STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XL, 1, 1995

PRECONDITIONING FOR THE FULFILMENT OF THE APPROXIMATION ASSUMPTION IN THE ALGEBRAIC MULTIGRID METHOD

Constantin POPA*

Received: November 15, 1994
MS subject classification: 65M55, 65M60, 65N55, 65F10

REZUMAT. - Precondiţionarea pentru îndeplinirea condiţiilor de aproximare în metoda multigrid algebrică. Se prezintă o metodă de precondiţionare pentru sisteme liniare simetrice şi pozitiv definite. Folosind un operator de interpolare se dovedeşte că se realizează îndeplinirea condiţiilor de aproximare, care de obicei cauzează cele mai multe dificultăţi în utilizarea algoritmilor algebrici multigrid [4], [17]. Astfel se obţine convergenţa V-cicluri de tip multigrid pentru sistemele simetrice generale pozitiv definite. Lucrarea se încheie cu prezentarea mai multor exemple numerice pentru ecuaţiile Dirichlet, precum şi Poisson şi Helmholtz anisotropice.

Abstract. In the last years a lot of papers ([1], [2], [3], [15], [20]) presented various preconditioning techniques for the improvement of the condition number of symmetric and positive definite M-matrices arrising from the discretization of elliptic partial differential equations. All of these techniques essentially use the "geometric" information offerend by the continuous problem ("good" properties of the partial differential operator, special types of regular discretizations etc.). Thus, even the ideas are quite general we cannot apply these methods for arbitrary systems.

In this paper we present a method of preconditioning for arbitrary symetric and positive definite linear systems. We don't obtain an improvement of the condition number of the system matrix (which is very hard in this general approach) but using a special construction of the interpolation operator we prove the fulfilment of the approximation

* *University of Constanţa, Department of Mathematics, 8700 Constanţa, Romania*

assumption (which usually causes the most troubles in the algebraic multigrid algorithms, see [4], [17]). Thus we obtain the convergence of the V-cycle type algebraic multigrid for general symetric and positive definite systems.

At the end of the paper we present numerical examples on Dirichlet, anisotropic Poisson and Helmholtz equations.

1. Introduction. In this section we shall use the notations, definitions and results from [17]. Let A be an n by n symmetric and positive definite matrix. For $b \in \mathbb{R}^n$ we consider the system

$$Au = b, \tag{1}$$

with the (unique) exact solution $u \in \mathbb{R}^n$. Let $q \geq 2$ be an integer and $C_1, C_2, \ldots, C_q$ a sequence of nonvoid subsets of $\{1, \ldots, n\}$ such that

$$\{1, \ldots, n\} = C_1 \supset C_2 \supset \ldots \supset C_q, \tag{2}$$

$$|C_m| = n_m, \; m = 1, \ldots, q, \tag{3}$$

$$n = n_1 > n_2 > \ldots > n_q > 1, \tag{4}$$

where by $|C_m|$ we denoted the number of elements in the set C_m. Furthermore for $m = 1, 2, \ldots, q-1$ we consider the linear operators

$$I_{m+1}^m : \mathbb{R}^{n_m} \to \mathbb{R}^{n_{m+1}}, \; I_m^{m+1} : \mathbb{R}^{n_{m+1}} \to \mathbb{R}^{n_m} \tag{5}$$

and the matrices A^{m+1} with the properties

$$I_m^{m+1} = (I_{m+1}^m)^t, \; A^1 = A, \; A^{m+1} = I_m^{m+1} A^m I_{m+1}^m. \tag{6}$$

For $m = 1, \ldots, q-1$ we define the coarse grid correction operators T^m by

$$T^m = I_m - I_{m+1}^m (A^{m+1})^{-1} I_m^{m+1} A^m \tag{7}$$

and the smoothing process of the form

$$u_{new}^{m} = G^{m} u_{old}^{m} + (I_m - G^{m})(A^{m})^{-1} b^{m}, \tag{8}$$

where I_m is the identity and

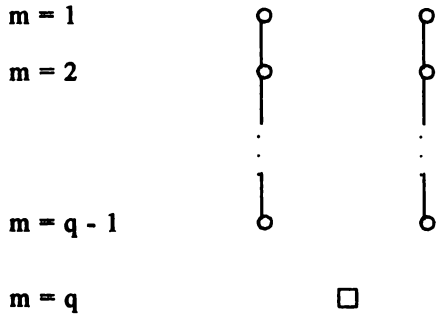
$$A^{m} u^{m} = b^{m} \tag{9}$$

are the systems corresponding to the coarse levels.

Remarks 1. The sets C_m, $m = 1, ..., q$ formally play the same role as coarse grids in the classical geometric multigrid ([3]), I_{m+1}^{m}, I_{m}^{m+1} are the interpolation and restriction operators, respectively and A^m the coarse grids matrices.

2. The form (8) of the smoothing process includes the classical relaxation schemes (ω-Jacobi, Gauss-Seidel, S O R, I L U - decomposition).

3. With all the above defined elements we consider a classical V - cycle type algorithm (with at least one smoothing step performed after each coarse grid correction step) looking like (e.g. [18])



(10)

where we suppose that on the last grid ($m = q$) the system (9) is solved exactly.

We introduce the matrix

$$D_m = \mathrm{diag}(A^{m}),\ m = 1, ..., q-1 \tag{11}$$

and define on each level the inner products

$$<u^m, v^m>_0 = <D_m u^m, v^m>, \quad <u^m, v^m>_1 = <A^m u^m, v^m>, \tag{12}$$

$$<u^m, v^m>_2 = <D_m^{-1} A^m u^m, A^m v^m>,$$

along with their corresponding norms $\|\cdot\|_i$, $i = 0, 1, 2$, where $<\cdot, \cdot>$ is the Euclidean inner product and $\|\cdot\|$ the Euclidean norm (on the spaces $\mathbb{R}^{n_m}$). We shall denote by $e^m = v^m - u^m$ the error on each level $m = 1, \ldots, q\text{-}1$. We know the following result concerning the convergence of the above defined V-cycle.

THEOREM 1. ([17]) *Assume that the interpolations* I_{m+1}^m, $m = 1, \ldots, q-1$ *have full rank and that there exists a constant* $\delta > 0$ *independently on m and* e^m *such that*

$$\|G^m e^m\|_1^2 \le \|e^m\|_1^2 - \delta \|T^m e^m\|_1^2, \quad m = 1, \ldots, q-1. \tag{13}$$

Then $\delta \le 1$ *and the V - cycle* (10) *to solve* (1) *has a convergence factor (in the energy norm* $\|\cdot\|_1$*) bounded above by* $\sqrt{1-\delta}$.

COROLLARY 1. ([17]) *If there exists constants* $\alpha, \beta > 0$ *independently of m and* e^m *such that*

$$\|G^m e^m\|_1^2 \le \|e^m\|_1^2 - \alpha \|e^m\|_2^2, \tag{14}$$

$$\|T^m e^m\|_1^2 \le \beta \|e^m\|_2^2, \tag{15}$$

for every $m = 1, \ldots, q\text{-}1$ *then we have* (13) *with*

$$\delta = \alpha/\beta \tag{16}$$

Remarks 1. Properties (14), (15) are called the smoothing assumption (SA) and the approximation assumption (AA), respectively ([17]).

2. (SA) is fulfilled by the classical relaxation schemes (see [4], [12], [17]).

3. The condition (AA) causes the most troubles. There are two weaker forms, namely

$$(AA_1) \qquad \|T^m e^m\|_1^2 \le \beta_1 \|T^m e^m\|_2^2, \tag{17}$$

$$(AA_2) \qquad \min\{\|e^m - I_{m+1}^m e^{m+1}\| + O^2,\ e^{m+1} \in \mathbb{R}^{n_{m+1}}\} \le \beta_2 \|e^m\|_1^2, \tag{18}$$

where the positive constants β_1 and β_2 are also independently on m and e^m. Following the result from [17] (AA_2) implies (AA_1) with $\beta_1 = \beta_2$ and one of them with the smoothing assumption (14) ensures the convergence of the two grid algorithm $(m, m+1)$. For the multilevel case $(q \ge 3)$ it is necesary that (15) holds. This is, in fact, our principal aim in the present paper.

2. Preconditioning - the two level case. We present in this section the method of preconditioning for a pair of two consecutive grids $(m, m+1)$ where $m \in \{1, ..., q-1\}$ is arbitrary fixed. In order to simplify the notations we shall write $n, p, I_p^n, I_n^p, C_p, A, A_p$ instead of $n_m, n_{m+1}, I_{m+1}^m, I_m^{m+1}, C_m, A^m, A^{m+1}$ respectively. We shall suppose also that the coarse grid C_p satisfies

$$C_p = \{n-p+1, n-p+2, ..., n\} \tag{19}$$

Accordingly to (19) we consider the block decomposition of A

$$A = \begin{bmatrix} A_1 & B \\ B^t & A_2 \end{bmatrix} \tag{20}$$

where A_1, A_2 are symmetric invertible matrices of dimension $n-p$ and p, respectively, with A_1 positive definite. Let $\bar{A}_1$ be another symmetric and positive definite matrix of dimension $n-p$.

We consider the Cholesky decompositions of A_1 and $\bar{A}_1$

$$A_1 = L_1 L_1^t, \ \bar{A}_1 = \bar{L}_1 \bar{L}_1^t \tag{21}$$

and we define the matrix $\bar{\Delta}_1$ (of dimension n) by

$$\bar{\Delta}_1 = \begin{bmatrix} \bar{L}_1 \bar{L}_1^{-1} & O \\ O & I_2 \end{bmatrix} \tag{22}$$

where I_2 is the identity on $\mathbb{R}^p$. We shall also denote by I_1 the identity on $\mathbb{R}^{n-p}$ and by $u = [u_1, u_2]$ a vector $u \in \mathbb{R}^n$ for the descomposition

$$\mathbb{R}^n = \mathbb{R}^{n-p} \oplus \mathbb{R}^p \tag{23}$$

We precondition the system (1) in the following way

$$(\bar{\Delta}_1 A \bar{\Delta}_1^t)((\bar{\Delta}_1^t)^{-1} u) = \bar{\Delta}_1 b. \tag{24}$$

Thus the system (1) becomes

$$\bar{A}\bar{u} = \bar{b}, \tag{25}$$

where

$$\bar{u} = (\bar{\Delta}_1^t)^{-1} u, \ \bar{b} = \bar{\Delta}_1 b, \tag{26}$$

and

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{B} \\ \bar{B}^t & A_2 \end{bmatrix} = \bar{\Delta}_1 A \bar{\Delta}_1^t, \tag{27}$$

with the $(n-p) \times p$ matrix $\bar{B}$ given by

$$\bar{B} = \bar{L}_1 L_1^{-1} B. \tag{28}$$

Remark. It is clear that u is the solution of (1) if and only if $\bar{u}$ from (26) is the

solution of (25). We also observe that the preconditioned matrix $\bar{A}$ from (27) is symmetric and positive definite. Thus we can define for $\bar{A}$ the inner products from (12) and the associated norms. These norms will be denoted by $|||\cdot|||_i$, $i = 0, 1, 2$. Accordingly with [13] and [14] we shall define the interpolation I_p^n by

$$I_p^n = \begin{bmatrix} -\bar{A}_1^{-1}\bar{B} \\ I_2 \end{bmatrix}. \tag{29}$$

Then I_p^n has full rank and from (21) and (28) we obtain

$$I_p^n = \begin{bmatrix} -(\bar{L}_1^t)^{-1} L_1^{-1} B \\ I_2 \end{bmatrix} \tag{30}$$

PROPOSITION 1. (i) *The coarse grid matrix A_p is given by*

$$A_p = A_2 - B^t A_1^{-1} B \tag{31}$$

and is independent on the matrix $\bar{A}_1$ of the preconditioning.

(ii) *The coarse grid correction operator, T, is given by*

$$T = \begin{bmatrix} I_1 & \bar{A}_1^{-1}\bar{B} \\ O & O \end{bmatrix}. \tag{32}$$

(iii) *If $\bar{e} = [\bar{e}_1, \bar{e}_2] \in \mathbb{R}^n$ is the error after the correction step then*

$$|||\bar{e}|||_1^2 = <\bar{A}\bar{e}, \bar{e}> = <\bar{A}_1\bar{e}_1, \bar{e}_1> \geq \lambda_{\min}(\bar{A}_1)\|\bar{e}\|^2, \tag{33}$$

where $\lambda_{\min}(\bar{A}_1)$ is the smallest eigenvalue of $\bar{A}_1$.

Proof. (i) Firstly we observe that from (29), (6) and (27) we obtain

$$I_m^p \bar{A} = [O \,|\, A_2 - \bar{B}^t \bar{A}_1^{-1} \bar{B}] = [O \,|\, \tilde{A}_2], \tag{34}$$

with $\tilde{A}_2$ given by

$$\tilde{A}_2 = A_2 - \bar{B}^t \bar{A}_1^{-1} \bar{B}.$$

Then

$$A_p = I_n^p \bar{A} I_p^n = [O \vdots \bar{A}_2] \begin{bmatrix} -\bar{A}_1^{-1}\bar{B} \\ I_2 \end{bmatrix} = \tilde{A}_2$$

But, using (21), (35) and (28) we have

$$\tilde{A}_2 = A_2 - B^t (L_1^{-1})^t \bar{L}_1^t (\bar{L}_1^t)^{-1} (\bar{L}_1)^{-1} \bar{L}_1 \bar{L}_1^{-1} B = A_2 - B^t (L_1^{-1})^t L_1^{-1} B = A_2 - B^t A_1^{-1} B,$$

which gives us (31).

(ii) Using (7), (34) and (31) we obtain

$$T = I - I_p^n A_p^{-1} I_n^p \bar{A} = I - I_p^n A_p^{-1} [O \vdots A_p] = I - \begin{bmatrix} -\bar{A}_1^{-1}\bar{B} \\ I_2 \end{bmatrix} [O \vdots I_2] =$$

$$= \begin{bmatrix} I_1 & O \\ O & I_2 \end{bmatrix} - \begin{bmatrix} O & -\bar{A}_1^{-1}\bar{B} \\ O & I_2 \end{bmatrix} = \begin{bmatrix} I_1 & \bar{A}_1^{-1}\bar{B} \\ O & O \end{bmatrix}$$

which is exactly (32).

(iii) If $\bar{e} = [\bar{e}_1, \bar{e}_2] \in \mathbb{R}^n$ is the error after the correction step we have ([8])

$$I_n^p \bar{A} \bar{e} = 0.$$

From (39), (36) we obtain

$$[O \vdots A_p] \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} = 0$$

thus

$$A_p \bar{e}_2 = 0 \Rightarrow \bar{e}_2 = 0,$$

because A_p is invertible. Then (33) is obvious.

We shall make now the following assumption: *there exists a constant* $c > 0$ *independently of the dimension n of the matrix A such that*

$$\|\bar{A}_1^{-1}\bar{B}\| \le c. \tag{42}$$

Then we obtain the following result concerning the fulfilment of (15) and (17).

THEOREM 2. *For every vector* $e = [e_1, e_2] \in \mathbb{R}^n$ *we have*

$$|||Te|||_1^2 \le \frac{\min\{\bar{a}_{ii}, 1 \le i \le n-p\}}{\lambda_{\min}(\bar{A}_1)} |||Te|||_2^2 \tag{43}$$

and

$$|||Te|||_1^2 \le c^2 \frac{\max\{\bar{a}_{ii}, 1 \le i \le n-p\} \cdot \min\{\bar{a}_{ii}, 1 \le i \le n-p\}}{\min\{a_{ii}, n-p+1 \le i \le n\} \cdot \lambda_{\min}(\bar{A}_1)} |||e|||_2^2, \tag{44}$$

where we denoted the elements of the matrix $\bar{A}_1$ *by* $\bar{a}_{ij}$.

Proof. We denote the vector

$$\bar{e} = Te \tag{45}$$

by $\bar{e} = [\bar{e}_1, \bar{e}_2]$ i.e. the error after the correction step. From Proposition 1 (iv)

$$\bar{e}_2 = 0, \tag{46}$$

thus, using (33) and (46),

$$|||\bar{e}|||_1^2 \ge \lambda_{\min}(\bar{A}_1)\|e_1\|^2 = \lambda_{\min}(\bar{A}_1)\|\bar{e}\|^2 \ge \frac{\lambda_{\min}(\bar{A}_1)}{\min\{\bar{a}_{ii}, 1 \le i \le n-p\}} |||\bar{e}|||_0^2. \tag{47}$$

But a simple calculation using Cauchy-Schwarz inequality (see also [4], [17]) yields for $\bar{e}$, using also (46),

$$|||\bar{e}|||_1^2 \le |||\bar{e}|||_2 \, |||\bar{e}|||_0. \tag{48}$$

Combining (47) with (48) we get (43).

For the second assertion, (44), we firstly observe that

$$\bar{A}T = T'\bar{A}, \tag{49}$$

which follows from (7) and the symmetry of $\bar{A}$. Then we have

$$|||\bar{e}|||_2^2 = |||Te|||_2^2 = <\bar{D}^{-1}\bar{A}Te, \bar{A}Te> \leq \|\bar{D}^{\frac{1}{2}}T\bar{D}^{-1}T'\bar{D}^{\frac{1}{2}}\| \cdot |||e|||_2^2 = \rho(EE')|||e|||_2^2, \tag{50}$$

where $\bar{D} = \mathrm{diag}(\bar{A})$ and E is the matrix given by

$$E = \bar{D}^{\frac{1}{2}}T(\bar{D})^{-\frac{1}{2}}. \tag{51}$$

But from (32) and (51) we obtain

$$E = \begin{bmatrix} I_1 & \bar{D}^{1/2}\bar{A}_1^{-1}\bar{B}(\bar{D})^{-1/2} \\ O & O \end{bmatrix} \tag{52}$$

then

$$EE' = \begin{bmatrix} I_1 + KK' & O \\ O & O \end{bmatrix} \tag{53}$$

where K is the matrix

$$K = \bar{D}_1^{\frac{1}{2}}\bar{A}_1^{-1}\bar{B}(\bar{D}_2)^{-\frac{1}{2}} \tag{54}$$

Then, using (42), it results that

$$\rho(EE') = \rho(I_1 + KK') = 1 + \rho(KK') \leq 1 + \|K\|^2 \leq 1 + \|\bar{D}_1\| \cdot \|\bar{A}_1^{-1}\bar{B}\| \cdot \|\bar{D}_2^{-1}\| \leq$$

$$\leq \frac{\max\{\bar{a}_{ii}, 1 \leq i \leq n-p\}}{\min\{a_{ii}, n-p+1 \leq i \leq n\}} \cdot c^2 \tag{55}$$

and from (50) and (55) we obtain

$$|||Te|||_2^2 = |||\bar{e}|||_2^2 \leq \frac{\max\{\bar{a}_{ii}, 1 \leq i \leq n-p\}}{\min\{a_{ii}, n-p+1 \leq i \leq n\}} c^2 \cdot |||e|||_2^2. \tag{56}$$

Now, using (43), (44) is obvious.

It remains now to see under what assumptions $\lambda_{\min}(\bar{A}_1)$ from (33) and c from (42) are

constants which not depend on the dimension of the matrices A or $\bar{A}$. In that sense we have the following result.

PROPOSITION 2. *Suppose that there exists a constant* $\gamma > 0$, *independently on the dimension n of A such that*

$$\lambda_{\min}(A_1) \geq \gamma, \lambda_{\min}(\bar{A}_1) \geq \gamma. \tag{57}$$

Then (42) *holds with* $c > 0$ *given by*

$$c = \frac{\|A\|_\infty}{\gamma} \tag{58}$$

where by $\|S\|_\infty$ *we denoted the number*

$$\|S\|_\infty = \max_i \sum_j |s_{ij}| \tag{59}$$

for an arbitrary matrix $S = (s_{ij})$.

Proof. From (30) we have

$$\bar{A}_1^{-1}\bar{B} = (\bar{L}_1^t)^{-1}(L_1^{-1})B$$

Thus

$$\|\bar{A}_1^{-1}\bar{B}\| \leq \|(\bar{L}_1^t)^{-1}\| \cdot \|L_1^{-1}\| \cdot \|B\| \tag{60}$$

But, because $\bar{L}_1$ and L_1 are Cholesky factors, we obtain

$$\|(\bar{L}_1^t)^{-1}\| = \sqrt{\rho(\bar{A}_1^{-1})} \leq \frac{1}{\sqrt{\gamma}} \tag{61}$$

and

$$\|L_1^{-1}\| = \sqrt{\rho(A_1^{-1})} \leq \frac{1}{\sqrt{\gamma}} \tag{62}$$

For $\|B\|$ we can write (using the symmetry of A)

$$\|B\| = \sqrt{\rho(B^tB)} \leq \sqrt{\|B^tB\|_\infty} \leq \sqrt{\|B^t\|_\infty \cdot \|B\|_\infty} \leq \|A\|_\infty \tag{63}$$

Then, introducing (61)-(63) in (60) we obtain (58).

We shall denote by β_m the positive constant

$$\beta_m = \frac{1}{\gamma^3} \frac{\max\{\bar{a}_{ii}, 1 \le i \le n-p\} \cdot \{\bar{a}_{ii}, 1 \le i \le n-p\}}{\min\{a_{ii}, n-p < i \le n\}} \cdot \|A\|_\infty^2 \tag{64}$$

where $m \in \{1, ..., q\text{-}1\}$ is the arbitrary level considered at the begining of this section. Accordingly to (44), (57), (58) and (64) we obtain

$$|||Te|||_1^2 \le \beta_m |||e|||_2^2, \tag{65}$$

i.e. the approximation assumption (15) (on the level m). Defining $\beta > 0$ by

$$\beta = \max\{\beta_m, 1 \le m \le q-1\}, \tag{66}$$

from (65) it results for every $e^m \in \mathbb{R}^{n_m}$,

$$|||T^m e^m|||_1^2 \le \beta |||e^m|||_2^2, \ (\forall)\, m = 1, ..., q-1, \tag{67}$$

where T^m is the same matrix with T from (38) (on the level m).

3. The smoothing assumption for the preconditioned system. We obtained in (67) the approximation assumption for the preconditioned system with respect to the norms $|||\cdot|||_i$, $i = 1, 2$ defined with the inner products from (12) for the preconditioned matrix $\bar{A}$. Thus, it is necessary that the smoothing assumption be also fulfilled with respect to these norms. This is the aim of the present section.

We shall mentain the notational conventions from the above section. Firstly we observe that a relaxation step of the type (8) can be written in the form

$$u_{new} = M^{-1}Nu_{old} + M^{-1}b, \tag{68}$$

where

$$A = M - N \tag{69}$$

is a splitting of the matrix A with M invertible and

$$\rho(M^{-1}N) < 1, \tag{70}$$

(indeed, it is sufficient to define $G = M^{-1}N$ and from (68) we get (8)). Suppose that relaxation (68) satisfies the smoothing assumption (14) (on the level m) with a constant $\alpha_m > 0$, i.e. $(\forall)e \in \mathbb{R}^r$

$$\|M^{-1}Ne\|_1^2 \leq \|e\|_1^2 - \alpha_m \|e\|_2^2 \tag{71}$$

We shall define now (only for theoretical purpose!) for the preconditioned system (25) a similar relaxation, i.e.

$$\bar{u}_{new} = \bar{M}^{-1}\bar{N}\bar{u}_{old} + \bar{M}^{-1}\bar{b}, \tag{72}$$

where the matrices $\bar{M}$ and $\bar{N}$ are given by

$$\bar{M} = \Delta_1 M \Delta_1^t, \ \bar{N} = \bar{\Delta}_1 N \bar{\Delta}_1^t. \tag{73}$$

We denote by e, $\bar{e}$ respectively the errors

$$e = u_{old} - u \tag{74}$$

and

$$\bar{e} = \bar{u}_{old} - \bar{u} \tag{75}$$

THEOREM 3. (i) *$\bar{M}$ is invertible, $\bar{A} = \bar{M} - \bar{N}$ and*

$$\rho(\bar{M}^{-1}\bar{N}) < 1. \tag{76}$$

(ii) *If*

$$\bar{u}_{old} = (\bar{\Delta}_1^t)^{-1} u_{old} \tag{77}$$

then

$$\bar{u}_{new} = (\bar{\Delta}_1^t)^{-1} u_{new}. \tag{78}$$

(iii) *The relaxation* (72) *satisfies the smoothing assumption with the same constant* α_m, i.e.

$$|||\bar{M}^{-1}\bar{N}\bar{e}|||_1^2 \leq |||\bar{e}|||_1^2 - \alpha_m |||\bar{e}|||_2^2. \tag{79}$$

Proof. (i) The first two statements are obvious. For the third, using the well known equality $\rho(AB) = \rho(BA)$ (see e.g. [19]) we obtain

$$\rho(\bar{M}^{-1}\bar{N}) = \rho((\bar{\Delta}_1^t)^{-1} M^{-1} N (\bar{\Delta}_1^t)) = \rho(M^{-1}N) < 1$$

(ii) It results by simple computations using (68), (72), (73), (25) and (26).

(iii) From (26) and (77) we have

$$\bar{e}_{old} = (\bar{\Delta}_1^t)^{-1} e_{old} \tag{80}$$

Then, it is sufficient to observe, using (73), that

$$<\bar{A}\bar{M}^{-1}\bar{N}\bar{e}_{old}, \bar{M}^{-1}\bar{N}\bar{e}_{old}> = <AM^{-1}Ne_{old}, M^{-1}Ne_{old}>,$$

$$<\bar{A}\bar{e}_{old}, \bar{e}_{old}> = <Ae_{old}, e_{old}>,$$

$$<\bar{D}^{-1}\bar{A}\bar{e}_{old}, \bar{A}\bar{e}_{old}> = <D^{-1}Ae_{old}, Ae_{old}>$$

and the prof is complete.

Remark. From the assertion (ii) of the above theorem we obtain the following usefull fact: computing $\bar{u}_{new}$ with (72) and a given approximation $\bar{u}_{old}$ is the same as computing u_{new} with (68) and u_{old} given by

$$u_{old} = \bar{\Delta}_1^t \bar{u}_{old} \tag{81}$$

and calculate

$$\bar{u}_{new} = (\bar{\Delta}_1')^{-1} u_{new}, \tag{82}$$

In this way, the relaxation proces (72), for the preconditioned system (25), can be carried out using a classical relaxation of the type (68) for the initial system and the relations (81)-(82).

Like in the previous section we can now define

$$\alpha = \min \{\alpha_m, 1 \le m \le q-1\} \tag{83}$$

Then, over denoting $\bar{G} = \bar{M}^{-1}\bar{N}$ from (79) by G^m and $\bar{e}$ by e^m we obtain

$$|||G^m e^m|||_1^2 \le |||e^m|||_1^2 - \alpha \cdot |||e^m|||_2^2, \ (\forall) m = 1, ..., q-1, \tag{84}$$

i.e. the smoothing assumption (14).

4. The convergence of the algebraic multigrid algorithm. Accordingly to the Theorem 1 we obtain that the V - cycle type multigrid algorithm defined in section 2 converges to the exact solution u of (1) and the convergence factor, in the energy norm of the preconditioned matrix A is bounded above by

$$\bar{\rho} = \sqrt{1 - \alpha/\beta} \tag{85}$$

with α and β from (83) and (66) respectively.

We have the possibility (see the next section) to obtain γ from (57) independently of the dimension and the spectrum of the matrices A and $\bar{A}$. Thus, the constants α_m and β_m from (64) and (79) will depend only on the coefficients of the matrices A^m and $\bar{A}^m$ ($\bar{A}^m$ is $\bar{A}$ on the level m). But, unfortunately, in the general case, α and β, and so $\bar{\rho}$ from (85), will

depend on the number of levels used in the V - cycle. It is very hard, even in particular cases, to find a theoretical value of the factor $\bar{\rho}$. The only way is to use an accurate coarsening process and to define an efficient interpolation such that the coarse grid matrices keep the properties of the initial matrix.

In our case an encouraging aspect comes to helps us. Indeed, from the relation (31) it results that the coarse grid matrix A_p, obtain with the Galerkin approach (6) and I_p^n from (29), don't depend on the preconditioning. More than that, A_p is the Schur complement of A_1 obtained with Gaussian elimination. But there exist results (see e.g. [9]) which say that, for example, A is (weakly) diagonally dominant, A_p keeps this property a.s.o. In this way we can controle the coefficients of A_p, their signs, absolute values, positions (i.e. the sparsity of the matrix). Thus, defining interpolations like in (29) the only problem is 'to properly choose' A_1 (and $\bar{A}_1$) for that the 'extra work' and the computational costs be not too expensive.

Remark. Choosing A_1 means, from (19) and (20), choosing the coarse grid C_p. Some facts related to this aspect can be found in the papers [4], [17], [16]. Concerning the (spectral) condition of the preconditioned matrix $\bar{A}$ denoted by $k(\bar{A})$, we can easly obtain some precise informations. Indeed, from (27) we have

$$k(\bar{A}) = \|\bar{A}\| \cdot \|\bar{A}^{-1}\| \leq k(A) \cdot \|\bar{\Delta}_1^t \bar{\Delta}_1\| \cdot \|(\bar{\Delta}_1^t)^{-1} \bar{\Delta}_1^{-1}\|. \tag{86}$$

From (20) and (57) we obtain

$$\|\bar{\Delta}_1^t \bar{\Delta}_1\| \leq \frac{\|\bar{A}_1\|}{\gamma}, \ \|(\bar{\Delta}_1^t)^{-1} \bar{\Delta}_1^{-1}\| \leq \frac{\|A_1\|}{\gamma}. \tag{87}$$

Then, (86), (87) and similar arguments with A instead $\bar{A}$ get

$$k(A)\frac{\gamma^2}{\|A_1\|\cdot\|\bar{A}_1\|} \le k(\bar{A}) \le \frac{\|A_1\|\cdot\|\bar{A}_1\|}{\gamma^2}k(A), \tag{88}$$

with $k(\bar{A})$ the spectral condition number of $\bar{A}$.

Then, for an accurate and realistic γ in (57), $k(A)$ is of the same order with $k(A)$ and the convergence in the norm $|||\cdot|||_1$ will not deteriorate the results.

5. Some particular cases.

I. $\bar{A}_1 = A_1$. Then $\bar{A} = A$ thus no preconditioning occurs. Condition (57) will hold if, for example A_1 is strictly diagonally dominant, i.e.

$$\nu_i = a_{ii} - \sum_{j=1, j\neq i}^{n-p} |a_{ij}| > 0, i = 1, \ldots, n-p. \tag{89}$$

Then, we can take γ from (57) to be (from Gershgorin's theorem, [19])

$$\gamma = \min\{\nu_i, i = 1, \ldots, n-p\}. \tag{90}$$

The interpolation operator will be given by (see also [13]).

$$I_p^n = \begin{bmatrix} -A_1^{-1}B \\ I_2 \end{bmatrix}. \tag{91}$$

The following result gives us a way for constructing I_p^n without inverting the matrix A_1. Firstly we have to observe that, the matrix A being positive definite (and symmetric) we can perform the Gaussian elimination algorithm without pivoting ([16]) and making 1 on the diagonal of A_1, for the first n-p columns. After that we obtain a matrix $\tilde{A}$ of the form (in block notation)

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{B} \\ O & \bar{A}_2 \end{bmatrix}. \tag{92}$$

or elementwise

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{a}_{12} & \tilde{a}_{13} & \dots & \tilde{a}_{1,n-p} & \tilde{a}_{1,n-p+1} & \dots & \tilde{a}_{1n} \\ o & 1 & \tilde{a}_{23} & \dots & \tilde{a}_{2,n-p} & \tilde{a}_{2,n-p+1} & \dots & \tilde{a}_{2n} \\ \dots & \dots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & \tilde{a}_{n-p,n-p+1} & \dots & \tilde{a}_{n-p,n} \\ 0 & 0 & 0 & \dots & 0 & \tilde{a}_{n-p+1,n-p+1} & \dots & \tilde{a}_{n-p+1,n} \\ \dots & \dots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & \tilde{a}_{n,n-p+1} & \dots & \tilde{a}_{nn} \end{bmatrix} \tag{93}$$

For $k = 1, \dots, n-p$ we define the matrices H_k of dimension $(n-k) \times (n-k+1)$ and H of dimension $p \times n$ by

$$H_k = \begin{bmatrix} -\tilde{a}_{k,k+1} & 1 & 0 & \dots & 0 \\ -\tilde{a}_{k,k+2} & 0 & 1 & \dots & 0 \\ -\tilde{a}_{k,n} & 0 & 0 & \dots & 1 \end{bmatrix} \tag{94}$$

and

$$H = H_{n-p} H_{n-p-1} \dots H_1. \tag{95}$$

Observation. The first column of H_k (without minus sign) is the k - row of the matrix $[\tilde{A}_1 \,|\, \tilde{B}]$ from (93) without the 1 on the diagonal.

THEOREM 4. *With the above considerations we have*

$$I_n^p = H. \tag{96}$$

Proof. It results from (94) and (95) that the matrix H has the structure

$$H = [\tilde{H} \,|\, I_2], \tag{97}$$

where I_2 is the identity on $\mathbb{R}^p$ and $\tilde{H}$ is a $p \times (n-p)$ real matrix. We observe that the first

column of H_1 is given by

$$\tilde{a}_{1k} = a_{1k}/a_{11},\ k = 2, \tag{98}$$

Thus, in block notation,

$$H_1 A = [O \,|\, A^{(1)}], \tag{99}$$

or elementwise

$$H_1 A = \begin{bmatrix} 0 & a_{22}-(a_{21}a_{12})/a_{11} & \dots & a_{2n}-(a_{21}a_{1n})/a_{11} \\ 0 & a_{32}-(a_{31}a_{12})/a_{11} & \dots & a_{3n}-(a_{31}a_{1n})/a_{11} \\ 0 & a_{n2}-(a_{n1}a_{12})/a_{11} & \dots & a_{nn}-(a_{n1}a_{1n})/a_{11} \end{bmatrix}. \tag{100}$$

From the symmetry of A it results that the matrix $A^{(1)}$ from (100) is the same with the square matrix of dimension $(n\text{-}1) \times (n\text{-}1)$ obtained after the first step of the Gaussian elimination by neglecting the first row and column. Recursively we obtain that

$$H_{n-p} H_{n-p-1} \dots H_1 A = [O \,|\, \tilde{A}_1]. \tag{101}$$

But, from (101), (20) and (97) it results

$$\tilde{H} A_1 + B^t = O, \tag{102}$$

which gives us

$$\tilde{H} = -B^t A_1^{-1}, \tag{103}$$

COROLLARY 2. *The interpolation operator I_p^n and the coarse grid matrix A_p are given by*

$$I_p^n = H_1^t H_2^t \dots H_{n-p}^t, \tag{104}$$

$$A_p = H_{n-p} \dots H_2 H_1 A H_1^t H_2^t \dots H_{n-p}^t. \tag{105}$$

Remark. We observe that for the construction of I_p^n (or I_n^p) we must perform the Gaussian elimination only on the matrix A_1 (i.e. only for the $n-p$ rows of A).

II. $\bar{A}_1 = \text{diag}(d_1, d_2, \ldots, d_{n-p})$ where we suppose that

$$d_i > 0,\ i = 1, \ldots, n-p. \tag{106}$$

Then

$$\bar{L}_1 = \bar{L}_1^t = \text{diag}\,(d_1^{\frac{1}{2}}, d_2^{\frac{1}{2}}, \ldots, d_{n-p}^{\frac{1}{2}}) \tag{107}$$

and the interpolation I_p^n is given by

$$I_p^n = \begin{bmatrix} -\bar{L}_1^{-1} L_1^{-1} B \\ I_2 \end{bmatrix}. \tag{108}$$

In order to obtain the product $L_1^{-1}B$ (with L_1 the Cholesky factor of A_1, from (21)) we make a Gaussian elimination (Without pisoting and making 1 on the diagonal) on the first $n-p$ rows of A. In this way we obtain the matrix

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{B} \\ B^t & A_2 \end{bmatrix} \tag{109}$$

where

$$A_1 = \tilde{L}_1 \tilde{A}_1 \tag{110}$$

is an (LU) - decomposition of A_1 ($\tilde{A}_1$ is upper triangular with 1 on his diagonal) and

$$\tilde{B} = \tilde{L}_1^{-1} B \tag{111}$$

Then, if $\tilde{D}_1 = \text{diag}\,(\tilde{L}_1) = \text{diag}\,(\tilde{l}_{11}, \tilde{l}_{22}, \ldots, \tilde{l}_{n-p,n-p})$ it is obvious that

$$L_1^t = \tilde{D}_1^{1/2} \tilde{A}_1 \tag{112}$$

Then elements of the matrix $\tilde{D}_1$ can be recursively obtained by the formulas

$$a_{11} = \tilde{l}_{11},\ a_{ii} = \tilde{l}_{ii} + \sum_{k=1}^{i-1} \tilde{l}_{kk} \cdot \tilde{a}_{ki}^2,\ i = 2, \ldots, n-p \tag{113}$$

(where $\tilde{a}_{ij}$ are the elements of $\tilde{A}_1$). Then we have

$$L_1^{-1} B = \tilde{D}_1^{1/2} \cdot \tilde{B} \tag{114}$$

The constant γ from (57) can be taken as

$$\gamma = \min \{\nu_i, d_i,\ i = 1, \ldots, n-p\} \tag{115}$$

III. $\bar{A}_1 = A_1 + R_1$ where

$$A_1 = \bar{A}_1 - R_1 \tag{116}$$

is an incomplete Cholesky decomposition of A_1 (if A_1 is supposed to be an M - matrix, cf. [11]). The factor $\bar{L}_1$ is obtained during this decomposition. We know from [11] that

$$\rho = \rho(\bar{A}_1^{-1} R_1) < 1 \tag{117}$$

From (116) we obtain

$$\bar{A}_1^{-1} A_1 = I - \bar{A}_1^{-1} R_1 \tag{118}$$

Thus, if $\lambda \in \mathbf{C}$ is an eigenvalue of $\bar{A}_1^{-1} A_1$, $1-\lambda$ will be an eigenvalue for $\bar{A}_1^{-1} R_1$ and (using (117))

$$|1 - |\lambda|| \le |1 - \lambda| \le \rho \tag{119}$$

From (119) it results that for every eigenvalue λ of $\bar{A}_1^{-1} A_1$

$$1 - \rho \le |\lambda| \le 1 + \rho \tag{120}$$

In particular

$$1-\rho \le \rho(\bar{A}_1^{-1} A_1) \le \|\bar{A}_1^{-1} A_1\| \le \|\bar{A}_1^{-1}\| \cdot \|A_1\| \tag{121}$$

and

$$\lambda_{\min}(\bar{A}_1) = \frac{1}{\|\bar{A}_1^{-1}\|} \leq \frac{\|A_1\|}{1-\rho} \leq \frac{\|A_1\|_\infty}{1-\rho} \tag{122}$$

Thus, γ from (57) can be taken as

$$\gamma = \min\left\{\min\{\nu_i, i = 1, \ldots, n-p\}, \frac{\|A_1\|_\infty}{1-\rho}\right\} \tag{123}$$

Remark. Relation (123) tells us that the number 1-ρ must not depend on the dimension of the matrix A_1. Thus, the ILU-decomposition (116) must not be 'too incomplete', i.e. the matrix R_1 must not have too much nonempty entries, the 'ideal' case being

$$R_1 = 0, \tag{124}$$

i.e. our particular case I.

6. Nummerical examples. We considered the following plane problems:

Dirichlet
$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Anisotropic Poisson
$$\begin{cases} -\varepsilon \cdot \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Helmholtz
$$\begin{cases} \Delta u + k^2 u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

with $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, discretized by a classical 5-point stencil finite differences (see e.g. [8]). We used two different initial (finest grid) discretizations (corresponding to meshsizes $h = 1/14$ and $h = 1/32$) and a 5 - grids V - cycle algebraic multigrid (see section 1). We

applied the preconditioning methods from cases I and II (section 5). As relexation we used the classical Gauss - Seidel method ([19]). The stopping criterion of the multigrid algorithm was

$$|||u^N - u|||_1 \leq 10^{-6} \tag{125}$$

where u is the exact solution and u^N the corresponding approximation (N is the minimum number of iteration such that (125) holds).

In tables 1-4 we indicated the worst norm reduction factor per iteration step, ρ, computed with the formula

$$\rho = \sup\left\{\frac{|||e^{j+1}|||_1}{|||e^j|||_1},\ j = 1, \ldots, N-1\right\} \tag{126}$$

for Dirichlet and anisotropic Poisson problems and

$$\rho = \sup\left\{\frac{\|e^{j+1}\|}{\|e^j\|},\ j = 1, \ldots, N-1\right\} \tag{127}$$

for Helmholtz equation ($e^j = u^j - u$ is the error at the j-th iteration of the multigrid algorithm).

Remarks 1. For coarsening we used the algorithm presented in the paper [16].

2. In the case of Helmoltz equation the algebraic system is symmetric but not more positive definite. But following the results of Mandel ([10]), the condition (33), with $\lambda_{min}(\bar{A}_1)$ not depending on the dimension of the initial matrix A, ensures the convergence of the two grid algorithm even in the indefinite case.

3. Some improvements in order to avoid the fill - in process appearing sometimes in

the coarses grids matrices were presented in [7].

4. The values of ε (table 2) and k^2 (tables 3 and 4) were selected accordingly to similar examples solved in papers [17] and [5] respectively.

Acknowledgement. The author thanks to Mr. Gabriel Golubovici, Departement of Mathematics, Univ. of Constanţa, for his help preparation of numerical examples.

h	1/14	1/32
ρ for case I	0.051	0.078
ρ for case II	0.19	0.4

Table 1. The Dirichlet problem

h		1/14	1/32
ρ for case I	$\varepsilon = 10^{-1}$	0.052	0.078
	$\varepsilon = 10^{-2}$	0.052	0.078
	$\varepsilon = 10^{-6}$	0.054	0.079
ρ for case II	$\varepsilon = 10^{-1}$	0.19	0.41
	$\varepsilon = 10^{-2}$	0.2	0.41
	$\varepsilon = 10^{-6}$	0.23	0.42

Table 2. The anisotropic Poisson problem

ρ for case I	$k^2 = 4$	0.054
	$k^2 = 19$	0.058
	$k^2 = 25$	0.09
	$k^2 = 30$	0.37
ρ for case II	$k^2 = 4$	0.21
	$k^2 = 10$	0.27
	$k^2 = 25$	0.48
	$k^2 = 30$	0.74

Table 3. The Helmmoltz problem, $h = 1/14$.

ρ for case I	$k^2 = 19$	0.077
	$k^2 = 55$	0.34
	$k^2 = 100$	0.83
ρ for case II	$k^2 = 19$	0.56
	$k^2 = 55$	0.8
	$k^2 = 100$	0.97

Table 4. The Helmholtz problem, $h = 1/32$.

REFERENCES

1. D. Adam, *Mesh independence of Galerkin approach by preconditioning.* Intern. J. Computer Math., vol. 28, pp. 227-236, 1989.
2. O. Axelsson, P.S. Vassilevski, *Algebraic multilevel preconditioning methods* I. Numer. Math, 56, pp. 157-177, 1989.
3. O. Axelsson, M. Neytcheva, *Algebraic multilevel iteration method for Stieltjes matrices.* Report 9102, Dept. of Math., Catholic Univ. Toemooiveld, 1991.
4. A. Brandt, *Algebraic multigrid theory: the symmetric case.* Preprint, Weizmann Inst. Of Science, Rehovot, Israel, 1983.
5. A. Brandt, S. Ta'assan, *Multigrid method for nearly singular and slighty indefinite problems.* Lecture Notes in Math., vol. 1228, Springer, Berlin, 1986.
6. G.H. Golub, C.F. van Loan, *Matrix computations.* John's Hopkins Univ. Press, Baltimore, 1983.
7. G. Golubovici, C. Popa, *Interpolation and related coarsening techniques for the algebraic multigrid method.* Paper presented at the 4th European Conference on Multigrid Methods, 6-9 July 1993, Amsterdam.
8. W. Hackbusch, *Multigrid methods and applications.* Springer, Berlin, 1985.
9. A.S. Honseholder, The Theory of matrices in numerical analysis. Blaisdell Publ. Comp., New York, 1964.
10. J. Mandel, *On multigrid and iterative aggregation methods for nonsymmetric problems.* Lecture Notes in Math., vol. 1228, Springer, Berlin, 1986.
11. J.A. Meijerink, H.A. van der Vorst, *An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix.* Math. of Comput., vol. 31, nr. 157, pp. 148-162, 1977.
12. C. Popa, *On smoothing properties of SOR relaxation for algebraic multigrid method.* Studii şi Cerc. Mat., vol. 5, pp. 399-406, 1989.
13. C. Popa, *ILU decomposition for coarse grid correction step on algebraic multigrid.* In GMD - Studien, nr. 189, 1991.
14. C. Popa, *The construction of the interpolation operator with ILU decomposition for algebraic positive definite systems.* In Proceed. of the Eight GAMM - Seminar, Notes on Numerical Fluid Mech., vol. 41, Vieweg, Braunschweig, 1993).
15. C. Popa, *Mest independence of the condition number of discrete Galerkin systems by preconditioning.* To appear in Intern, J. Computer Math., London, 1993.

16. C. Popa, *A coarsening algorithm for the algebraic method.* To appear in Studii şi Cerc. Matem., Romanian Academy of Sciences, 1994.
17. J. Ruge, K. Stuben, *Algebraic multigrid (AMG).* In "Multigrid Methods", S. McCormick eed., SIAM, 1987.
18. K. Stuben, U. Trottenberg, *Multigrid methods. Fundamental algorithms, model problem analysis and apllications.* In Lecture Notes in Math., vol. 960, Springer, Berlin, 1982.
19. R. Varga, *Matrix iterative analysis.* Pretince - Hall, New Jersey, 1962.
20. H. Yserentant, *Hierarchical bases of finite - elements spaces in the discretization of nonsymmetric elliptic b.v.p.* Computing, vol. 35, pp. 39-49, 1985.