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A NUMERICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF m-TH ORDER USING SPLINE FUNCTIONS

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REZUMAT. - O soluţie numerică pentru ecuaţia diferenţială de ordinul *m* folosind funcţii spline. Se construieşte un procedeu numeric folosind funcţii spline polinomiale pentru rezolvarea unei clase de ecuaţii diferenţiale neliniare de ordin m cu condiţii iniţiale. Se estimează eroarea şi se investighează stabilitatea metodei propuse.

I. Introduction. In the last years, the problem of approximating the solution of non linear differential equations by spline functions has been of growing interest. Many authors [1]-[6] have proposed various methods to approximate the solution by means of spline.

Recently, J. Gyorvari and Cs. Mihalyko [3] gave a spline algorithm to solve numerically a differential equation with initial conditions. In this paper, using the idea of T. Fawzy in [1], [2] an improved algorithm is constructed using spline functions and in addition, the stability of the proposed method is given.

Consider the differential equation with initial condition

$$z^{(m)}(x) = f(x, z(x), z'(x), \dots, z^{(m-1)}(x)),\ x \in [0,b],\ b > 0 \tag{1.1}$$

$$z^{(j)}(0) = z_0^{(j)}, \quad j = \overline{0, m-1}$$

where $f \in C^r([0,b] \times \mathbf{R}^r)$ and $r \in N$.

We assume that f satisfies the following Lipschitz conditions

$$|f^{(q)}(x,u) - f^{(q)}(x,v)| \le L_f \|u - v\| \tag{1.2}$$

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$x \in [0,b],\ \ u,v \in \mathbf{R}^r,\ q = \overline{0,r}$

The differential equation (1.1) can be reduced to a system of m differential equations of first degree as follows:

One denote: $y_0(x) := z(x),\ y_1(x) := z'(x), \ldots, y_{m-1}(x) := z^{(m-1)}(x)$

Then (1.1) is equivalent to

$$y'(x) = F(x, y(x)),\ \ x \in [0,b] \tag{1.3}$$

$$y = (y_0, \ldots, y_{m-1}) : [0,b] \rightarrow \mathbf{R} \text{ and}$$

$$F(x, y(x)) = (y_1(x), \ldots, y_{m-1}(x), f(x, y(x))).$$

One have $F^{(q)}(x,y(x)) = (y_1^{(q)}(x), \ldots, y_{m-1}^{(q)}(x), f^{(q)}(x,y(x)))$ so the Lipschitz conditions for f holds for F too:

$$\| F^{(q)}(x,u) - F^{(q)}(x,v) \| \leq L \| u - v \| \tag{1.4}$$

$x \in [0,b],\ \ u,v \in \mathbf{R}^r,\ q = \overline{0,r}.$

One consider for the system (1.3) the initial conditions

$$y(0) = y_0$$

On $[0,b]$ we define an uniform partition by the knots

$$\Delta : 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \quad n \in \mathbf{N}^*$$

with the step $h = x_{k+1} - x_k,\ \ k = \overline{0,n-1}$ and one denote $y_k^{(j)} = y^{(j)}(x_k),\ k = \overline{0,n},\ j = \overline{0,r}.$

II. The first approximation process. Let y be the exact solution of Cauchy problem for the system (1.3). By integrating from x_k to x we get

$$y(x) = y_k + \int_{x_k}^{x} F(t, y(t))dt,\ \ x \in [x_k, x_{k+1}] \tag{2.1}$$

and for $x := x_{k+1}$ we get

$$y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} F(t, y(t))dt \tag{2.2}$$

This equality may be approximated with

$$\bar{y}_{k+1} = \bar{y}_k + \int_{x_k}^{x_{k+1}} F(t, y_k^*(t))dt \tag{2.3}$$

where

$$y_k^*(t) = \sum_{j=0}^{r+1} (t - x_k)^j \frac{\bar{y}_k^{(j)}}{j!}, \; t \in [x_k, x_{k+1}] \tag{2.3}$$

which corresponds to the Taylor expansion:

$$y(t) = \sum_{j=0}^{r} (t - x_k)^j \frac{y_k^{(j)}}{j!} + \frac{y^{(r+1)}(\xi_k)}{(r+1)!}(t - x_k)^{r+1}, \tag{2.4}$$

$t \in [x_k, x_{k+1}]$, $x_k < \xi_k < x_{k+1}$.

Now, we assume that the function f has the modulus of continuity $\omega_r(h)$ associated to the above defined mesh of points.

One will also use: $\bar{y}_0 = y_0$, $\bar{y}_0' = y_0'$, ..., $\bar{y}_0^{(r+1)} = y_0^{(r+1)}$.

LEMMA 2.1 *The inequality*

$$\| y_{k+1} - \bar{y}_{k+1} \| \le \| y_k - \bar{y}_k \| (1 + c_0 h) + c_1 \omega_r(h) h$$

holds for $k = \overline{0, n-1}$, *where* c_0 *and* c_1 *are positive and independent of* h.

Proof.

$$\| y_{k+1} - \bar{y}_{k+1} \| \le \| y_k - \bar{y}_k \| + L \int_{x_k}^{x_{k+1}} \| y(t) - y_k^*(t) \| \, dt \le$$

$$\le \| y_k - \bar{y}_k \| + L \int_{x_k}^{x_{k+1}} \| \sum_{j=0}^{r} \frac{y_k^{(j)}}{j!}(t - x_k)^j + \frac{y^{(r+1)}(\xi_n)}{(r+1)!}(t - x_k)^{r+1} + \sum_{j=0}^{r+1} \frac{y_k^{(j)}}{j!}(t - x_k)^j \| \, dt \le$$

$$\le \| y_k - \bar{y}_k \| + L \sum_{j=0}^{r+1} \frac{\| y_k^{(j)} - \bar{y}_k^{(j)} \|}{(j+1)!} h^{j+1} + L \frac{h^{r+2}}{(r+2)!} \omega_r(h) = \| y_k - \bar{y}_k \| + Lh \| y_k - \bar{y}_k \| +$$

$$+ L \sum_{q=0}^{r} \frac{\| F^{(q)}(x_k, y_k) - F^{(q)}(x_k, \bar{y}_k) \|}{(q+2)!} h^{q+2} + L \frac{h^{r+2}}{(r+2)!} \omega_r(h) =$$

$$= \| y_k - \bar{y}_k \| + \| y_k - \bar{y}_k \| \cdot Lh + \| y_k - \bar{y}_k \| \cdot L^q \sum_{q=0}^{r} \frac{h^{q+2}}{(q+2)!} + \frac{L}{(r+2)!} \omega_r(h) h^{r+2} \le$$

$$\leq \|y_k - \bar{y}_k\| (1 + c_0 h) + c_1 \omega_r(h) h^{r+2}.$$

THEOREM 2.2 *The convergence of the approximate value* $\bar{y}_{k+1}$ *to the exact value* y_{k+1} *is given by the inequality*

$$\|y_{k+1} - \bar{y}_{k+1}\| \leq c_3 \omega_r(h) h^{r+1}.$$

Proof. One apply succesively Lemma 2.1:

$$\|y_{k+1} - \bar{y}_{k+1}\| \leq \|y_k - \bar{y}_k\| \cdot (1 + c_0 h) + c_1 \omega_r(h) h^{r+2}$$

$$\|y_{k+1} - \bar{y}_{k+1}\| (1 + c_0 h) \leq \|y_k - \bar{y}_k\| \cdot (1 + c_0 h)^2 + c_1 \omega_r(h) h^{r+2} (1 + c_0 h)$$

$$\|y_{k+1} - \bar{y}_{k+1}\| (1 + c_0 h)^k \leq \|y_k - \bar{y}_k\| \cdot (1 + c_0 h)^{k+1} + c_1 \omega_r(h) h^{r+2} (1 + c h)^k.$$

Adding the inequalities above one obtain

$$\|y_{k+1} - \bar{y}_{k+1}\| \leq c_1 \omega_r(h) h^{r+2} \sum_{q=0}^{k} (1 + c_0 h)^2 = c_1 \omega_r(h) h^{r+2} \frac{(1 + c_0 h)^{k+1} - 1}{c_0 h}.$$

Because $(1 + c_0 h)^{k+1} = \left(1 + \frac{b c_0}{n}\right)^{k+1} \leq \left(1 + \frac{b c_0}{n}\right)^n \leq e^{b c_0} =$ cosnstant, $(1 + c_0 h)^{k+1}$ is bounded, so $\|y_{k+1} - \bar{y}_{k+1}\| \leq c_3 \omega_r(h) h^{r+1}$.

THEOREM 2.3 *The error for* $\bar{y}_{k+1}^{(q+1)}$ *is given by the inequality*

$$\|y_{k+1}^{(q+1)} - \bar{y}_{k+1}^{(q+1)}\| c_4 \omega_r(h) h^{r+1}, \quad q = \overline{0, r}$$

Proof. $\|y_{k+1}^{(q+1)} - \bar{y}_{k+1}^{(q+1)}\| = \|F^{(q)}(x_{k+1}, y_{k+1}) - F^{(q)}(x_{k+1}, \bar{y}_{k+1})\| \leq$

$$\leq L \cdot \|y_{k+1} - \bar{y}_{k+1}\| \leq c_4 \omega_r(h) h^{r+1}.$$

So, one obtained the approximative values $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n \in \mathbb{R}^r$ corresponding to the mesh of points $0 = x_0 < x_1 < \ldots < x_n = b$.

In x_k one obtained the following approximations for the solution of (1.3): $y_k^{(q)} = (y_{k,1}^{(q)}, y_{k,2}^{(q)}, \ldots, y_{k,m}^{(q)})$ for $y_k^{(q)}$, $q = \overline{0, r+1}$ which correspond in (1.1) to $(z, z', \ldots, z^{(m-1)})$.

One denote: $\bar{z}_k := \bar{y}_{k,1}$, $\bar{z}'_k := \bar{y}_{k,2}, \ldots,$ $\bar{z}_k^{(m-1)} := \bar{y}_{k,m}$, $\bar{z}_k^{(m)} := \bar{y}'_{k,m}$, $\bar{z}_k^{(r+m+1)} := \bar{y}_{k,m}^{(r+1)}$

THEOREM 2.4 *The convergence of the approximative value* $\bar{z}_{k+1}^{(j)}$ *to the exact value* $z_{k+1}^{(j)}$ *is given by the inequality*

$$|z_{k+1}^{(j)} - \bar{z}_{k+1}^{(j)}| \le c_5 \omega_r(h) h^{r+1}, \; j = \overline{0, r+m+1}$$

Proof. This is a direct consequence of Theorems 2.1 and 2.3.

III. The second approximation process. One obtain the following sets of approximate values:

$$\bar{Z}^{(q)}: \bar{z}_0^{(q)}, \ldots, \bar{z}_n^{(q)}, \; q = \overline{0, r+m}$$

which correspond respectively to

$$Z^{(q)}: z_0^{(q)}, z_1^{(q)}, \ldots, z_n^{q)}, \; q = \overline{0, r+m}$$

We are going to construct a spline function S_Δ interpolated to the set $\bar{Z}$ on the mash Δ and approximating the solution of (1.1).

THEOREM 3.1 *For a given mesh of points*

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b, \; x_{k+1} - x_k = h, \; k = \overline{0, n-1}$$

and for the given sets of values $\bar{Z}^{(q)}: \bar{z}_0^{(q)}, \bar{z}_1^{(q)}, \ldots, \bar{z}_n^{(q)}, \; q = \overline{0, r+m}$ *there is a unique spline function* S_Δ *interpolated to the set* $\bar{Z}$ *on the mesh and satisfying the following conditions:*

(i) $\quad S_\Delta(\bar{z}, x) = S_\Delta(x) \in C^{r+m}\,[0, b].$

(ii) $\quad S_k^{(q)}(x_k) = \bar{z}_k^{(q)}$ *for* $q = \overline{0, r+m}, \; k = \overline{0, n}$

(iii) $\quad$ For $x_k \le x \le x_{k+1}, \; k = \overline{0, n-1}$

$$S_\Delta(x) = \sum_{j=1}^{r+m} \frac{\bar{z}_k^{(j)}}{j!}(x - x_k)^j + \sum_{p=1}^{r+m+1} a_p^{(k)} (x - x_k)^{p+r+m}.$$

Proof. From the continuity condition (i), for $x = x_{k+1}$, using (ii) we get

$$S_k^{(i)}(x_{k+1}) = S_{k+1}^{(i)}(x_{k+1}) = \bar{z}_{k+1}^{(i)}. \tag{3.1}$$

Substituting from (3.1) in (iii) we get the following linear system of equations:

$$\sum_{p=1}^{r+m+1} t!\,C^{t}_{r+m+p}\,a^{(k)}_{p}\,h^{p-1} = h^{t-r-m-1}\left(\overline{z}^{(t)}_{k+1} - \sum_{j=0}^{r+m-t}\frac{\overline{z}^{(j+t)}_{n}}{j!}h^{j}\right),\quad t = \overline{0, r+m} \tag{3.2}$$

for the unknowns $a^{(k)}_{p}$, $p = \overline{1, r+m+1}$. One denote

$$F^{(k)}_{t} = h^{t-r-m-1}\left(\overline{z}^{(t)}_{k+1} - \sum_{j=0}^{r+m-1}\frac{\overline{z}^{(j+t)}_{k}}{j!}h\right). \tag{3.3}$$

The system (3.2) has always (for $h = 0$) a unique solution because its determinant is

$$D_r = \begin{vmatrix} 1 & & h^{p-1} & & h^{r+m} \\ C^{1}_{r+m+1}\cdot 1! & & C^{1}_{r+m+p}\cdot 1!\,h^{p-1} & & C_{2r+2m+1}\cdot 1!\,h^{r+m} \\ C^{2}_{r+m+1}\cdot 2! & & C^{2}_{r+m+p}\cdot 2!\,h^{p-1} & & C^{2}_{2r+2m+1}\cdot 2!\,h^{r+m} \\ & & & & \\ C^{r+m}_{r+m+1}(r+m)! & \dots & C^{r+m}_{r+m+p}(r+m)!\,h^{p-1} & \dots & C^{r+m}_{2r+2m+1}(r+m)!\,h^{r+m} \end{vmatrix}$$

$$\prod_{t=0}^{r+m} t!\,h^{1+2+\dots+(r+m)}\,1 = h^{\frac{1}{2}(r+m)(r+m+1)}\prod_{t=0}^{r+m} t! \neq 0\,.$$

So $D_r \neq 0$ and the system (3.2) has always a unique solution for $h > 0$ i.e. the spline function approximating the solution of (1.1) exists and is unique determined.

The coefficients are determined as follows.

One replace the column p in D_r by the column

$$(F^{(k)}_{0}, F^{(k)}_{1}, \dots, F^{(k)}_{r+m})$$

and we denote the determinant obtained by D^{P}_{r}. Then, the solution of system (3.2) will be

$$a^{(k)}_{p} = \frac{D^{p}_{r}}{D_r},\quad p = \overline{1, r+m+1}.$$

By factorising D^{P}_{r} in terms of $F^{(k)}_{0}, \dots, F^{(k)}_{r+m}$ we get

$$a^{(k)}_{p} = \frac{1}{h^{p-1}}\sum_{i=0}^{r+m} cp_i F^{(k)}_{i} \tag{3.4}$$

where $1/h^{p-1}$ is a factor put in front of the sum so the coefficients c_{pi} be independent of h.

Now we shall discuss the convergence of the spline function to the solution.

LEMMA 3.2 *The inequalities* $|a^{(k)}_{p}| \leq \frac{A_p}{h^{p}}\,\omega_r(h)$ *hold* $p = \overline{1, r+m+1}$ *where* A_p *are* *constants independent of h.*

Proof. One estimate

$$|F_t^{(k)}| = h^{t-r-m-1} \left| \bar{z}_{k+1}^{(t)} - \sum_{j=0}^{r+m-t} \frac{\bar{z}_k^{(j+t)}}{j!} h^j \right|.$$

One have the following Taylor expansion for $z^{(t)}(x)$, for $x_k \le x \le x_{k+1}$.

$$z^{(t)}(x) = \sum_{j=0}^{r+m-1-t} \frac{z_k^{(j+t)}}{j!}(x-x_k)^j + \frac{z^{(r+m)}(\xi_{kt})}{(r+m-t)!}(x-x_k)^{r+m-t},\ t = \overline{0,r+m}.$$

and for $x = x_{k+1}$:

$$z_{k+1}^{(t)} = \sum_{j=0}^{r+m+1-t} \frac{z_k^{(j+t)}}{j!} h^j + \frac{z^{(r+m)}(\xi_{kt})}{(r+m-t)!} h^{r+m-t},\ t = \overline{0,r+m}.$$

Using (3.5) and the t-th equation in the system (3.2) we get

$$|F_k^{(t)}| \le h^{t-r-m-1} \left[|z_{k+1}^{(t)} - \bar{z}_{k+1}^{(t)}| + \sum_{j=0}^{r+m-t} \frac{|z_k^{(j+t)} - \bar{z}_k^{(j+t)}|}{j!} h^j \right.$$

$$\left. + \frac{|z^{(r+m)}(\xi_{k,t}) - z^{(r+m)}|}{(r+m+t)!} \right] \le h^{t-r-m-1}\, [c_t^* \omega_r(h)\, h^{r+m-t}],$$

with $c_t^* > 0$, $t = \overline{0,r+m}$, independent of h, so

$$F_t^{(k)} \le c_t^* \frac{\omega_r(h)}{h},\ t = \overline{0,r+m}$$

One substitute (3.6) in (3.4) and one obtain

$$a_p^{(k)} = \frac{1}{h^{p-1}} \sum_{i=0}^{r+m} c_i F_i^{(k)} \le \frac{1}{h^{p-1}} \sum_{i=0}^{r+m} c_i c_i^* \omega_r(h) \frac{1}{h} =$$

$$= \frac{1}{h^p} \omega_r(h) \sum_{i=0}^{r+m} c_i\ c_i^* = A_p \frac{\omega_r(h)}{h^p},$$ where $A_p := \sum_{i=0}^{r+m} c_i\ c_i^*$ is a cosstant independent of h.

THEOREM 3.3 *Let z be the exact solution of* (1.1). *If S_Δ is the spline function constructed in Theorem* 3.1 *then there exists a constant E independent of h for which the inequalities*

$$|z^{(q)}(x) - S_\Delta^{(q)}(x)| \le E\,\omega_r(h)\, h^{r+m-q},\ q = \overline{0,r+m}$$

hold for any $x \in [0,b]$.

Proof. Using the Taylor expansion previously constructed for $z^{(q)}(x)$ and condition (iii) in Theorem 3.1 we get

$$|z^{(q)}(x) - S_\Delta^{(q)}(x)| = |\sum_{j=0}^{r+m+1-q} \frac{z_k^{(j+q)}}{j!}(x-x_k)^j + \frac{z^{(r+m)}(\xi_{kq})}{(r+m-q)!}(x-x_k)^{r+m-q} -$$

$$- \sum_{j=0}^{r+m+1-q} \frac{\bar{z}_k^{(j+q)}}{j!}(x-x_k)^j - \frac{\bar{z}_k^{(r+m)}}{(r+m-q)!}(x-x_k)^{r+m-q} - \sum_{p=1}^{r+m+1-q} q!\, c_{p+r+m}^{q}\, a_p^{(k)}(x-x_k)^{(p+r+m-q}|$$

$$\leq \sum_{j=0}^{r+m+1-q} \frac{|z_k^{(j+q)} - \bar{z}_k^{(j+q)}|}{j!} h^j + \frac{|z^{(r+m)}(\xi_{k,q} - \bar{z}_k^{(r+m)}|}{(r+m-2)!} h^{r+m-q} + \sum_{p=1}^{r+m-q-1} q!\, c_{p+r+m}^{q}\, a_p^{(q)} h^{p+r+m-q} \leq$$

$$\leq c_q^{**}\omega_r(h)\, h^{r+m-q}.$$

Taking $E = \max\{c_q^{**};\ q = \overline{0, m+r}\}$, the theorem is proved.

THEOREM 3.4 *If we denote by* $S_\Delta^{(m)}$ *the function* $s_\Delta^{(m)}(x) = f(x, S_\Delta(x), S_\Delta'(x), \ldots, S_\Delta^{(m-1)}(x))$, $x \in [0,b]$ *and if* $\bar{S}_\Delta$ *is the spline function defined in Theorem* 3.1 *then for any* $x \in [0,b]$

$$|\bar{S}_\Delta^{(m)}(x) - S_\Delta^{(m)}(x)| \leq M\omega_r(h)\, h^r.$$

where M is a positive constant independent of h (i.e. *the spline function verifies the equation while* $n \to \infty$ *or* $h \to 0$).

Proof. $|\bar{S}_\Delta^{(m)}(x) - S_\Delta^{(m)}(x)| \leq |\bar{S}_\Delta^{(m)}(x) - z^{(m)}(x)| + |x^{(m)}(x) - S_\Delta^{(m)}(x)| =$

$$= |f(x, S_\Delta(x), \ldots, S_\Delta^{(m-1)}(x)) - f(x, z(x), \ldots, z^{(m-1)}(x))| + |z^{(m)}(x) - S_\Delta^{(m)}(x)| \leq$$

$$\leq LK|S_\Delta(x) - z(x)| + LK|S'(x) - z'(x)| + \ldots +$$

$$+ LK|S_\Delta^{(m-1)}(x) - z^{(m-1)}(x)| + |z^{(m)}(x) - S_\Delta^{(m)}(x)| \leq$$

$$\leq LKE\omega_r(h)h^{r+m} + LKE\omega_r(h)h^{r+m-1} + \ldots + LKE\omega_r(h)h^{r+1} + E\omega_r(h)h^r =$$

$$= (LKEh^m + LKEh^{m-1} + \ldots + LKEh + E)h^r\omega_r(h) \leq M\omega_r(h)h^r,$$

where $M > 0$ is independent of h.

Remark. If $f \in C^\infty([0,b] \times \mathbf{R}^m)$, as the error is $O(h^{r+m})$ we may choose $r \in \mathbf{N}$

suitable so that the method is available.

IV. The stability of the method. A change in one of the calculated values from $\bar{y}_k$ to $\bar{u}_k$ will lead us to solve

$$\bar{u}_{i+1} = \bar{u}_i + \int_{x_i}^{x_{i+1}} F(t, u_i^*(t))\, dt. \tag{4.1}$$

Let $\varepsilon_k := \|\bar{u}_k - \bar{y}_k\|$, the introduced error.

THEOREM 4.1 *If any of the calculated values $\bar{y}_k$ is changed into $\bar{u}_k$ then the inequality $\|u_i^{(t)} - y_i^{(t)}\| \le c_8 \varepsilon_k$ holds for any $i = \overline{k+1, n}$ and $t = \overline{0, r+1}$.*

Proof. Substracting (2.3) from (4.1) and proceeding as in the proof of Lemma 2.1 we get

$$\varepsilon_{i+1} \le \varepsilon_i(1 + c_6 h) \le (1 + c_6 h)^{i-k} \varepsilon_k \le e^{c_6 b} \varepsilon_k \le c \varepsilon_k$$

where c is independent of h. Also, for $q = \overline{0, r}$ we get

$$\|\bar{u}_i^{(q+1)} - \bar{y}_i^{(q+1)}\| = \|F^{(q)}(x_i, \bar{u}_i) - F^{(q)}(x_i, \bar{y}_i)\| \le L\|\bar{u}_i - \bar{y}_i\| \le Lc\varepsilon_k \le c_7 \varepsilon_k$$

so

$$\|u_i^{(t)} - y_i^{(t)}\| \le c_8 \varepsilon_k, \quad t = \overline{0, r+1}.$$

As we did in paragrapf II., we shall denote

$$\bar{v}_k := \bar{u}_{k,1},\ \bar{v}_k' := \bar{u}_{k,2}, \ldots, \bar{v}_k^{(m-1)} := \bar{u}_{k,m},\ \bar{v}_k^{(m)} := \bar{u}_{k,m}',\ \bar{v}_k^{(r+m+1)} := \bar{u}_{k,m}^{(r+m)}$$

So

$$|\bar{v}_i^{(t)} - \bar{z}_i^{(t)}| \le \|\bar{u}_i - \bar{y}_i\| \le c_8 \varepsilon_k \quad \text{for} \quad t = \overline{0, m-1}$$

$$|\bar{v}_i^{(t)} - \bar{z}_i^{(t)}| \le \|\bar{u}_i^{(m-1-t)} - \bar{y}_i^{(m-1-t)}\| \le c_8 \varepsilon_k \quad \text{for} \quad t = \overline{m, m+r+1}.$$

and thus the theorem is proved.

THEOREM 4.2 *If any of the calculated values $\bar{y}_k$ is changed into $\bar{u}_k$ and consequently, the spline function approximating the solution of (1.1) is changed from S into* *s, then for any* $x \in [x_i, x_{i+1}]$, $i = \overline{k, n-1}$, *the inequality*

$$|s_i(x) - S_i(x)| \le c_{10}\varepsilon_k \text{ holds.}$$

Proof. Consider the interval $[x_i, x_{i+1}]$ where $i = \overline{k, n+1}$. Then, analogously to the spline function S_Δ introduced Theorem 3.1, the new spline function due to the variation of $\bar{y}_k$ to $\bar{u}_k$ will be

$$s_i(x) = \sum_{j=0}^{r+m} \frac{\bar{v}_i^{(j)}}{j!}(x - x_i)^j + \sum_{p=1}^{r+m+1} b_p^{(i)}(x - x_i)^{p+r+m} \tag{4.3}$$

and will satisfy the conditions

$$s_i^{(t)}(x_{i+1}) = s_{i+1}^{(t)}(x_{i+1}) = \bar{v}_{i+1}^{(t)},\ s_{n-1}^{(t)}(x_n) = \bar{v}_n^{(t)} \tag{4.4}$$

for $i = \overline{k, n-2}$.

Then the linear system corresponding to (3.2) will be

$$\sum_{p=1}^{r+m+1} t!\, C_{r+m+p}^{t}\, b_p^{(i)} h^{p-1} = G_t^{(i)},\ t = \overline{0, r+m} \tag{4.5}$$

where

$$G_t^{(i)} = h^{t-r-m-1}\left(\bar{v}_{i+1}^{(t)} - \sum_{j=0}^{r+m-t} \frac{\bar{v}_i^{(j+t)}}{j!} h^j\right),\ t = \overline{0, r+m} \tag{4.6}$$

and corresponding to (3.4) we get

$$b_p^{(i)} = \frac{1}{h^{p-1}} c_{pt} G_t^{(i)}. \tag{4.7}$$

$$|s_i(x) - S_i(x)| = |\sum_{j=0}^{r+m} \frac{\bar{v}_i^{(j)}}{j!}(x - x_i)^j + \sum_{p=1}^{r+m+1} b_p^{(i)}(x - x_i)^{p+r+m} -$$

$$\sum_{j=0}^{r+m} \frac{\bar{z}_i^{(j)}}{j!}(x - x_i)^j - \sum_{p=1}^{r+m+1} a_p^{(i)}(x - x_i)^{p+r+m}| \le$$

$$\le \sum_{j=0}^{r+m} \frac{|\bar{v}_i^{(j)} - \bar{z}_i^{(j)}|}{j!} h^j + \sum_{p=1}^{r+m+1} |b_p^{(i)} - a_p^{(i)}|\, h^{p+r+m}.$$

From (3.4) and (4.7) we get

$$|b_p^{(i)} - a_p^{(i)}| \le \frac{1}{h^{p-1}} \sum_{t=0}^{r+m} c_{pt}\, |G_t^{(i)} - F_t^{(i)}|$$

From (3.3) and (4.6) we get

$$|G_i^{(t)} - F_i^{(t)}| = h^{t-r-m-1} \left|\bar{v}_{i+1}^{(t)} - \sum_{j=0}^{r+m-t} \frac{\bar{v}_i^{(j+t)}}{j!} h^j - \bar{z}_{i+1}^{(t)} + \sum_{j=0}^{r+m-t} \frac{\bar{z}_i^{(j+t)}}{j!} h^j\right| \le$$

$$\le h^{t-r-m-1} |\bar{v}_{i+1}^{(t)} - z_{i+1}^{(t)}| + h^{t-r-m-1} \sum_{j=0}^{r+m-t} \frac{|\bar{v}_i^{(j+t)} - \bar{z}_i^{(j+t)}|}{j!} h^j \le$$

$$\le h^{t-r-m-1} \left(c_8 \varepsilon_k + \sum_{j=0}^{r+m-t} c_8 \varepsilon_k \frac{h^j}{j!} \right) \le c_9 \varepsilon_k h^{t-r-m-1}$$

and so we get

$$|b_p^{(i)} - a_p^{(i)}| \le \frac{1}{h^{p-1}} \sum_{t=0}^{r+m} c_9 c_{pt} \varepsilon_k h^{t-r-m-1} .$$

Using Theorem 4.1 we get

$$|s_i(x) - S_i(x)| \le \sum_{j=0}^{r+m} c_8 \varepsilon_k \frac{h^j}{j!} + \sum_{p=1}^{r+m+1} h^{p+r+m} \frac{1}{h^{p-1}} \sum_{t=0}^{r+1} c_9 c_{pt} \varepsilon_k h^{t-r-m-1} =$$

$$= c_8 \varepsilon_k \frac{h^j}{j!} + c_9 \varepsilon_k \sum_{p=1}^{r+m+1} \sum_{t=0}^{r+1} c_{pt} h^t \le c_{10} \varepsilon_k$$

which is a bounded multiple of the introduced error.

THEOREM 4.3 *Under the assumptions of Theorem* 4.2 *the inequalities*

$$|s_i^{(t)}(x) - S_i^{(t)}(x)| \le c_{11} \varepsilon_k$$

hold for any $t = \overline{0,m}$ *and* $i = \overline{k, n-1}$.

Proof. Following the same procedure as in Theorem 4.2 one obtain the requested inequalities.

Conclusion. As any variation of the calculated error is a bounded multiple of the introduced error, the method is stable.

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