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Darboux problem for fractional partial hyperbolic differential inclusions on unbounded domains with delay

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Abstract. In this paper we investigate the existence of solutions of initial value problems (IVP for short), for partial hyperbolic functional and neutral differential inclusions of fractional order involving Caputo fractional derivative with finite delay by using the nonlinear alternative of Frigon type for multivalued admissible contraction in Fréchet spaces.

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1. Introduction

In this paper we are concerned with the existence of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(t,x)}),\ \text{if } (t,x) \in J, \tag{1.1}$$

$$u(t,x) = \phi(t,x),\ \text{if } (t,x) \in \tilde{J}, \tag{1.2}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ (t,x) \in J, \tag{1.3}$$

where $\varphi(0) = \psi(0)$, $J := [0,\infty) \times [0,\infty)$, $\tilde{J} := [-\alpha,+\infty) \times [-\beta,+\infty)\backslash[0,\infty) \times [0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $F : J \times C([-\alpha,0] \times [-\beta,0],\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact valued, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $\phi \in C := C([-\alpha,0] \times [-\beta,0],\mathbb{R}^n)$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J, \varphi : [0,\infty) \to \mathbb{R}^n, \psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous

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functions and C is the space of continuous functions on $[-\alpha, 0] \times [-\beta, 0]$.
We denote by $u_{(t,x)}$ the element of C defined by

$$u_{(t,x)}(s, \tau) = u(t + s, x + \tau);\ (s, \tau) \in [-\alpha, 0] \times [-\beta, 0],$$

here $u_{(t,x)}(\cdot, \cdot)$ represents the history of the state u.

Next we consider the following system of partial neutral hyperbolic differential inclusion of fractional order

$$^cD_0^r[u(t, x) - g(t, x, u_{(t,x)})] \in F(t, x, u_{(t,x)}),\ \text{if } (t, x) \in J, \tag{1.4}$$

$$u(t, x) = \phi(t, x),\ \text{if } (t, x) \in \tilde{J}, \tag{1.5}$$

$$u(t, 0) = \varphi(t),\ u(0, x) = \psi(x),\ (t, x) \in J, \tag{1.6}$$

where F, ϕ, φ, ψ are as in problem (1.1)-(1.3) and $g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a given continuous function.

It is well known that differential equations and inclusions of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics, viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [14, 20, 21, 22]). The theory of differential equations and inclusions of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations and inclusions, for example see the monographs of Kilbas *et al.* [16], Lakshmikantham *et al.* [18], and the papers by Belarbi *et al.* [3], Benchohra *et al.* [4, 5, 6, 7] and the references therein.

Differential delay equations and inclusions, or functional differential equations and inclusions, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Lakshmikantham *et al.* [19], Wu [25] and the papers [8, 13, 23].

In this paper, we present existence result for the problems (1.1)-(1.3) and (1.4)-(1.6). Our aim here is to give global existence results for the above problem. The fundamental tools applied here are essentially multi-valued version of nonlinear alternative of Frigon type [10].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $n \in \mathbb{N}$ and $J_0 = [0, n] \times [0, n]$. By $C(J_0, \mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J_0} \|u(t, x)\|,$$

where $\| \cdot \|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J_0, \mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0, \mathbb{R})$ is the space of Lebesgue-integrable functions $u : J_0 \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^n \int_0^n \|u(t, x)\| dt dx.$$

Definition 2.1. [24] Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty), \theta = (0, 0)$ and $u \in L^1(J, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t, x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s, \tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t, x) = u(t, x), \ (I_\theta^\sigma u)(t, x) = \int_0^t \int_0^x u(s, \tau) d\tau ds; \text{ for almost all } (t, x) \in J,$$

where $\sigma = (1, 1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty) \times (0, \infty)$, when $u \in L^1(J, \mathbb{R}^n)$. Note also that when $u \in C(J, \mathbb{R}^n)$, then $(I_\theta^r u) \in C(J, \mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t, 0) = (I_\theta^r u)(0, x) = 0; \ (t, x) \in J.$$

Example 2.2. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \text{ for almost all } (t, x) \in J.$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.3. [24] Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1(J, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t, x) = (D_{tx}^2 I_\theta^{1-r} u)(t, x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^c D_0^r u)(t, x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t, x).$$

The case $\sigma = (1, 1)$ is included and we have

$$(D_\theta^\sigma u)(t, x) = ({}^c D_\theta^\sigma u)(t, x) = (D_{tx}^2 u)(t, x), \text{ for almost all } (t, x) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t, x) \in J.$$

3. Some properties of set-valued maps

Let $(X, \|\cdot\|)$ be a Banach space. Denote

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

For each $u \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S_{F \circ u} = \{f \in L^1(J, \mathbb{R}^n) : f(t,x) \in F(t,x,u(t,x)) \ a.e. \ (t,x) \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$, $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [17]).

Definition 3.1. A multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(t, x) \longmapsto F(t, x, u)$ is measurable for each $u \in \mathbb{R}^n$;

(ii) $u \longmapsto F(t, x, u)$ is upper semicontinuous for almost all $(t, x) \in J$.

F is said to be L^1-Carathéodory if $(i), (ii)$ and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned} \|F(t,x,u)\|_{\mathcal{P}} &= \sup\{\|f\| : f \in F(t,x,u)\} \\ &\leq \sigma_c(t,x) \ \text{ for all } \|u\| \leq c \text{ and for } a.e. \ (t,x) \in J. \end{aligned}$$

For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [9], Gorniewicz [12], Hu and Papageorgiou [15] and Kisielewiecz [17].

4. Some properties in Fréchet spaces

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|u\|_1 \leq \|u\|_2 \leq \|u\|_3 \leq ... \quad for \ every \ u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|v\|_n \leq \overline{M}_n \quad for \ all \ v \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $u \in X$, we denote $[u]_n$ the equivalence class of u of subset X^n and we defined $Y^n = \{[u]_n : u \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [11].

Definition 4.1. A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n \in \mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that

(i) $H_d(F(u), F(v)) \leq k_n ||u - v||_n$ for all $u, v \in X$.

(ii) For every $u \in X$ and every $\varepsilon \in (0,\infty)^n$, there exists $v \in F(u)$ such that

$$||u-v||_n \leq ||u-F(u)||_n + \varepsilon_n \quad \text{for every } n \in \mathbb{N}.$$

Theorem 4.2. *(Nonlinear alternative of Frigon type)* [10] *Let X be a Fréchet space and U an open neighborhood of the origin in X, and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements is holds:*

$(C1)$ *N has at least one fixed point;*

$(C2)$ *There exist $\lambda \in [0,1)$ and $u \in \partial U$ such that $u \in \lambda N(u)$.*

5. Existence of solutions

In this section, we give our main existence result for the problems (1.1)-(1.3) and (1.4)-(1.5). For each $n \in \mathbb{N}$ we set

$$C_n = C([-\alpha,n] \times [-\beta,n], \mathbb{R}^n)$$

and we define seminorms in $C_0 := C([-\alpha,\infty) \times [-\beta,\infty), \mathbb{R}^n)$ by:

$$\|u\|_n = \{\sup \|u(t,x)\| : -\alpha \leq t \leq n, -\beta \leq x \leq n\}.$$

Then C_0 is a Fréchet space with the family $\{\| \cdot \|_n\}$. of seminorms.

5.1. The functional case

Now we are able to state and prove our main theorem for the problem (1.1)-(1.3).

Before starting and proving this result, we give what we mean by a solution of the problem (1.1)-(1.3).

Definition 5.1. A function $u \in C_0$ is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J,\mathbb{R}^n)$ with $f(t,x) \in F(t,x,u_{(t,x)})$ such that $({}^cD_0^r u)(t,x) = f(t,x)$ and u satisfies equations (1.3) on J and the condition (1.2) on $\tilde{J}$.

For the existence of solutions for the problem (1.1)-(1.3), we need the following lemma:

Lemma 5.2. *A function $u \in C_0$ is a solution of problem* (1.1)-(1.3) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\,d\tau ds$$

for all $(t,x) \in J$ and the condition (1.2) *on $\tilde{J}$, where*

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Our main existence result in this section is based on the nonlinear alternative of Frigon. We will need to introduce the following hypothesis:

$(H1)$ $F : J \times C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n) \to \mathcal{P}_{cp,c}(\mathbb{R}^n)$ is a L^1-Carathéodory map.

$(H2)$ For each $n \in \mathbb{N}$, there exist $p_n \in L^1(J,\mathbb{R}_+)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|F(t,x,u)\|_{\mathcal{P}} \leq p_n(t,x)\Psi(||u||), \text{ for a.e. } (t,x) \in J_0 \text{ and each } u \in C,$$

$(H3)$ For each $n \in \mathbb{N}$, there exists $\ell_n \in L^1(J_0, \mathbb{R}^+)$ such that

$$H_d(F(t,x,u), F(t,x,v)) \le \ell_n(t,x)|u - v|, \text{ for all } u, v \in C,$$

and

$$d(0, (F(t,x,0)) \le \ell_n(t,x), \text{ a.e. } (t,x) \in J_0.$$

Where $C := C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$.

$(H4)$ For each $n \in \mathbb{N}$, there exists a numbre $M_n > 0$ such that

$$\frac{M_n}{||z||_n + \frac{\Psi(M_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1, \tag{5.1}$$

where $p_n^* = \sup_{(t,x) \in J_0} p_n(t,x)$.

Theorem 5.3. *Assume that hypotheses* $(H1)$-$(H4)$ *hold. If*

$$\frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.2}$$

where

$$\ell_n^* = \sup_{(t,x) \in J_0} \ell_n(t,x),$$

then the IVP (1.1)-(1.3) *has at least one solution on* $[-\alpha, \infty] \times [-\beta, \infty]$.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $N : C_0 \to \mathcal{P}(C_0)$ defined by,

$$(Nu)(t,x) = h \in C_0$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F,u}$.

Remark 5.4. For each $u \in C_0$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a mesurable selection.

Let u be a possible solution of the inclusion $u \in \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$,

$$\begin{aligned} ||u(t,x)|| &= \lambda ||z(t,x)|| + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad ||f(s,\tau)|| d\tau ds \\ &\le ||z(t,x)|| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad p_n(s,\tau)\Psi(||u_{(s,\tau)}||) d\tau ds \\ &\le ||z||_n + \frac{\Psi(||u||_n) p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

This implies by $(H4)$ that, for each $(t,x)\in J_0$, we have

$$\frac{||u||_n}{||z||_n+\frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}}\leq 1.$$

Then by condition (5.1) we have a contradiction, so there exists M_n such that $||u||_n\neq M_n$. Since for every $(t,x)\in J_0$, we have

$$\|u\|_n\leq \max(\|\phi\|_C, M_n^*) := R_n.$$

Set

$$U=\{u\in C_0 : \|u\|_n\leq R_n+1 \; for\ all\ n\in\mathbb{N}\}.$$

We shall show that $N : U \to \mathcal{P}(U)$ is a contraction and an admissible operator. First, we prove that N is a contraction; that is, there exists $\gamma<1$, such that

$$H_d(N(u)-N(u^*))\leq \gamma||u-u^*||_n, \qquad for\ u,u^*\in U.$$

Let $u,u^*\in U$ and $h\in N(u)$. Then there exists $f(t,x)\in F(t,x,u_{(t,x)})$ such that for each $(t,x)\in J_0$,

$$h(t,x)=z(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds.$$

From $(H3)$ it follows that

$$H_d(F(t,x,u_{(t,x)})-F(t,x,u^*_{(t,x)}))\leq \ell_n(t,x)||u_{(s,\tau)}-u^*_{(s,\tau)}||.$$

Hence there is exists $f^*\in F(t,x,u^*_{(t,x)})$ such that

$$|f(t,x)-f^*(t,x)|\leq \ell_n(t,x)||u_{(t,x)}-u^*_{(t,x)}||, \quad \forall (t,x)\in J_0.$$

Let us define for each $(t,x)\in J_0$,

$$h^*(t,x)=z(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f^*(s,\tau)d\tau ds.$$

Then we have

$$\begin{aligned}
|h(t,x) \;&-\; h^*(t,x)|\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\\
&\times\; |f(s,\tau)-f^*(s,\tau)|d\tau ds\\
&\leq\; \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_n(s,\tau)||u-u^*||\\
&\leq\; \frac{\ell_n^*||u-u^*||_n}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds,
\end{aligned}$$

where $\ell_n^*=\sup\limits_{(s,\tau)\in J_0}\ell_n(s,\tau)$. Therefore

$$\|h-h^*\|_n\leq \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|u-u^*\|_n.$$

By an analogous relation, obtained by interchanging the roles of u and u^*, it follows that

$$H_d(N(u) - N(u^*)) \leq \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} ||u - u^*||_n.$$

Hence by (5.2), N is a contraction.
Now, $N : C_n \to \mathcal{P}_{cp}(C_n)$ is given by,

$$(Nu)(t,x) = h \in C_n$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J_0, \end{cases}$$

where $f \in S^n_{F,u} = \{f \in L^1(J_0, \mathbb{R}^n) : f(t,x) \in F(t,x,u_{(t,x)})\ a.e.\ (t,x) \in J_0\}$. From $(H2)$-$(H3)$ and since F is compact valued, we can prove that for every $u \in C_n, N(u) \in \mathcal{P}_{cp}(C_n)$, and there exists $u^* \in C_n$ such that $u^* \in N(u^*)$. (For the proof see *Benchohra et al.* [4]). Let $h \in C_n, u \in U$ and $\varepsilon > 0$. Now, if $\tilde{u} \in N(u^*)$, then we have

$$||u^* - \tilde{u}||_n \leq ||u^* - h||_n + ||\tilde{u} - h||_n.$$

Since h is arbitrary we may suppose that $h \in B(\tilde{u}, \varepsilon) = \{k \in C_n : ||k - \tilde{u}||_n \leq \varepsilon\}$. Therefore,

$$||u^* - \tilde{u}||_n \leq ||u^* - N(u^*)||_n + \varepsilon.$$

On the other hand, if $\tilde{u} \notin N(u^*)$, then $||\tilde{u} - N(u^*)||_n \neq 0$. Since $N(u^*)$ is compact, there exists $v \in N(u^*)$ such that $||\tilde{u} - N(u^*)||_n = ||\tilde{u} - v||_n$. Then we have

$$||u^* - v||_n \leq ||u^* - h||_n + ||v - h||_n.$$

Therefore,

$$||u^* - v||_n \leq ||u^* - N(u^*)||_n + \varepsilon.$$

So, N is an admissible operator contraction. By our choice of U, there is no $u \in \partial U$ such that $u \in \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Frigon type, we deduce that N has a fixed point which is a solution to problem (1.1)-(1.3).

5.2. The neutral type case

Now, we present the existence of solutions to fractional order IVP (1.4)-(1.6).

Definition 5.5. A function $u \in C_0$ is said to be a solution of (1.4)-(1.6) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ with $f(t,x) \in F(t,x,u_{(t,x)})$ such that

$${}^cD_0^r[u(t,x) - g(t,x,u_{(t,x)})] = f(t,x)$$

and u satisfies equations (1.6) on J and the condition (1.5) on $\tilde{J}$.

For the existence of solutions for the problem (1.4)-(1.6), we need the following lemma:

Lemma 5.6. *A function $u \in C_0$ is a solution of problem* (1.4)-(1.6) *if and only if u satisfies the equation*

$$\begin{aligned} u(t,x) &= z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) \\ & \quad -g(0,x,u_{(0,x)}) + g(0,0,u_{(0,0)}) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds \end{aligned}$$

for all $(t,x) \in J$ and the condition (1.5) *on $\tilde{J}$, where*

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Theorem 5.7. *Assume $(H1)$-$(H3)$ and the following hypothesis holds.*

$(H5)$ *For each $n \in \mathbb{N}$, there exists $d_n \in C(J_0, \mathbb{R}^n)$ such that for each $(t,x) \in J_0$ we have*

$$\|g(t,x,u) - g(t,x,v)\| \le d_n \|u - v\|, \; for \; each \; u \in C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n).$$

$(H6)$ *For each $n \in \mathbb{N}$, there exists an numbre $M_n > 0$ such that*

$$\frac{M_n}{||z||_n + 4g^* + 4d_n M_n + \frac{\Psi(M_n) p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1, \tag{5.3}$$

where $p_n^* = \sup_{(t,x)\in J} p_n(t,x)$ and $g^* = \sup_{(s,\tau)\in J_0} ||g(t,x,0)||$.

If

$$4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}, \tag{5.4}$$

where

$$\ell_n^* = \sup_{(t,x)\in J_0} \ell_n(t,x),$$

then there exists at least one solution for IVP (1.4)-(1.6) *on $[-\alpha, \infty) \times [-\beta, \infty)$.*

Proof. Transform the problem (1.4)-(1.6) into a fixed point problem. Consider the operator $N_1 : C_0 \to C_0$ defined by

$$(N_1 u)(t,x) = h_1 \in C_0$$

such that

$$h_1(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + g(t,x,u_{(t,x)}) - & \\ g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u) + & \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F,u}$.

Remark 5.8. For each $u \in C_0$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a mesurable selection.

Let u be a possible solution of the inclusion $u \in \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$,

$$\begin{aligned} u(t,x) &= \lambda[z(t,x) + g(t,x,u_{(t,x)}) + g(t,0,u_{(t,0)}) + g(0,x,u_{(0,x)}) + g(0,0,u)] \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} ||f(s,\tau)|| d\tau ds, \end{aligned}$$

then, we have

$$\begin{aligned} ||u(t,x)|| &\leq ||z(t,x)|| + ||g(t,x,u_{(t,x)})|| + ||g(t,0,u_{(t,0)})|| \\ &\quad + ||g(0,x,u_{(0,x)})|| + ||g(0,0,u)|| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} ||f(s,\tau)|| d\tau ds. \end{aligned}$$

This implies by $(H2)$ and $(H5)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned} ||u(t,x)|| &\leq ||z(t,x)|| + 4d_n||u|| + ||g(t,x,0)|| \\ &\quad + ||g(t,0,0)|| + ||g(0,x,0)|| + ||g(0,0,0)|| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad p_n(s,\tau)\Psi(||u_{(s,\tau)}||) d\tau ds \\ &\leq ||z||_n + 4g^* + 4d_n||u||_n + \frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

This implies by $(H6)$ that, for each $(t,x) \in J_0$, we have

$$\frac{||u||_n}{||z||_n + 4g^* + 4d_n||u||_n + \frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} \leq 1.$$

Then by condition (5.3) we have a contradiction, so there exists M_n such that $||u||_n \neq M_n$. Since for every $(t,x) \in J_0$, we have

$$\|u\|_n \leq \max(\|\phi\|_C, M_n^*) := R_n'.$$

Set

$$U = \{u \in C_0 : \|u\|_n \leq R_n' + 1 \ for\ all\ n \in \mathbb{N}\}.$$

We shall show that $N_1 : U \to \mathcal{P}(U)$ is a contraction and an admissible operator. First, we prove that N_1 is a contraction; that is, there exists $\gamma < 1$, such that

$$H_d(N_1(u) - N_1(u^*)) \leq \gamma ||u - u^*||_n, \quad for\ u, u^* \in U.$$

Let $u, u^* \in U$ and $h \in N_1(u)$. Then there exists $f(t,x) \in F(t,x,u_{(s,\tau)})$ such that for each $(t,x) \in J_0$,

$$\begin{aligned} h_1(t,x) &= z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) \\ &\quad - g(0,x,u_{(0,x)}) + g(0,0,u) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds. \end{aligned}$$

From $(H3)$ it follows that

$$H_d(F(t,x,u_{(t,x)}) - F(t,x,u^*_{(t,x)})) \leq \ell_n(t,x)||u_{(t,x)} - u^*_{(t,x)}||.$$

Hence there is exists $f^* \in F(t,x,u^*_{(t,x)})$ such that

$$|f(t,x) - f^*(t,x)| \leq \ell_n(t,x)||u_{(t,x)} - u^*_{(t,x)}||, \quad \forall (t,x) \in J_0.$$

Let us define $\forall (t,x) \in J_0$,

$$\begin{aligned} h_1^*(t,x) &= z(t,x) + g(t,x,u^*_{(t,x)}) - g(t,0,u^*_{(t,0)}) - g(0,x,u^*_{(0,x)}) + g(0,0,u^*) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds. \end{aligned}$$

Then we have

$$||h_1(t,x) - h_1^*(t,x)|| \leq$$

$$\begin{aligned} &\leq \|g(t,x,u_{(t,x)}) - g(t,x,u^*_{(t,x)})\| + \|g(t,0,u_{(t,0)}) - g(t,0,u^*_{(t,0)})\| \\ &\quad +\|g(0,x,u_{(0,x)}) - g(0,x,u^*_{(0,x)})\| + \|g(0,0,u) - g(0,0,u^*)\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau) - f^*(s,\tau)||d\tau ds \\ &\leq d_n(\|u_{(t,x)} - u^*_{(t,x)}\|_n + \|u_{(t,0)} - u^*_{(t,0)}\|_n \\ &\quad +\|u_{(0,x)} - u^*_{(0,x)}\|_n + \|u - u^*\|_n) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_n(s,\tau)||u - u^*||d\tau ds \\ &\leq 4d_n\|u - u^*\|_n \\ &+ \frac{\ell_n^*||u - u^*||_n}{\Gamma(r_1+1)\Gamma(r_2+1)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds \\ &\leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n, \end{aligned}$$

where $\ell_n^* = \sup_{(s,\tau)\in J_0} \ell_n(s,\tau)$. Therefore

$$\|h_1 - h_1^*\|_n \leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n.$$

By an analogous relation, obtained by interchanging the roles of u and u^*, it follows that

$$H_d(N_1(u) - N_1(u^*)) \leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n.$$

Hence by (5.4), N_1 is a contraction.

Now, $N_1 : C_n \to \mathcal{P}_{cp}(C_n)$ is given by

$$(N_1u)(t,x) = h_1 \in C_n$$

such that

$$h_1(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + g(t,x,u_{(t,x)}) & \\ -g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u) + & \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S^n_{F,u} = \{f \in L^1(J_0, \mathbb{R}^n) : f(t,x) \in F(t,x,u_{(t,x)})\ a.e.\ (t,x) \in J_0\}$. From $(H2)$-$(H3)$ and since F is compact valued, we can prove that for every $u \in C_n, N_1(u) \in \mathcal{P}_{cp}(C_n)$, and there exists $u^* \in C_n$ such that $u^* \in N_1(u^*)$. Let $h_1 \in C_n, u \in U$ and $\varepsilon > 0$. Now, if $\tilde{u} \in N_1(u^*)$, then we have

$$||u^* - \tilde{u}||_n \leq ||u^* - h_1||_n + ||\tilde{u} - h_1||_n.$$

Since h_1 is arbitrary we may supose that $h_1 \in B(\tilde{u}, \varepsilon) = \{k \in C_n : ||k - \tilde{u}||_n \leq \varepsilon\}$. Therefore,

$$||u^* - \tilde{u}||_n \leq ||u^* - N_1(u^*)||_n + \varepsilon.$$

On the other hand, if $\tilde{u} \notin N_1(u^*)$, then $||\tilde{u} - N_1(u^*)||_n \neq 0$. Since $N_1(u^*)$ is compact, there exists $v \in N_1(u^*)$ such that $||\tilde{u} - N_1(u^*)||_n = ||\tilde{u} - v||_n$. Then we have

$$||u^* - v||_n \leq ||u^* - h_1||_n + ||v - h_1||_n.$$

Therefore,

$$||u^* - v||_n \leq ||u^* - N_1(u^*)||_n + \varepsilon.$$

So, N_1 is an admissible operator contraction. By our choice of U, there is no $u \in \partial U$ such that $u \in \lambda N_1(u)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Frigon type, we deduce that N_1 has a fixed point which is a solution to problem (1.4)-(1.6).

6. Examples

As an application of our results we consider the following hyperbolic functional differential inclusions of the form

$$({}^cD^r_0 u)(t,x) \in F(t-1, x-2, u), \quad if\ (t,x) \in J := [0,\infty) \times [0,\infty), \tag{6.1}$$

$$u(t,0) = t,\ u(0,x) = x^2, \quad (t,x) \in J, \tag{6.2}$$

$$u(t,x) = t + x^2, \quad (t,x) \in \tilde{J} := [-1,\infty) \times [-2,\infty) \backslash [0,\infty) \times [0,\infty), \tag{6.3}$$

where $F : J \times C([-1,0] \times [-2,0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact valued, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$.

Thus under appropriate conditions on the function F as those mentioned in the hypotheses $(H1)$-$(H4)$ implies that problem (6.1)-(6.3) has at least one solution defined on $[-1,\infty) \times [-2,\infty)$.

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