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Analysis of quasistatic viscoelastic viscoplastic piezoelectric contact problem with friction and adhesion

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Abstract. In this paper we study the process of bilateral contact with adhesion and friction between a piezoelectric body and an insulator obstacle, the so-called foundation. The material's behavior is assumed to be electro-viscoelastic-viscoplastic; the process is quasistatic, the contact is modeled by a general nonlocal friction law with adhesion. The adhesion process is modeled by a bonding field on the contact surface. We derive a variational formulation for the problem and then, under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution to the model.The proofs are based on a general results on elliptic variational inequalities and fixed point arguments.

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1. Introduction

A piezoelectric body is one that produces an electric charge when a mechanical stress is applied (the body is squeezed or stretched). Conversely, a mechanical deformation (the body shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials and those for which the mechanical properties are viscoplastic are also called electro-viscoplastic materials. Therfore, a viscoelastic-viscoplastic piezoelectric contact problems are considered. Different models have been developed to describe the

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interaction between the electrical and mechanical fields (see, e.g. [2, 14, 18] and the references therein). A static frictional contact problem for electric-elastic material was considered in [3], under the assumption that the foundation is insulated. Electro-elastic-visco-plastic and elastic-visco-plastic contact problems were recently studied in [13, 15].

Adhesion may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesive contact is modeled by a bonding field on the contact surface, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [11], [12], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. Basic modelling can be found in [11, 12]. Analysis of models for adhesive contact can be found in [7, 4, 6].

In this work we continue in this line of research, where we extend the result established in [8]. The novelty here lies in the fact that we consider a viscoelastic-viscoplastic piezoelectric body, the contact is bilateral and the friction is described by a nonlocal version of Coulomb's law of dry friction with adhesion. A similar boundary conditions are used in [20], where the constitutive law of the material is viscoelastic.

This paper is structured as follows. In Section 2 we present the viscoelastic-viscoplaastic piezoelectric contact model with friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness result, Theorem (4.1), which states the unique weak solvability of the contact problem under a smallness assumption on the coefficient of friction.

2. The model

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν. The body is acted upon by body forces of density f_0 and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\ \Gamma_1 > 0$ and $meas\ \Gamma_a > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in adhesive and frictional contact with an insulator obstacle, the so-called foundation.

We are interested in the deformation of the body on the time interval $[0, T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on x or t. Here and everywhere in this paper, i, j, k, $l = 1, ..., d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of x. The dot above variable represents the time derivatives.

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ and by ".", $\|.\|$ the inner product and the norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $u.v = u_i v_i$, $\|v\| = (v.v)^{1/2}$ for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$, and $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\sigma\| = (\sigma.\sigma)^{1/2}$ for $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$. With these assumptions, the classical model for the process is the following.

Problem ($\mathcal{P}$). Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0, T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Omega \times [0, T] \to \mathbb{R}$ *such that*

$$\begin{aligned} \sigma(x,t) = \mathcal{A}\varepsilon(\dot{u}(x,t)) + \mathcal{F}\varepsilon(u(x,t)) \\ + \int_0^t \mathcal{G}(\sigma(x,s), \varepsilon(u(x,s))ds - \mathcal{E}^*\mathbf{E}(\varphi(x,t)) \end{aligned} \qquad in\ \Omega \times (0,T), \tag{2.1}$$

$$D = \mathcal{B}\mathbf{E}(\varphi) + \mathcal{E}\varepsilon(u) \qquad in\ \Omega \times (0,T), \tag{2.2}$$

$$Div\sigma + f_0 = 0 \qquad in\ \Omega \times (0,T), \tag{2.3}$$

$$divD = q_0 \qquad in\ \Omega \times (0,T), \tag{2.4}$$

$$u = 0 \qquad on\ \Gamma_1 \times (0,T), \tag{2.5}$$

$$\sigma\nu = f_2 \qquad on\ \Gamma_2 \times (0,T), \tag{2.6}$$

$$u_\nu = 0, \qquad on\ \Gamma_3 \times (0,T), \tag{2.7}$$

$$\left\{ \begin{array}{l} \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| \leq \mu p(|R\sigma_\nu|), \\ \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| < \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \dot{u}_\tau = 0, \\ \bullet\ \|\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)\| = \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \exists\ \lambda > 0, \text{such that:} \\ \quad \sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = -\lambda\ \dot{u}_\tau, \end{array} \right. \qquad on\ \Gamma_3 \times (0,T), \tag{2.8}$$

$$\dot{\beta}(t) = -(\beta(t)\gamma_\tau\|R_\tau(u_\tau(t))\|^2 - \varepsilon_a)_+ \qquad on\ \Gamma_3 \times (0,T), \tag{2.9}$$

$$\varphi = 0 \qquad on\ \Gamma_a \times (0,T), \tag{2.10}$$

$$D.\nu = q_2 \qquad on\ \Gamma_b \times (0,T), \tag{2.11}$$

$$D.\nu = 0 \qquad on\ \Gamma_3 \times (0,T), \tag{2.12}$$

$$u(0) = u_0 \qquad in\ \Omega, \tag{2.13}$$

$$\beta(0) = \beta_0 \qquad on\ \Gamma_3. \tag{2.14}$$

Equations (2.1) and (2.2) represent the electro-viscoelastic-viscoplastic constitutive law of the material in which $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon(u) = (\varepsilon_{ij}(u))$ denotes the linearized strain tensor, $\mathcal{A}$ and $\mathcal{F}$ are the elasticity and viscosity tensors, respectivelly, $\mathcal{G}$ denotes a viscoplastic function, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$ is its transpose such that:

$$\mathcal{E}\sigma.\upsilon = \sigma.\mathcal{E}^*\upsilon \quad \forall\sigma \in \mathbb{S}^d,\ \upsilon \in \mathbb{R}^d, \tag{2.15}$$

$D = (D_1, ..., D_d)$ is the electric displacement vector and $\mathcal{B} =(\mathcal{B}_{ij})$ denotes the electric permittivity tensor. Equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "*Div*" and "*div*" denote the divergence operators for tensor and vector valued functions, respectively. Conditions (2.5) and (2.6) are the displacement and traction boundary conditions in which $\sigma\nu$ represents the Cauchy stress vector, whereas (2.10) and (2.11) represent the electric boundary conditions. Note that we need to impose assumption (2.12) for physical reasons. Indeed, this condition models the case when the obstacle is a perfect insulator and was used in [3, 9]. Condition (2.7) represents the bilateral contact, where u_ν represents the normal displacement. Conditions (2.8) is a non local *Coulomb's* law of friction coupled with adhesion in which μ denotes the coefficient of friction and γ_τ is a given adhesion coefficients, u_τ and σ_τ are tangential components of vector u and tensor σ, respectively, σ_ν represents the normal stress, $\dot{u}_\tau$ is the tangential velocity on the bondary, the operator $R : H^{-\frac{1}{2}} \to L^2(\Gamma)$ (see e.eg. [10]) is a linear continuous operator used to regularize the normal trace of stress which is too rough on Γ, p is a non-negative function, the so-called friction bound, and R_τ is the truncation operator defined by

$$R_\tau(\upsilon) = \begin{cases} \upsilon & \text{if } \|\upsilon\| \le L, \\ L\dfrac{\upsilon}{\|\upsilon\|} & \text{if } \|\upsilon\| > L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see e.eg. [19]). The evolution of the bonding field is governed by the differential equation (2.9) with given positive adhesion coefficients γ_τ and ε_a where $r_+ = \max\{0, r\}$. Finally, (2.13) and (2.14) represent the initial conditions in which u_0 and β_0 are the prescribed initial displacement and bonding fields, respectively.

3. Preliminaries and variational formulation

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminaries. We use the notation H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces

$$H = \{\upsilon = (\upsilon_i)\ |\upsilon_i \in L^2(\Omega), i = \overline{1, d}\}, \quad H_1 = \{\upsilon = (\upsilon_i)\ |\ \varepsilon(\upsilon) \in \mathcal{H}\},$$
$$\mathcal{H} = \{\tau = (\tau_{ij})\ |\tau_{ij} = \tau_{ji} \in L^2(\Omega), i, j = \overline{1, d}\},\ \mathcal{H}_1 = \{\tau \in \mathcal{H}\ |Div\tau \in H\}.$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real *Hilbert* spaces endowed with the canonical inner products given by

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v)_{\mathcal{H}},.$$
$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H,$$

such that $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are the deformation and divergence operators, respectively defined by

$$\begin{array}{l} \varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall v \in H_1, \\ Div\,\tau = (\tau_{ij,j}) \quad \forall \tau \in \mathcal{H}_1. \end{array}$$

and the associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. We recall that for every element $v \in H_1$ we denote by v the trace γv of v on Γ. If $\sigma \in C^1(\overline{\Omega})^{\mathbb{N}\times\mathbb{N}}$ then, the following *Green's* formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu \cdot v \, da, \quad \forall v \in H_1. \tag{3.1}$$

For every real *Hilbert* space X we employ the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [0,\infty]$, $k = 1, 2, ...$

We now list the assumptions on the problem's data.

$$\left\{ \begin{array}{ll} \text{(a)} & \mathcal{A} = (a_{ijkl}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ such that} \\ & \mathcal{A}(x,\tau) = (a_{ijkl}(x)\tau_{kl}) \ \forall\ \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \\ \text{(b)} & a_{ijkl} = a_{jikl} = a_{klij} \in L^\infty(\Omega), 1 \le i,j,k,l \le d. \\ \text{(c)} & \text{there exists } m_{\mathcal{A}} > 0 \text{ such that:} \\ & a_{ijkl}\tau_{ij}\tau_{kl} \ge m_{\mathcal{A}}||\tau||^2 \ \ \forall\ \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.2}$$

$$\left\{ \begin{array}{ll} \text{(a)} & \mathcal{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ such that:} \\ & \mathcal{F}(x,\tau) = (f_{ijkl}(x)\tau_{kl}) \ \forall\ \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \\ \text{(b)} & f_{ijkl} = f_{jikl} = f_{klij} \in L^\infty(\Omega), 1 \le i,j,k,l \le d. \\ \text{(c)} & \text{there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & f_{ijkl}\tau_{ij}\tau_{kl} \ge m_{\mathcal{F}}||\tau||^2 \ \ \forall\ \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.3}$$

$$\left\{ \begin{array}{ll} \text{(a)} & \mathcal{E} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}^d \text{ such that:} \\ & \mathcal{E}(x,\varepsilon) = (e_{ijk}(x)\varepsilon_{jk}) \quad \forall \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega, \\ \text{(b)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{ll} \text{(a)} & \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \text{ such that:} \\ & \mathcal{B}(x,\mathbf{E}) = (\mathcal{B}_{ij}(x)E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \mathbf{x} \in \Omega, \\ \text{(c)} & \mathcal{B}_{ij} = \mathcal{B}_{ji} \in L^\infty(\Omega), \\ \text{(d)} & \text{there exists } m_{\mathcal{B}} > 0 \text{ such that } \mathcal{B}_{ij}(x)E_iE_j \ge m_{\mathcal{B}}\|\mathbf{E}\|^2 \\ & \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.5}$$

$$\begin{cases} \text{(a)} & p:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+. \\ \text{(b)} & \text{there exists } L_p>0 \text{ such that} \\ & |p(x,r_1)-p(x,r_2)|\le L_p|r_1-r_2|, \\ & \forall r_1,\ r_2\in\mathbb{R},\ \text{a.e. } x\in\Gamma_3, \\ \text{(c)} & x\longmapsto p(x,r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \text{(d)} & \text{the mapping } x\longmapsto p(x,0)\in L^2(\Gamma_3). \end{cases} \tag{3.6}$$

$$\begin{cases} \text{(a)} & \mathcal{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\longrightarrow\mathbb{S}^d \\ \text{(b)} & \text{there exists } L_{\mathcal{G}}>0 \text{ such that} \\ & \|\mathcal{G}(x,\sigma_1,\varepsilon_1)-\mathcal{G}(x,\sigma_2,\varepsilon_2)\|\le L_{\mathcal{G}}\|\sigma_1-\sigma_2\| \\ & \forall\ \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ \text{a.e. } x\in\Omega, \\ \text{(b)} & \text{for any } \sigma,\varepsilon\in\mathbb{S}^d,\ x\longmapsto\mathcal{G}(x,\sigma,\varepsilon) \text{ is measurable}, \\ \text{(c)} & \text{the mapping } x\longmapsto\mathcal{G}(x,0,0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.7}$$

The forces, tractions, volume and surface free charge densities satisfy

$$f_0\in W^{1,2}(0,T;H),\quad f_2\in W^{1,2}(0,T;L^2(\Gamma_2)^d), \tag{3.8}$$

$$q_0\in W^{1,2}(0,T;L^2(\Omega)),\quad q_2\in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{3.9}$$

The adhesion coefficient γ_τ and the limit bound ε_a satisfy the conditions

$$\gamma_\tau\in L^\infty(\Gamma_3),\quad \epsilon_a\in L^2(\Gamma_3),\quad \gamma_\tau,\ \epsilon_a\ge 0\quad \text{a.e. on } \Gamma_3. \tag{3.10}$$

Also, we assume that the initial bonding field satisfies the condition

$$\beta_0\in L^2(\Gamma_3),\ 0\le\beta_0\le 1\ \ \text{a.e. on } \Gamma_3, \tag{3.11}$$

Finally, the coefficient of friction μ is assumed to satisfy

$$\mu\in L^\infty(\Gamma_3),\quad \mu(x)\ge 0\quad \text{a.e. on } \Gamma_3. \tag{3.12}$$

Let now consider the closed subspace of H_1 defined by

$$V=\{\ v\in H_1\mid v=0 \text{ on } \Gamma_1,\ v_\nu=0 \text{ on } \Gamma_3\}. \tag{3.13}$$

Since $meas\,(\Gamma_1)>0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}}\ge C_K\|v\|_{H_1}\quad \forall v\in V, \tag{3.14}$$

where the proof my be found in [16] (p. 79). Equiping V with the inner product

$$(u,v)_V=(\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \tag{3.15}$$

and let $\|\cdot\|_V$ be the associated norm. We deduce from *Korn's* inequality that $\|.\|_{H_1}$ and $\|.\|_V$ are eauivalente norme on V. Then $(V,\|.\|_V)$ is a real *Hilbert* space. Next, we assume that the initial displacement satisfies the condition

$$u_0\in\ V. \tag{3.16}$$

We also introduce the following spaces

$$W=\{\ \psi\in H^1(\Omega)\mid \psi=0 \text{ on } \Gamma_a\ \}, \tag{3.17}$$

$$\mathcal{W}=\{\ D=(D_i)\mid D_i\in L^2(\Omega),\ div\ D\in L^2(\Omega)\}. \tag{3.18}$$

Since $meas\,(\Gamma_a)>0$ it is well known that W is a real *Hilbert* space endowed with the inner product

$$(\varphi,\psi)_W=(\nabla\varphi,\nabla\psi)_{L^2(\Omega)^d}, \tag{3.19}$$

and the associated norm is $\| \cdot \|_W$. Also we have the following *Friedrichs-Poincaré* inequality

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq C_F \, \|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in W, \tag{3.20}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. The space $\mathcal{W}$ is a real *Hilbert* space endowed with the inner product

$$(D, \mathbf{E})_{\mathcal{W}} = \int_\Omega D \cdot \mathbf{E}\, dx + \int_\Omega divD \cdot div\mathbf{E}\, dx,$$

and the associated norm is $\|\cdot\|_{\mathcal{W}}$. Moreover, by the *Sobolev* trace theorem, there exist two positive constants C_0 and $\tilde{C}_0$ depending only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0 \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq \tilde{C}_0 \|\psi\|_W \quad \forall\, \psi \in W. \tag{3.21}$$

It follows from proprieties of R that there existe a constant C_R depending only on Ω, Γ_3 and R such that

$$\|R\sigma_\nu\|_{L^2(\Gamma_3)} \leq C_R \|\sigma_\nu\|_{\mathcal{H}_1} \quad \forall \sigma \in \mathcal{H}_1. \tag{3.22}$$

Next, we define the two mappings $f : [0,T] \longrightarrow V$ and $q : [0,T] \longrightarrow W$, respectively, by

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v\, dx + \int_{\Gamma_2} f_2(t) \cdot v\, da, \tag{3.23}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi\, dx - \int_{\Gamma_b} q_2(t) \psi\, da, \tag{3.24}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. We note that the definitions of f and q are based on the *Riesz* representation theorem. Moreover, it follows from assumptions (3.8) and (3.9) that

$$f \in W^{1,2}(0,T;V), \tag{3.25}$$

$$q \in W^{1,2}(0,T;W). \tag{3.26}$$

Also, we introduce the set

$$\mathcal{Q} = \{\beta \in L^\infty(0,T;L^2(\Gamma_3)) \;/\; 0 \leq \beta(t) \leq 1 \;\forall\, t \in [0,T],\; a.e.\ on\ \Gamma_3\}. \tag{3.27}$$

Now, let us define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ and the friction functional $j_{fr} : \mathcal{H}_1 \times V \longrightarrow \mathbb{R}$, respectivelly, by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau da, \tag{3.28}$$

$$j_{fr}(\sigma, v) = \int_{\Gamma_3} \mu p(|R\sigma_\nu|) \cdot \|v_\tau\| da. \tag{3.29}$$

Using a standard procedure based on *Green's formulas* (see (3.1)) we can derive the following variational formulation of the problem (2.1)–(2.14).

Problem ($\mathcal{P}^V$). Find a displacement field $u : [0,T] \to V$, a stress field $\sigma : \Omega \times [0,T] \to \mathcal{H}$, an electric potential $\varphi : [0,T] \to W$, *and a bonding field* $\beta : [0,T] \to L^2(\Gamma_3)$ *such that*

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{F}\varepsilon(u(t)) + \int_0^t \mathcal{G}(\sigma(x,s), \varepsilon(u(x,s))ds - \mathcal{E}^*\boldsymbol{E}(\varphi(t)) \tag{3.30}$$

$$(\sigma(t), \varepsilon(\omega) - \varepsilon(\dot{u}(t))_{\mathcal{H}} + j_{ad}(\beta(t), u(t), \omega - \dot{u}(t)) \tag{3.31}$$
$$+j_{fr}(\sigma(t), \omega) - j_{fr}(\sigma(t), \dot{u}(t)) \geq (f(t), \omega - \dot{u}(t))_V,$$
$$\forall v \in V, \quad \forall t \in [0,T],$$

$$(\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t), \psi)_W, \tag{3.32}$$
$$\forall \psi \in W, \ \forall t \in [0,T],$$

$$\dot{\beta}(t) = -(\gamma_\tau \beta(t) \left\| R_\tau(u_\tau(t)) \right\|^2) - \epsilon_a)_+ \ , \ \text{a.e. } t \in (0,T), \tag{3.33}$$

$$u(0) = u_0 \tag{3.34}$$

$$\beta(0) = \beta_0. \tag{3.35}$$

4. Existence and uniqueness result

Theorem 4.1. *Assume that* (3.2)*–*(3.12) *and* (3.16) *hold. Then, there exists a constant* $\mu_0 > 0$ *such that Problem* $\mathcal{P}^V$ *has a unique solution* $(u, \sigma, \varphi, \beta)$ *if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. *Moreover, the solution satisfies*

$$\boldsymbol{u} \ \in W^{2,2}(0,T;V), \tag{4.1}$$

$$\sigma \ \in W^{1,2}(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\varphi \ \in W^{1,2}(0,T;W). \tag{4.3}$$

$$\beta \ \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.4}$$

A quintuple of functions $(u,\ \sigma,\ \varphi,\ D,\ \beta)$ which satisfies (2.1), (2.2) and (3.30), (3.35) is called a *weak solution* of the contact Problem ($\mathcal{P}$). We conclude by Theorem (4.1) that, under the assumptions (3.2)–(3.12) and (3.16), there exists a unique weak solution of Problem ($\mathcal{P}$). To precise the regularity of the weak solution we note that the constitutive relations (2.2), the assumptions (3.4)–(3.5) and the regularity (4.3) implies that $\boldsymbol{D} \in W^{1,2}(0,T;L^2(\Omega)^d)$. Moreover, using again (2.2) combined with (3.32) and the notation (3.24) and choosing $\psi \in C_0^\infty(\Omega)$ we find that *div* $D(t) = q_0(t)$ for all $t \in [0,T]$. It follows now from the regularities (3.9) that $divD \in W^{1,2}(0,T;L^2(\Omega))$, which shows that

$$D \in W^{1,2}(0,T;\mathcal{W}). \tag{4.5}$$

We conclude that the weak solution $(u, \sigma, \varphi, D, \beta)$ of the piezoelectric contact problem ($\mathcal{P}$) has the regularity (4.1)–(4.5).

The proof of Theorem(4.1) will be carried out in several steps. We assume in the following that the conditions, (3.2)–(3.12) and (3.16), of Theorem(4.1) hold and below we denote by "c" a generic positive constant which is independent of time and whose value may change from place to place. In the first step, let $\eta \in W^{1,2}(0,T;V)$,

$\kappa \in L^2(0,T;\mathcal{H})$ and $\lambda \in W^{1,2}(0,T;\mathcal{H}_1)$ be a given functions. We introduce the function $z_\kappa \in W^{1,2}(0,T;\mathcal{H})$ defined by

$$z_\kappa(t) = \int_0^t \kappa(s)ds \quad \forall t \in [0,T], \tag{4.6}$$

and we consider the following intermediate problem.

Problem ($\mathcal{P}_1^V$). Find $u_{\kappa\eta\lambda} : [0,T] \to V$ and $\sigma_{\kappa\eta\lambda} : [0,T] \to \mathcal{H}_1$ *such that*

$$\sigma_{\kappa\eta\lambda}(t) = \mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)) + \mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)) + z_\kappa(t) + \varepsilon(\eta(t)). \tag{4.7}$$

$$\begin{aligned}(\mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} \\ +(z_\kappa(t), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + (\varepsilon(\eta(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} \\ +j_{fr}(\lambda(t), \omega) - j_{fr}(\lambda(t), \dot{u}_{\kappa\eta\lambda}(t)) \geq (f(t), \omega - \dot{u}_{\kappa\eta\lambda}(t))_V \\ \forall\, \omega \in V, \quad \forall t \in [0,T],\end{aligned} \tag{4.8}$$

$$u_{\kappa\eta\lambda}(0) = u_0. \tag{4.9}$$

Lemma 4.1. *Problem $\mathcal{P}_1^V$ has a unique solution $(u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$. Moreover, the solution satisfies*

$$\begin{aligned}&\mathbf{a})u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V), \\ &\mathbf{b})\sigma_{\kappa\eta\lambda} \in W^{1,2}(0,T;\mathcal{H}_1), \\ &\mathbf{c})Div\sigma_{\kappa\eta\lambda} + f_0 = 0.\end{aligned} \tag{4.10}$$

Proof. We denote by $\tilde{\sigma}_{\kappa\eta\lambda}$ and j_λ the elements given by

$$\tilde{\sigma}_{\kappa\eta\lambda}(t) \quad = \sigma_{\kappa\eta\lambda}(t) - z_\kappa(t) - \varepsilon(\eta(t)). \tag{4.11}$$

$$j_\lambda(\omega) \quad = j_{fr}(\lambda, \omega) \quad \forall \omega \in V. \tag{4.12}$$

By (3.15) and Riesz's representation theorem we deduce that there exists an element $f_{\kappa\eta} \in W^{1,2}(0,T;V)$ such that

$$(f_{\kappa\eta}(t), \upsilon)_V = (f(t) - \eta(t), \upsilon)_V + (z_\kappa(t), \varepsilon(\upsilon))_{\mathcal{H}}. \tag{4.13}$$

Since $f, \eta \in W^{1,2}(0,T;V)$ and $z_\kappa \in W^{1,2}(0,T;\mathcal{H})$ we deduce that $f_{\kappa\eta} \in W^{1,2}(0,T;V)$. Moreover, using (4.7), (4.8), (4.9), (4.11) and (4.12) leads us to consider the following variational problem.

Problem ($\mathcal{P}_2^V$). Find $u_{\kappa\eta\lambda} : [0,T] \to V$ and $\tilde{\sigma}_{\kappa\eta\lambda} : [0,T] \to \mathcal{H}_1$ *such that*

$$\tilde{\sigma}_{\kappa\eta\lambda}(t) = \mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)) + \mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)). \tag{4.14}$$

$$\begin{aligned}(\tilde{\sigma}_{\kappa\eta\lambda}(t), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + j_\lambda(\omega) - j_\lambda(\dot{u}_{\kappa\eta\lambda}(t)) \\ \geq (f_{\kappa\eta}(t), \omega - \dot{u}_{\kappa\eta\lambda}(t))_V \quad \forall\, \omega \in V, \quad \forall t \in [0,T],\end{aligned}$$

$$u_{\kappa\eta\lambda}(0) = u_0, \tag{4.15}$$

Note that V is a closed subspace of H_1 and the fonctional j_λ is convex lower semicontinuous on V such that $j \neq +\infty$. By a classical results for elliptic variational inequalities (see e.g. [5], Theorem (4.1) page 348) there exists a unique solution $(u_{\kappa\eta\lambda}, \tilde{\sigma}_{\kappa\eta\lambda})$ for the variational problem $\mathcal{P}_2^V$ stisfying the regularity condition

$$u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V), \ \tilde{\sigma}_{\kappa\eta\lambda} \in W^{1,2}(0,T;\mathcal{H}_1). \tag{4.16}$$

Next, kepping in mind (4.7) we put $\omega = \dot{u}_{\kappa\eta\lambda}(t) \pm v$ where $v \in \mathcal{D}(\Omega)^d$ in (4.8) to obtain $Div\sigma_{\kappa\eta\lambda} + f_0 = 0$.

Finally, we deduce that $(u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$ is the unique solution of the variational problem $\mathcal{P}_1^V$ stisfying condition (4.10), which concludes the proof of Lemma (4.1). □

In the second step we use the displacement field $u_{\kappa\eta\lambda}$ obtained in Lemma(4.1) to obtain the following existence and uniqueness result for the electric potential field.

Lemma 4.2. *There exists a unique function $\varphi_{\kappa\eta\lambda} \in W^{1,2}(0,T;W)$ such that*

$$\begin{gathered}(\mathcal{B}\nabla\varphi_{\kappa\eta\lambda}(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{\kappa\eta\lambda}(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \\ \forall\psi \in W, \quad \forall\, t \in [0\ T]\,,\end{gathered} \tag{4.17}$$

Moreover, if φ_1 and φ_2 are the solution of (4.17) for $u_1, u_2 \in W^{2,2}(0,T;V)$, respectivelly, then we have

$$\begin{gathered}\|\varphi_1(t) - \varphi_2(t)\|_W \le c\|u_1(t) - u_2(t)\|_V ds, \\ \forall\, t \in [0,T], \quad a.e.\ on\ \Gamma_3\end{gathered} \tag{4.18}$$

Proof. Let $u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V)(0,T;V)$ be the function defined in Lemma (4.1). As in [1], using *Riesz's* representation theorem we may define the operator $\mathcal{L}_{\kappa\eta\lambda} : W \longrightarrow W$ by

$$\begin{gathered}(\mathcal{L}_{\kappa\eta\lambda}(\varphi(t)), \psi)_W = (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{\eta\lambda}(t)), \nabla\psi)_{L^2(\Omega)^d} \\ \forall\psi \in W,\ \forall\, t \in [0,T].\end{gathered} \tag{4.19}$$

It follows from assumptions (3.4) and (3.5) that the operator $\mathcal{L}_{\kappa\eta\lambda}$ is stongly monotone Lipschitz continuous on W. Then, we deduce that there exists a unique element $\varphi_{\kappa\eta\lambda}(t) \in W$ satisfies,

$$\mathcal{L}_{\kappa\eta\lambda}(\varphi_{\kappa\eta\lambda}(t)) = q(t) \quad \forall t\ \in [0,T]. \tag{4.20}$$

Thus, it follows from (4.19) and (4.20) that $\varphi_{\kappa\eta\lambda}(t) \in W$ is the unique solution of equation (4.17). Let now $t_1, t_2 \in [0,T]$ and for the sake of simplicity we use the notations $\varphi_i = \varphi_{\kappa\eta\lambda}(t_i)$, $u_i = u_{\kappa\eta\lambda}(t_i)$, $q_i = q(t_i)$ for $i = 1,2$. Using (4.17), (3.4) and (3.5) we find that

$$\|\varphi_1 - \varphi_2\|_W \le c(\|u_1 - u_2\|_V + \|q_1 - q_2\|_W),$$

the previous inequality yields

$$\|\varphi_{\kappa\eta\lambda}(t_1) - \varphi_{\kappa\eta\lambda}(t_2)\|_W \le c(\|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{4.21}$$

Since $u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V)$ and $q \in W^{1,2}(0,T;W)$, it follows that

$$\varphi_{\kappa\eta\lambda} \in W^{1,2}(0,T;W).$$

Assume now that φ_1 and φ_2 are the solution of (4.17) for $u_1, u_2 \in W^{2,2}(0,T;V)$, respectively. Arguments similar to those used in proof of (4.21) leads to (4.18), which concludes the proof of Lemma (4.2). □

In the third step, for $u_{\kappa\eta\lambda}$ obtained in Lemma (4.1), we solve equation (3.33) for the adhesion field.

Problem ($\mathcal{P}^{\beta_{\kappa\eta\lambda}}$). Find a bonding field $\beta_{\kappa\eta\lambda} : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_{\kappa\eta\lambda}(t) = -(\gamma_\tau \beta_{\kappa\eta\lambda}(t) \|R_\tau(u_{\kappa\eta\lambda\tau}(t))\|^2 - \varepsilon_a)_+ \text{ a.e. } t \in (0,T), \tag{4.22}$$

$$\beta_{\kappa\eta\lambda}(0) = \beta_0. \tag{4.23}$$

Lemma 4.3. *There exists a unique solution $\beta_{\kappa\eta\lambda}$ to Problem $\mathcal{P}^{\beta_{\kappa\eta\lambda}}$ satisfing $\beta_{\kappa\eta\lambda} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$. Moreover, if β_1 and β_2 are the solution of (4.22)-(4.23) for $u_1, u_2 \in W^{2,2}(0,T;V)$, respectivelly, then we have*

$$\begin{gathered} \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|u_1(s) - u_2(s)\|_V ds, \\ \forall\ t \in [0,T], \quad a.e.\ on\ \Gamma_3 \end{gathered} \tag{4.24}$$

Proof. The proof of Lemma 4.3 is based on a version of *Cauchy-Lipschitz* theorem (see, e.g., [17], page 48), by arguments similar to those used in [7]. □

In the fourth step, for $\eta \in W^{1,2}(0,T;V)$, $\kappa \in L^2(0,T;\mathcal{H})$ and $\lambda \in W^{1,2}(0,T;\mathcal{H}_1)$ we denote by $u_{\kappa\eta\lambda}$, $\varphi_{\kappa\eta\lambda}$ and $\beta_{\kappa\eta\lambda}$ the functions obtained in Lemmas (4.1), (4.2) and (4.3), respectively. We now define the operator $\Lambda_{\kappa\eta} : L^2(0,T;\mathcal{H}_1) \longrightarrow L^2(0,T;\mathcal{H}_1)$ by

$$\Lambda_{\kappa\eta}\lambda = \sigma_{\kappa\eta\lambda}. \tag{4.25}$$

Lemma 4.4. *For all $\lambda \in L^2(0,T;\mathcal{H}_1)$ the function $\Lambda_{\kappa\eta}\lambda$ belongs to $W^{1,2}(0,T;\mathcal{H}_1)$. Moreover, The operator $\Lambda_{\kappa\eta}$ has a unique fixed point $\lambda_{\kappa\eta} \in W^{1,2}(0,T;\mathcal{H}_1)$.*

Proof. Let $t_1, t_2 \in [0,T]$. Keeping in mind (3.2), (3.3), (3.15) and using (4.7) written for $t = t_1$ and $t = t_2$ we find that

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V). \end{aligned} \tag{4.26}$$

On the other hand, we have

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}_1} \leq{}& \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}} \\ &+ \|Div\sigma_{\kappa\eta\lambda}(t_1) - Div\sigma_{\kappa\eta\lambda}(t_2)\|_H, \end{aligned}$$

using (4.10)(c), (4.26) and the previous inequality we obtain

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}_1} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V) \\ &+ \|f_0(t_1) - f_0(t_2)\|_H. \end{aligned} \tag{4.27}$$

Now, we get from (4.25) that

$$\begin{aligned} \|\Lambda_{\kappa\eta}\lambda(t_1) - \Lambda_{\kappa\eta}\lambda(t_2)\|_{\mathcal{H}_1} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V) \\ &+ \|f_0(t_1) - f_0(t_2)\|_H. \end{aligned} \tag{4.28}$$

Since

$$\dot{u}_{\kappa\eta\lambda} \in W^{1,2}(0,T;V),\ u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V),\ z_\kappa \in W^{1,2}(0,T;\mathcal{H}),\ \eta \in W^{1,2}(0,T;V)$$

and $f_0 \in W^{1,2}(0,T;H)$, it follows that

$$\Lambda_{\kappa\eta}\lambda \in W^{1,2}(0,T;\mathcal{H}_1) \tag{4.29}$$

Let now $\lambda_1, \lambda_2 \in L^2(0,T;\mathcal{H}_1)$ and let $t \in [0,T]$. We use the notation $u_i = u_{\kappa\eta\lambda_i}$, $\sigma_i = \sigma_{\kappa\eta\lambda_i}$ $\dot{u}_i = \dot{u}_{\kappa\eta\lambda_i}$ for $i = 1,2$. In (4.8) written for $\lambda = \lambda_1$, we take $\omega = \dot{u}_2$, and also written for $\lambda = \lambda_2$, we take $\omega = \dot{u}_1$. After adding the resulting inequalities and using (3.2), (3.3), (3.6), (3.12), (3.15), (3.21), (3.22), (3.29) with some elementary calculus we find that

$$\begin{aligned}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le \frac{L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)}}{m_{\mathcal{A}}} \|\lambda_1(t) - \lambda_2(t)\|_{\mathcal{H}_1} \\ + \frac{C_{\mathcal{F}}}{m_{\mathcal{A}}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds,\end{aligned} \tag{4.30}$$

and, after a Gronwall argument, we obtain

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le \frac{L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)}}{m_{\mathcal{A}}} \|\lambda_1(t) - \lambda_2(t)\|_{\mathcal{H}_1}. \tag{4.31}$$

Next, from (4.10)(c) we have $Div\sigma_1(t) = Div\sigma_2(t)$. Moreover, using (4.7), (3.2), (3.3), (3.15) and (3.21) we obtain

$$\begin{aligned}\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \le c(\|\dot{u}_1(t) - \dot{u}_2(t\|_V \\ + \|u_1(t) - u_2(t)\|_V).\end{aligned} \tag{4.32}$$

Now, using using (4.32) and Young's inequality we obtain

$$\|\sigma_1(t) - \sigma_2(t)\|^2_{\mathcal{H}_1} \le c(\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_V + \|u_1(t) - u_2(t)\|^2_V),$$

where, we deduce by using (4.25) that

$$\begin{aligned}\|\Lambda_{\kappa\eta}\lambda_1(t) - \Lambda_{\kappa\eta}\lambda_2(t)\|^2_{\mathcal{H}_1} \le c(\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_V \\ + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|^2_V ds).\end{aligned} \tag{4.33}$$

We combine now (4.31) and (4.33) to obtain

$$\|\Lambda_{\kappa\eta}\lambda_1(t) - \Lambda_{\kappa\eta}\lambda_2(t)\|^2_{\mathcal{H}_1} \le c(\|\lambda_1(t) - \lambda_2(t)\|^2_{\mathcal{H}_1} + \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{\mathcal{H}_1} ds),$$

and, reiterating this inequality m times, yields

$$\|\Lambda^m_{\kappa\eta}\lambda_1 - \Lambda^m_{\kappa\eta}\lambda_2\|^2_{L^2(0,T;\mathcal{H}_1)} \le \frac{c^m(m+T)^m}{m!} \|\lambda_1 - \lambda_2\|^2_{L^2(0,T;\mathcal{H}_1)},$$

which implies that for m sufficiently large, $\Lambda^m_{\kappa\eta}$ is contraction on the Banach space $L^2(0,T;\mathcal{H}_1)$. Therefore, there exists a unique $\lambda_{\kappa\eta} \in L^2(0,T;\mathcal{H}_1)$ such that $\Lambda^m_{\kappa\eta}\lambda_{\kappa\eta} = \lambda_{\kappa\eta}$ where we deduce that $\lambda_{\kappa\eta}$ is the unique fixed point of $\Lambda_{\kappa\eta}$. Moreover, equality (4.25) implies that $\lambda_{\kappa\eta} \in W^{1,2}(0,T;\mathcal{H}_1)$, which concludes the proof of Lemma (4.4). □

Now, let $\lambda_{\kappa\eta}$ the fixed point of the operator $\Lambda_{\kappa\eta}$. We use *Riesz*'s representation theorem to define the operator $\Lambda_\kappa : L^2(0,T;V) \longrightarrow L^2(0,T;V)$ by

$$(\Lambda_\kappa \eta(t), v)_V = j(\beta_{\kappa\eta\lambda_{\kappa\eta}}(t), u_{\kappa\eta\lambda_{\kappa\eta}}(t), v) + (\mathcal{E}^* \boldsymbol{E}(\varphi_{\kappa\eta\lambda_{\kappa\eta}}(t)), \varepsilon(v)), \tag{4.34}$$

for all $v \in V$ and $t \in [0,T]$. We have the following result.

Lemma 4.5. *For all $\eta \in L^2(0,T;V)$ the function $\Lambda_\kappa \eta$ belongs to $W^{1,2}(0,T;V)$. Moreover, there exists a constant $\mu_0 > 0$ such that the operator Λ_κ has a unique fixed point $\eta_\kappa \in W^{1,2}(0,T;V)$ if $\|\mu\|_{L^\infty(\Gamma_3)} \leq \mu_0$.*

Proof. Let $\eta \in L^2(0,T;V)$ and let $t_1, t_2 \in [0,T]$. Using (4.34), (3.28), (3.21) and keeping in mind the inequality $0 \leq \beta_{\kappa\eta\lambda_{\kappa\eta}}(t) \leq 1$ and the properies of the operators R_ν, R_τ and $\mathcal{E}^*$ we find that

$$\begin{aligned} \|\Lambda_\kappa \eta(t_1) - \Lambda_\kappa \eta(t_2)\|_V \leq c(&\|u_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - u_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_V \\ &+ \|\beta_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - \beta_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_{L^2(\Gamma_3)} \\ &+ \|\varphi_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - \varphi_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_W). \end{aligned} \tag{4.35}$$

Since

$$u_{\kappa\eta\lambda_{\kappa\eta}} \in W^{2,2}(0,T;V), \ \beta_{\kappa\eta\lambda_{\kappa\eta}} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$$

and $\varphi_{\kappa\eta\lambda_{\kappa\eta}} \in W^{1,2}(0,T;W)$ we deduce that $\Lambda_\kappa \eta \in W^{1,2}(0,T;V)$.
Let now $\eta_1, \eta_2 \in L^2(0,T;V)$ and let $u_i = u_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\dot{u}_i = \dot{u}_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\beta_i = \beta_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\varphi_i = \varphi_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\sigma_i = \sigma_{\kappa\eta_i\lambda_{\kappa\eta_i}}$ for $i = 1,2$. Arguments similar to those used in the proof of (4.35) lead to

$$\begin{aligned} \|\Lambda_\kappa \eta_1(t) - \Lambda_\kappa \eta_2(t)\|_V \leq c(&\|u_1(t) - u_2(t)\|_V \\ &+ \|\beta_1(t - \beta_2(t)\|_{L^2(\Gamma_3)} + \|\varphi_1(t) - \varphi_2(t)\|_W). \end{aligned} \tag{4.36}$$

We combine now (4.18), (4.24) and (4.36) to obtain

$$\|\Lambda_\kappa \eta_1(t) - \Lambda_\kappa \eta_2(t)\|_V \leq c(\|u_1(t) - u_2(t)\|_V + \int_0^t \|u_1(s) - u_2(s)\|_V ds). \tag{4.37}$$

Moreover, since $u_1(0) = u_2(0) = u_0$ we have

$$\|u_1(t) - u_2(t)\|_V \leq c \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds. \tag{4.38}$$

From (4.37) and (4.38) we find

$$\|\Lambda_\kappa \eta_1(t) - \Lambda_\kappa \eta_2(t)\|_V \leq c \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds \qquad \forall\, t \in [0\ T]\,. \tag{4.39}$$

On the other hand, keeping in mind that $\lambda_{\kappa\eta_i} = \sigma_i$, using (4.8) and by arguments similar to those used in (4.30) we find that

$$m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} + \|\eta_1(t) - \eta_2(t)\|_V + C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds,$$

and, after a Gronwall argument, we obtain

$$m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}}, + \|\eta_1(t) - \eta_2(t)\|_V. \tag{4.40}$$

Now, by (4.10)(c) it follows that $Div\sigma_1(t) = Div\sigma_2(t)$. Then, from (4.7), (3.2), (3.3) and (3.15) we find that

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \le C_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V + C_{\mathcal{F}}\|u_1(t) - u_2(t)\|_V + \|\eta_1(t) - \eta_2(t)\|_V, \tag{4.41}$$

where we deduce that

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \le C_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V + \|\eta_1(t) - \eta_2(t)\|_V + C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds, \tag{4.42}$$

We combine now (4.40) and (4.42) to obtain

$$\begin{aligned} m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le\ & C_{\mathcal{A}} L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\dot{u}_1(t) - \dot{u}_2(t)\|_V \\ & + (L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} + 1)\|\eta_1(t) - \eta_2(t)\|_V \\ & + L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds. \end{aligned} \tag{4.43}$$

Now, we take $\|\mu\|_{L^\infty(\Gamma_3)} \le \mu_0$ such that

$$\mu_0 = \frac{m_{\mathcal{A}}}{C_{\mathcal{A}} L_p C_0 C_R}. \tag{4.44}$$

Using (4.43) and after a Gronwall argument we find that

$$\begin{aligned} & (m_{\mathcal{A}} - C_{\mathcal{A}} L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)})\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \\ & \le (L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} + 1)\|\eta_1(t) - \eta_2(t)\|_V, \end{aligned}$$

where, we deduce that for $\|\mu\|_{L^\infty(\Gamma_3)} \le \mu_0$ we have

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le c\|\eta_1(t) - \eta_2(t)\|_V. \tag{4.45}$$

We combine now (4.45) and (4.39) to see that

$$\|\Lambda_\kappa \eta_1(t) - \Lambda_\kappa \eta_2(t)\|_V \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \qquad \forall\ t \in [0, T] \tag{4.46}$$

and by Cauchy-Schwartz inequality we deduce that

$$\|\Lambda_\kappa\eta_1(t)-\Lambda_\kappa\eta_2(t)\|_V^2\le c\int_0^t\|\eta_1(s)-\eta_2(s)\|_V^2ds\qquad\forall\, t\in[0,T] \tag{4.47}$$

Reiterating this inequality m times yields

$$\|\Lambda_\kappa^m\eta_1-\Lambda_\kappa^m\eta_2\|^2_{L^2(0,T;V)}\le\frac{c^mT^m}{m!}\|\eta_1-\eta_2\|^2_{L^2(0,T;V)}.$$

which implies that, for $\|\mu\|_{L^\infty(\Gamma_3)}\le\mu_0$ and m sufficiently large, a power Λ_κ^m of Λ_κ is a contraction in the *Banach* space $L^2(0,T;V)$. Thus, there exists a unique element $\eta_\kappa\in L^2(0,T;V)$ such that $\Lambda_\kappa^m\eta_\kappa=\eta_\kappa$ and η_κ is also the unique fixed point of Λ_κ, i.e $\Lambda_\kappa\eta_\kappa=\eta_\kappa$. The regularity $\eta_\kappa\in W^{1,2}(0,T;V)$ follows from the regularity $\Lambda_\kappa\eta_\kappa\in W^{1,2}(0,T;V)$, which concludes the proof of Lemma (4.5). □

Next, let $\|\mu\|_{L^\infty(\Gamma_3)}\le\mu_0$ and $\lambda_{\kappa\eta}$, η_κ the fixed points of operators $\Lambda_{\kappa\eta}$, Λ_κ respectivelly. We put $u_k=u_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$, $\sigma_k=\sigma_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$, $\varphi_k=\varphi_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$ and $\beta_k=\beta_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$ for the solutions obtened in lemmas (4.1), (4.2), (4.3). Moreover, we define the operator $\mathit{\Lambda}:L^2(0,T;\mathcal{H})\longrightarrow L^2(0,T;\mathcal{H})$ by

$$\mathit{\Lambda}\kappa=\mathcal{G}(\sigma_\kappa,\varepsilon(u_\kappa)), \tag{4.48}$$

such that

$$\sigma_\kappa(t)=\mathcal{A}\varepsilon(\dot u_\kappa(t))+\mathcal{F}\varepsilon(u_\kappa(t))+z_\kappa(t)+\mathcal{E}^*\boldsymbol{E}(\varphi_\kappa(t)). \tag{4.49}$$

$$(\sigma_\kappa(t),\varepsilon(\omega)-\varepsilon(\dot u_\kappa(t))_{\mathcal{H}}+j_{ad}(\beta_\kappa(t),u_\kappa(t),\omega-\dot u_\kappa(t)) \tag{4.50}$$

$$+j_{fr}(\sigma_\kappa(t),\omega)-j_{fr}(\sigma_\kappa(t),\dot u_\kappa(t))\ge(f_\kappa(t),\omega-\dot u_\kappa(t))_V$$

$$\forall\,\omega\in V,\quad\forall t\in[0,T].$$

$$(f_\kappa(t),\upsilon)_V=(f(t),\upsilon)_V+(z_\kappa(t),\varepsilon(\upsilon))_{\mathcal{H}}. \tag{4.51}$$

Lemma 4.6. *The function $\mathit{\Lambda}\kappa$ belongs to $W^{1,2}(0,T;\mathcal{H})$ and the operator $\mathit{\Lambda}$ has a unique fixed point $\kappa^*\in L^2(0,T;\mathcal{H})$.*

Proof. Let $\kappa\in L^2(0,T;\mathcal{H})$ and let $t_1,t_2\in[0,T]$. Using (4.48), (3.7) and (3.15) we find that

$$\|\mathit{\Lambda}\kappa(t_1)-\mathit{\Lambda}\kappa(t_2)\|_{\mathcal{H}}\le L_{\mathcal{G}}(\|\sigma_\kappa(t_1)-\sigma_\kappa(t_2)\|_{\mathcal{H}}+\|u_\kappa(t_1)-u_\kappa(t_2)\|_V).$$

Since $u_\kappa\in W^{2,2}(0,T;V)$, $\sigma_\kappa\in W^{1,2}(0,T;\mathcal{H}_1)$ we deduce that $\Lambda\kappa\in W^{1,2}(0,T;\mathcal{H})$.

Next, let $\kappa_1,\kappa_2\in L^2(0,T;\mathcal{H})$. For the sake of simplicity, we put $u_i=u_{\kappa_i}$, $\sigma_i=\sigma_{\kappa_i}$, $\beta_i=\beta_{\kappa_i}$, $\varphi_i=\varphi_{\kappa_i}$ and $z_i=z_{\kappa_i}$. Usin again (4.48), (3.7) and (3.15) we obtain

$$\|\mathit{\Lambda}\kappa_1(t)-\mathit{\Lambda}\kappa_2(t)\|_{\mathcal{H}}\le L_{\mathcal{G}}(\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}}+\|u_1(t)-u_2(t)\|_V). \tag{4.52}$$

On the other hand, by arguments similar to those used in (4.30) the inequality (4.50) leads to

$$\begin{aligned}&(\sigma_1(t)-\sigma_2(t),\varepsilon(\dot u_1)-\varepsilon(\dot u_2)_{\mathcal H}\le\\&+j_{ad}(\beta_1,u_1,\dot u_2-\dot u_1)+j_{ad}(\beta_1,u_1,\dot u_2-\dot u_1)\\&(\mathcal E^*\boldsymbol E(\varphi_1(t))-\mathcal E^*\boldsymbol E(\varphi_2(t)),\varepsilon(\dot u_1)-\varepsilon(\dot u_2))_{\mathcal H}\\&+j_{fr}(\sigma_1(t),\dot u_2(t))-j_{fr}(\sigma_1(t),\dot u_1(t))\\&+j_{fr}(\sigma_2(t),\dot u_1(t))-j_{fr}(\sigma_2(t),\dot u_2(t)).\end{aligned}\tag{4.53}$$

Using (4.49), (3.2), (3.3), (3.6), (3.21), (3.22), (3.12), (3.28), (3.29) and the previous inequality and after some algebric manipulation we find that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V^2\le{}&(c\|u_1(t)-u_2(t)\|_V+\|z_1(t)-z_2(t)\|_V\\&+c\|\varphi_1(t)-\varphi_2(t)\|_W+c\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\ \Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H})\|\dot u_1(t)-\dot u_2(t)\|_V,\end{aligned}$$

where we deduce that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&c\|u_1(t)-u_2(t)\|_V+\|z_1(t)-z_2(t)\|_{\mathcal H}\\&+c\|\varphi_1(t)-\varphi_2(t)\|_W+c\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\ \Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}.\end{aligned}$$

We combine now (4.18), (4.24) and the previous inequality to obtain

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&L_pC_0C_R\|\mu\|_{L^\infty(\ \Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}\\&+\|z_1(t)-z_2(t)\|_{\mathcal H}+c\|u_1(t)-u_2(t)\|_V\\&+\int_0^t\|u_1(s)-u_2(s)\|_Vds.\end{aligned}\tag{4.54}$$

Moreover, since $u_1(0)=u_2(0)=u_0$ we have

$$\|u_1(t)-u_2(t)\|_V\le\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_Vds.\tag{4.55}$$

From (4.54) and (4.55) we find

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&L_pC_0C_R\|\mu\|_{L^\infty(\ \Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}\\&+\|z_1(t)-z_2(t)\|_{\mathcal H}+c\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_Vds,\end{aligned}$$

and after a Gronwall argument we find that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&\|z_1(t)-z_2(t)\|_{\mathcal H}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}.\end{aligned}\tag{4.56}$$

On the other hand, using (4.49), (3.2), (3.3) we find that

$$\begin{aligned}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}} \le C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V + C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V \\ + \|z_1(t)-z_2(t)\|_V + c\|\varphi_1(t)-\varphi_2(t)\|_W,\end{aligned} \tag{4.57}$$

where, we deduce from (4.18) that

$$\begin{aligned}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}} \le \|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ +C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V + C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V\end{aligned} \tag{4.58}$$

Combining (4.56) and (4.58) we obtain

$$\begin{aligned}m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V \le \|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ +L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V \\ +L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ +L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V.\end{aligned}$$

It follows now from the previous inequality that

$$\begin{aligned}m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V \le (1+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)})\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ + C_{\mathcal{A}}L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\dot u_1(t)-\dot u_2(t)\|_V \\ + L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{F}}\int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V ds,\end{aligned}$$

and after a Gronwall argument we find that

$$\begin{aligned}m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V \le (1+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)})\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ + C_{\mathcal{A}}L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\dot u_1(t)-\dot u_2(t)\|_V.\end{aligned}$$

Since, $\|\mu\|_{L^\infty(\Gamma_3)} \le \mu_0$ the previous inequality leads to

$$\|\dot u_1(t)-\dot u_2(t)\|_V \le c\|z_1(t)-z_2(t)\|_{\mathcal{H}} \tag{4.59}$$

Moreover, since $u_1(0)=u_2(0)=u_0$ we have

$$\|u_1(t)-u_2(t)\|_V \le \int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V ds \le c\int_0^t \|z_1(s)-z_2(s)\|_{\mathcal{H}} ds. \tag{4.60}$$

Combining, (4.58), (4.59) and (4.60) we find

$$\|\Lambda\kappa_1(t)-\Lambda\kappa_2(t)\|_{\mathcal{H}} \le c\|z_1(t)-z_2(t)\|_{\mathcal{H}} + c\int_0^t \|z_1(s)-z_2(s)\|_{\mathcal{H}} ds. \tag{4.61}$$

Now, from (4.6) we have $z_1(0)=z_2(0)=0$. Then,

$$\|z_1(t)-z_2(t)\|_V \le \int_0^t \|\dot z_1(s)-\dot z_2(s)\|_{\mathcal{H}} ds. \tag{4.62}$$

Therefore, combining (4.61) and (4.62) we obtain

$$\|\Lambda\kappa_1(t) - \Lambda\kappa_2(t)\|_{\mathcal{H}} \le c \int_0^t \|\dot{z}_1(s) - \dot{z}_2(s)\|_{\mathcal{H}} ds. \tag{4.63}$$

Finally, using (4.6) and Cauchy-Schwartz inequality we find

$$\|\Lambda\kappa_1(t) - \Lambda\kappa_2(t)\|_{\mathcal{H}}^2 \le c \int_0^t \|\kappa_1(s) - \kappa_2(s)\|_{\mathcal{H}}^2 ds.$$

Reiterating this inequality m times yields

$$\|\Lambda^m\kappa_1 - \Lambda^m\kappa_2\|_{L^2(0,T;\mathcal{H})}^2 \le \frac{c^m T^m}{m!} \|\kappa_1 - \kappa_2\|_{L^2(0,T;\mathcal{H})}^2.$$

which implies that, for m sufficiently large, a power Λ^m of Λ is a contraction in the *Banach* space $L^2(0,T;\mathcal{H})$. Thus, there exists a unique element $\kappa^* \in L^2(0,T;\mathcal{H})$ such that $\Lambda^m\kappa^* = \kappa^*$ and κ^* is also the unique fixed point of Λ, i.e $\Lambda\kappa^* = \kappa^*$, which concludes the proof of Lemma (4.6). □

Now, we have all the ingredients necessary to prove Theorem 4.1.

Existence: Let κ^*, η_κ, $\lambda_{\kappa\eta}$ be the fixed points of operators Λ, Λ_κ, $\Lambda_{\kappa\eta}$, respectively, and $(u,\sigma) = (u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$ the solution of the variational problem $\mathcal{P}_1^V$ with $\kappa = \kappa^*$, $\eta = \eta_\kappa$, $\lambda = \lambda_{\kappa\eta}$. We also denote by $\varphi = \varphi_{\kappa\eta\lambda}$ and $\beta = \beta_{\kappa\eta\lambda}$ the solution of problems (4.17) and $\mathcal{P}^{\beta_{\kappa\eta\lambda}}$, respectively, with $\kappa = \kappa^*$, $\eta = \eta_\kappa$, $\lambda = \lambda_{\kappa\eta}$. Clearly, it follows from (4.6), (4.25), (4.34) and (4.48) that (3.30)-(3.35) holds. We conclude that (u, σ, φ, D, β) is a solution of Problem $\mathcal{P}^V$ and it satisfies (4.1)-(4.5).

Uniqueness: The uniqueness of the solution follows from the uniqueness of the fixed points of Λ, Λ_κ, $\Lambda_{\kappa\eta}$ and from the uniqueness part of Lemmas (4.1), (4.2) and (4.3).

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