decreasing sequences of integers such that $\lambda_1=\mu_1=1$, $\lambda_{m+1}-\lambda_m\le 1$, and $\mu_{n+1}-\mu_n\le 1$.

The mean

$$V_{m,n}^{\lambda,\mu}(f;x,y)=\frac{1}{\lambda_m\mu_n}\sum_{k=m-\lambda_m}^{m-1}\sum_{k=n-\mu_n}^{n-1}s_{k,\ell}(x,y),\quad (m,n\ge 1), \tag{3.8}$$

is called the (mn)-th deferred generalized de la Vallée-Poussin mean of the sequence $(s_{k,\ell}(x,y))$ generated by sequences (λ_m) and (μ_n).

The (total) modulus of continuity of a continuous function $f(x,y)$, 2π-periodic in each variable, in symbols $f\in C([-\pi,\pi]^2)$, is defined by (see [12], page 283)

$$\omega_1(f,\delta_1,\delta_2)=\sup_{x,y}\sup_{|u|\le\delta_1,|v|\le\delta_2}|f(x+u,y+v)-f(x,y)|,\quad \delta_1,\delta_2\ge 0.$$

To estimate the deviation

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)\right|,$$

which is the main result of this subsection, first we denote

$$\begin{aligned}\phi_{xy}(s,t):=f(x+s,y+t)&+f(x-s,y+t)\\&+f(x+s,y-t)+f(x-s,y-t)-4f(x,y).\end{aligned}$$

Now, we are in able to prove the following.

Theorem 3.6. *Let $f\in C([-\pi,\pi]^2)$, $\omega_1(f,s,t)=\mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t\to+0$, and $F_1(s),F_2(t)\ge 0$ two mediate functions. Then*

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)\right|=\mathcal{O}\left(\frac{1}{\lambda_m\lambda_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\lambda_n}\right)\right).$$

Proof. After some transforms we get:

$$V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)=\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\phi_{xy}(2s,2t)K_{mn}^{\lambda,\mu}(s,t)dsdt, \tag{3.9}$$

where

$$K_{mn}^{\lambda,\mu}(s,t):=\frac{1}{\lambda_m\mu_n}\sum_{k=m-\lambda_m}^{m-1}\sum_{\ell=n-\mu_n}^{n-1}\frac{\sin(2k+1)s}{\sin s}\frac{\sin(2\ell+1)t}{\sin t}.$$

Without difficulty the quantity $K_{mn}^{\lambda,\mu}(s,t)$ can be written as

$$K_{mn}^{\lambda,\mu}(s,t)=\frac{\sin(\lambda_m s)\sin[(2m-\lambda_m)s]\sin(\mu_n t)\sin[(2n-\mu_n)t]}{\lambda_m\mu_n\sin^2 s\sin^2 t}.$$

Therefore, we have:

$$\begin{aligned}|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)|&\le\left(\frac{4}{\pi}\right)^2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\omega_1(f,s,t)|K_{mn}^{\lambda,\mu}(s,t)|dsdt\\&=\mathcal{O}\left(\int_0^{\frac{\pi}{2\lambda_m}}\int_0^{\frac{\pi}{2\mu_n}}+\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2\mu_n}}+\int_0^{\frac{\pi}{2\lambda_m}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}}+\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}}\right)\\&:=\mathcal{O}\left(\mathbb{S}_1+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\right).\end{aligned} \tag{3.10}$$

Using Jordan's inequality $\sin\nu \geq \frac{2}{\pi}\nu$ for $0 < \nu \leq \frac{\pi}{2}$, given assumptions, and Lemma 2.1, we obtain:

$$\begin{aligned} \mathbb{S}_1 &= \mathcal{O}(1)\int_0^{\frac{\pi}{2\lambda_m}}\int_0^{\frac{\pi}{2\mu_n}} s^{-1}t^{-1}\omega_1(f,s,t)dsdt \qquad (3.11)\\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right). \end{aligned}$$

Using the same arguments and Lemma 2.2, we also obtain:

$$\begin{aligned} \mathbb{S}_2 &= \mathcal{O}(1)\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2\mu_n}} s^{-2}t^{-1}\omega_1(f,s,t)dsdt \qquad (3.12)\\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right). \end{aligned}$$

With very similar reasoning, we get:

$$\begin{aligned} \mathbb{S}_3 &= \mathcal{O}(1)\int_0^{\frac{\pi}{2\lambda_m}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}} s^{-1}t^{-2}\omega_1(f,s,t)dsdt \qquad (3.13)\\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right). \end{aligned}$$

Finally, based on given assumptions, and Lemma 2.2 twice, we have:

$$\begin{aligned} \mathbb{S}_4 &= \mathcal{O}(1)\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}} s^{-2}t^{-2}\omega_1(f,s,t)dsdt \qquad (3.14)\\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right). \end{aligned}$$

Subsequently, inserting (3.11), (3.12),(3.13), and (3.14) into (3.9), the requested estimation follows.
The proof is completed. □

Specifying functions $F_i(z)$, $(i=1,2)$, by:

$$F_i(z) = \begin{cases} z^{\gamma_i-1}, & 0<\gamma_i<1;\\ \log\left(\frac{\pi}{z}\right), & \gamma_i = 1\end{cases}$$

then Theorem 3.6 implies:

Corollary 3.7. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$. Then*

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y) - f(x,y)\right| = \begin{cases} \mathcal{O}\left(\frac{1}{\lambda_m^{\gamma_1}\mu_n^{\gamma_2}}\right), & 0<\gamma_1,\gamma_2<1;\\ \mathcal{O}\left(\frac{\log(2\mu_n)}{\lambda_m^{\gamma_1}\mu_n}\right), & 0<\gamma_1<1, \gamma_2=1;\\ \mathcal{O}\left(\frac{\log(2\lambda_m)}{\lambda_m\mu_n^{\gamma_2}}\right), & \gamma_1 = 1, 0<\gamma_2<1;\\ \mathcal{O}\left(\frac{\log(2\lambda_m)\log(2\mu_n)}{\lambda_m\mu_n}\right), & \gamma_1=\gamma_2=1\end{cases}$$

In particular case, it is clear that $V^{m,n}_{m+1,n+1}(f;x,y) \equiv \sigma_{m,n}(f;x,y)$, which is the double Fejèr mean of the sequence $(s_{k,\ell}(x,y))$. Thus, Theorem 3.6 also implies:

Corollary 3.8. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$. Then*

$$\max_{(x,y)\in Q} |\sigma_{m,n}(f;x,y) - f(x,y)| = \begin{cases} \mathcal{O}\left(\frac{1}{(m+1)^{\gamma_1}(n+1)^{\gamma_2}}\right), & 0<\gamma_1,\gamma_2<1; \\ \mathcal{O}\left(\frac{\log(2(n+1))}{(m+1)^{\gamma_1}(n+1)}\right), & 0<\gamma_1<1, \gamma_2=1; \\ \mathcal{O}\left(\frac{\log(2(m+1))}{(m+1)(n+1)^{\gamma_2}}\right), & \gamma_1=1, 0<\gamma_2<1; \\ \mathcal{O}\left(\frac{\log(2(m+1))\log(2(n+1))}{(m+1)(n+1)}\right), & \gamma_1=\gamma_2=1 \end{cases}$$

Let $a := (a_n)$, $b := (b_n)$, $c := (c_n)$, and $d := (d_n)$ be sequences of non-negative integers with conditions

$$1 \le b_m - a_m + \lambda_m, \quad 1 \le d_n - c_n + \mu_n, \quad (m,n = 1,2,\dots). \tag{3.15}$$

The mean $V^{\lambda,\mu}_{m,n}(f;x,y)$ can be generalized further by

$$V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) = \frac{1}{\lambda_m\mu_n}\sum_{k=a_m-\lambda_m}^{b_m-1}\sum_{k=c_n-\mu_n}^{d_n-1} s_{k,\ell}(x,y), \quad (m,n\ge 1), \tag{3.16}$$

is called the (mn)-th double deferred generalized de la Vallée Poussin mean of the sequence $(s_{k,\ell}(x,y))$ generated by sequences (λ_m) and (μ_n).

Remark 3.9. Note that for $a_m = b_m = m$ and $c_n = d_n = n$, for all $m,n \ge 1$, we obtain

$$V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) \equiv V^{\lambda,\mu}_{m,n}(f;x,y),$$

and

$$V^{m,n}_{m+1,n+1}(a,b,c,d;f;x,y) \equiv \sigma_{m,n}(f;x,y).$$

The mean $V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y)$ given by (3.16) can be used to prove the following general theorem.

Theorem 3.10. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$, and $F_1(s), F_2(t) \ge 0$ two mediate functions. Then*

$$\begin{aligned} &\max_{(x,y)\in Q} \left|V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) - f(x,y)\right| \\ &\qquad = \mathcal{O}\Bigg(\frac{1}{(b_m-a_m+\lambda_m)(d_n-c_n+\mu_n)} \\ &\qquad\quad \times F_1\left(\frac{\pi}{2(b_m-a_m+\lambda_m)}\right)F_2\left(\frac{\pi}{2(d_n-c_n+\mu_n)}\right)\Bigg). \end{aligned}$$

Proof. Because of the similarity with the proof of Theorem 3.6 we omit the proof of this theorem. ∎

Remark 3.11. One should note that Theorem 3.6 is a particular case of Theorem 3.10 (when $a_m = b_m$ and $c_n = d_n$; $\forall m, n \geq 1$). Moreover, it covers Corollary 3.7 and Corollary 3.8 as well (when $a_m = b_m$, $c_n = d_n$, $\lambda_m = m$, and $\mu_n = n$; $\forall m, n \geq 1$).

Further, let $a := (a_m)$, $b := (b_m)$, $c := (c_n)$, and $d := (d_n)$ be sequences of non-negative integers with conditions

$$a_m < b_m, \quad c_n < d_n, \quad (m, n = 1, 2, \dots), \tag{3.17}$$

and

$$\lim_{m\to\infty} b_m = +\infty, \quad \lim_{n\to\infty} d_n = +\infty. \tag{3.18}$$

If $\lambda_m = 1$ and $\mu_n = 1$ for all $m, n \geq 1$, then the double deferred de la Vallée Poussin mean $V_{m,n}^{\lambda,\mu}(a+2, b+1, c+2, d+1; f; x, y)$ reduces to

$$D_{a,c}^{b,d}(f; x, y) := \frac{1}{(b_m - a_m)(d_n - c_n)} \sum_{k=a_m+1}^{b_m} \sum_{\ell=c_n+1}^{d_n} s_{k,\ell}(f; x, y),$$

which is the double deferred Cesàro mean of the sum $s_{k,\ell}(f; x, y)$ introduced implicitly in [13]. It was shown there, that (3.17) and (3.18) are conditions of regularity for $D_{a,c}^{b,d}$. Therefore, if conditions (3.17) and (3.18)) are satisfied, then Theorem 3.10 implies the following.

Corollary 3.12. *Let $f \in C([-\pi, \pi]^2)$, $\omega_1(f, s, t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s, t \to +0$, and $F_1(s), F_2(t) \geq 0$ two mediate functions. Then*

$$\max_{(x,y)\in Q} \left|D_{a,c}^{b,d}(f; x, y) - f(x, y)\right|$$
$$= \mathcal{O}\left(\frac{1}{(b_m - a_m)(d_n - c_n)} F_1\left(\frac{\pi}{2(b_m - a_m)}\right) F_2\left(\frac{\pi}{2(d_n - c_n)}\right)\right).$$

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