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COMMON FIXED POINTS OF COMPATIBLE MAPPINGS OF TYPE (A)

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REZUMAT. - Puncte fixe comune pentru funcţii compatibile de tipul (A). În această lucrare vom da unele teoreme de punct fix pentru funcţii compatibile de tipul (A) extinzând unele rezultate din [4]-[7].

Abstract. In this paper, we give some common fixed point theorems for compatible mappings of type (A) extinding some results from [4] - [7].

Rhoades [8] summarized contractive mappings of some types and discussed on fixed points. Wang, Li Gao and Iseki [10] proved some fixed point theorems on expansion mappings, which correspond some contractive mappings. Rhoades [9] generalized the results for pairs of mappings. Recently, Popa [4] -[7] proved some theorems on unique fixed point for expansion mappings.

The purpose of this paper is to prove some fixed point theorems on expansion mappings extending some results from [4], [5], [6] and [7] for compatible mappings of type (A).

Let R_+ be the set all non-negative reals numbers and $\psi: R_+^3 \to R_+$ be a real function. Throughout this paper, (X,d) denotes a metric space.

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DEFINITION 1. $\psi: R_+^3 \to R_+$ satisfies property (h) if there exists $h \geq 1$ such that for every $u,v \in R_+$ with $u \geq \psi(v,u,v)$ or $u \geq \psi(v,v,u)$, we have $u \geq hv$.

DEFINITION 2 ([6]). $\psi: R_+^3 \to R_+$ satisfies property (u) if $\psi(u,0,0) > 0$, $u > 0$.

DEFINITION 3 ([1]). Let $S,T: (X,d) \to (X,d)$ be mappings, S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

DEFINITION 4 ([2]). Let $S,T: (X,d) \to (X,d)$ be mappings, S and T are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0 \text{ and } \lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Remark. By ex. 2.1 of [2], it follows that the notions of "compatible mappings" and "compatible mappings of type (A)" are independent.

LEMMA 1 ([2)]. *Let $S, T: (X,d) \to (X,d)$ be compatible mappings of type* (A). *If one of S and T is continuous, then S and T are compatible.*

LEMMA 2 ([1]). *Let S and T be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Then $\lim_{n\to\infty} TSx_n = St$ if S is continuous.*

LEMMA 3 ([2]). *Let $S, T: (X,d) \to (X,d)$ be mappings. If S and T are compatible of type* (A) *and $S(t) = T(t)$ for some $t \in X$, then $ST(t) = TT(t) = SS(t) = TS(t)$.*

LEMMA 4 ([7]). *Let (X,d) be a metric space, A, B, S, T four mappings of X satisfying the inequality*

$$d(Ax, By) \geq \psi(d(Sx, Ty), d(Ax, Sx), d(By, Ty)) \tag{1}$$

for all x,y in X, where ψ *satisfies property* (u). *Then A, B, S, T have at most one common fixed point.*

THEOREM 1. *Let A, B, S and T be mappings from a complete metric space* (X,d) *into itself satisfying the conditions:*

(1°) *A and B are surjective,*

(2°) *One of A, B, S, T is continuous,*

(3°) *A and S as well B and T are compatible of type* (A),

(4°) *The inequality* (1) *holds for all x,y in X, where* ψ *satisfied property* (h) *with* $h > 1$.

If property (u) *holds and* ψ *is continuous, then A, B, S and T have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. By (1°) we choose a point x_1 in X such that $Ax_1 = Tx_0 = y_0$ and for this point x_1, there exists a point x_2 in X such that $Bx_2 = Sx_1 = y_1$. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$Ax_{2n+1} = Tx_{2n} = y_{2n} \text{ and } Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}. \tag{2}$$

By (1) and (2) we have

$$d(y_0,y_1) = d(Ax_1,Bx_2) \geq \psi(d(Sx_1,Tx_2),d(Sx_1,Ax_1),d(Tx_2,Bx_2))$$
$$= \psi(d(y_1,y_2),d(y_1,y_0),d(y_2,y_1)).$$

Then by property (h), we have

$$d(y_0,y_1) \geq h \cdot d(y_2,y_1), \text{ where } h > 1.$$

Thus $d(y_2,y_1) \leq \frac{1}{h} d(y_0,y_1)$. Similarly, we have

$$d(y_n,y_{n+1}) \leq \left(\frac{1}{h}\right)^n \cdot d(y_0,y_1).$$

Then by a routine calculation we can show that $\{y_n\}$ is a Cauchy sequence and since X is complete, there is a $z \in X$ such that $\lim y_n = z$. Consequently, the subsequences $\{Ax_{2n+1}\}$, $\{Bx_{2n}\}$, $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ converges to z.

Now, suppose that A is continuous. Since A and S are compatible of type (A) and A is continuous by Lemma 1 A and S are compatible. Lemma 2 implies $A^2x_{2n+1} \to Az$ and $SAx_{2n+1} \to Az$ as $n \to \infty$. By (1), we have

$$d(A^2x_{2n+1}, Bx_{2n}) \geq \psi(d(SAx_{2n+1}, Tx_{2n}), d(SAx_{2n+1}, A^2x_{2n+1}), d(Tx_{2n}, Bx_{2n})).$$

Letting n tend to infinity we have by continuity of ψ

$$d(Az, z) \geq \psi(d(Az, z), 0, 0).$$

By property (u) follows $d(Az, z) > d(Az, z)$ if $Az \neq z$. Thus $z = Az$. By (1) we have

$$d(Az, Bx_{2n}) \geq \psi(d(Sx, Tx_{2n}), d(Sz, Az), d(Tx_{2n}, Bx_{2n})).$$

Letting n tend to infinity we have by continuity of ψ

$$0 \geq d(Az, z) \geq \psi(d(Sz, z), d(Sz, z), 0).$$

By definition (1) we have $0 \geq h\, d(Sz,z)$ which implies $z = Sz$. Let $z = Bu$ for some $u \in X$. Then we have by (1)

$$d(A^2x_{2n+1}, Bu) \geq \psi(d(SAx_{2n+1}, Tu), d(SAx_{2n+1}, A^2x_{2n+1}), d(Tu, Bu)).$$

Letting n tend to infinity we have by continuity of ψ

$$0 = d(Az, Bu) \geq \psi(d(Az, Tu), 0, d(Tu, Bu)) = \psi(d(z, Tu), 0, d(z, Tu)).$$

By definition (1) we have $0 \geq h \cdot d(z, Tu)$ which implies $z = Tu$. Since B and T are compatible of type (A) and $Bu = Tu = z$ Lemma 3 $Bz = BTu = TBu = Tz$, moreover by (1), we have

$$d(Ax_{2n+1}, Bz) \geq \psi(d(Sx_{2n+1}, Tz), d(Sx_{2n+1}, Ax_{2n+1}), d(Tz, Bz)).$$

Letting n tend to infinity we have by continuity of ψ

$$d(z, Tz) \geq \psi(d(z, Tz), 0, 0).$$

From property (u) it follows that $d(z, Tz) > d(z, Tz)$ if $z \neq Tz$. Thus $z = Tz$. Therefore, z is a common fixed point of A, B, S, T. Similarly, we can complete the proof in the case of the continuity of B.

Next, suppose that S is continuous. Since A and S are compatibly of type (A) and S is continuous by Lemma 1 A and S are compatible. Lemma 2 implies $S^2x_{2n+1} \to Sz$ and $ASx_{2n+1} \to Sz$ as $n \to \infty$. By (1), we have

$$d(ASx_{2n+1}, Bx_{2n}) \geq \psi(d(S^2x_{2n+1}, Tx_{2n}), d(S^2x_{2n+1}, ASx_{2n+1}), d(Tx_{2n}, Bx_{2n})).$$

Letting n tend to infinity we have by continuity of ψ

$$d(Sz, z) \geq \psi(d(Sz, z), 0, 0).$$

By property (u) we have $d(Sz, z) > d(Sz, z)$ if $z \neq Sz$. Thus $z = Sz$. Let $z = Av$ and $z = Bw$ for some v and w in X, respectively. Then by (1) we have

$$d(ASx_{2n+1}, Bw) \geq \psi(d(S^2x_{2n+1}, Tw), d(S^2x_{2n+1}, ASx_{2n+1}), d(Tw, Bw))$$

Letting n tend to infinity we have by continuity of ψ

$$0 = d(Sz, z) \geq \psi(d(Sz, Tw), 0, d(Bw, Tw)) = \psi(d(z, Tw), 0, d(z, Tw)).$$

By Definition (1) we have $0 \geq h \cdot d(z, Tw)$ which implies $z = Tw$. Since B and T are compatible of type (A) and $Bw = Tw = Tz$ by Lemma 3 $Bz = BTw = TBw = Tz$. Moreover, by (1), we have

$$d(Ax_{2n+1}, Bz) \geq \psi(d(Sx_{2n+1}, Tz), d(Sx_{2n+1}, Ax_{2n+1}), d(Bz, Tz)).$$

Letting n tend to infinity we have by continuity of ψ

$$d(z, Tz) = d(z, Bz) \geq \psi(d(z, Tz), 0, 0).$$

By property (u), it follows that $d(z, Tz) > d(z, Tz)$ if $z \neq Tz$. Thus $z = Tz$. Further, we have by (1)

$$d(Av, Bz) \geq \psi(d(Sv, Tz), d(Av, Sv), d(Tz, Bz)) \text{ and}$$

$0 = d(z,z) \geq \psi(d(Sv,z), d(z,Sv), 0)$. By Definition 1 we have $0 \geq h \cdot d(Sv,z)$ and thus $Sv = z$. Since A and S are compatible of type (A) and $Av = Sv = z$ by Lemma 3 $Az = ASv = SAv = Sz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can complete the proof in the case of continuity of T.

From Lemma 4 it follows that z is the unique common fixed point of A, B, S and T.

DEFINITION 5 ([3]). $\psi: R_+^3 \to R_+$ satisfies property (B) if for every $u,v \in R_+$ such $u \geq \psi(v,u,v)$ we have $u \geq hv$, where $\psi(1,1,1) = h \geq 1$.

DEFINITION 6 ([7]). $\psi: R_+^3 \to R_+$ satisfies property (B*) if for every $u,v \in R_+$ such that $u \geq \psi(v,v,u)$, we have $u \geq hv$, where $\psi(1,1,1) = h \geq 1$.

COROLLARY 1. *Let A,B,S and T be mappings from a complete metric space* (X,d) *into itself satisfying:*

(1) *the conditions* (1°), (2°), (3°) *of Theorem* 1,

(2) *The inequality* (1) *holds for all x,y in X where* ψ *satisfies property* (B) *and* (B*) *with* $h \geq 1$.

If property (u) *holds and* ψ *is continuous, then A,B,S and T have a unique common fixed point.*

THEOREM 2. *Let A,B,S and T be mappings from a complete metric space* (X,d) *into itself satisfying conditions* (1°), (2°) *and* (3°) *of Theorem* 1. *If there exist non negative reals* a,b,c,d *with* $a+b+c+d > 1$ *such that*

$$d^k(Ax,By) \geq a \cdot d^k(Sx,Ty) + b \cdot d^m(Ax,Sx) \cdot d^{k-m}(By,Ty) + c \cdot d^{k-p}(Sx,Ty) \cdot d^p(Ax,Sx) + d \cdot d^q(By,Ty) \cdot d^{k-q}(Sx,Ty) \tag{3}$$

where $k \geq 1$, $q \geq 0$, $m \geq 0$, $p \geq 0$ *and* $q \leq k$, $p \leq k$, $m \leq k$ *hold for all x and y in X, then A,B,S and T have a common unique fixed point if* $a > 1$.

Proof. Let

$$\psi(t_1,t_2,t_3) = \left[a\cdot t_1^{k} + b\cdot t_2^{m}\cdot t_3^{k-m} + c\cdot t_2^{p}\cdot t_1^{k-p} + d\cdot t_3^{q}\cdot t_1^{k-q}\right]^{1/k}.$$

Let u,v such that $u \geq \psi(v,u,v)$, then

$$a \geq \left[a, v^{k} + bu^{m} v^{k-m+\text{cup},k-p} + dv^{k}\right]^{1/k} \text{ and}$$

$$a_k \geq av^{k} + bu^{m} v^{k-m} + cu^{p}\cdot v^{k-p} + dv^{k}.$$

Thus $(a+d)\cdot t^{k} + b\cdot t^{m} + c\cdot t^{p} - 1 \leq 0$ where $t = v/u$.

Let $g_1(t)$: $[0,\infty) \to R$ be the function $g_1(t) = (a+d)t^{k} + bt^{m} + ct^{p} - 1$. Then $g_1'(t) > 0$ for $t > 0$, $g_1(0) < 0$ and $g_1(1) = a + b + c + d - 1 > 0$. Let $r_1 \in (0,1)$ be the root of the equation $g_1(t) = 0$, then $g_1(t) < 0$ for $t < r_1$. Let u,v be such that $u \geq \psi(v,v,u)$, then

$$u \geq \left[av^{k} + bv^{k-m}u^{m} + cv^{k} + du^{q,k-q}\right]^{1/k}.$$

Similarly, we have

$$(a+c)t^{k} + bt^{k-m} + dt^{k-q} - 1 \leq 0$$

where $t = v/u$. Let g_2: $[0,\infty) \to R$ be the function $g_2(t) = (a+c)t^{k} + b\cdot t^{k-m} + dt^{k-q} - 1$. Let $r_2 \in (0,1)$ the root of the equation $g_2(t) = 0$, then $g_2(t) < 0$ for $t < r_2$. Thus $g_1(t) < 0$ and $g_2(t) < 0$ for $t < \min\{r_1,r_2\} = r$, $r \in (0,1)$. Then $(v/u) < r$ and $u > (1/r)v$. Thus $h = (1/r) > 1$ and $u \geq hv$ with $h > 1$.

On the other hand we have $\psi(u,0,0) = a^{1/k}u > u$. By Theorem 1, it follows that A,B,S and T have a unique common fixed point.

COROLLARY 2 ([4]). *Let (X,d) be a complete metric space and f: $(X,d) \to (X,d)$ a surjective mapping. If there exist non-negative reals a,b,c,d with $a+b+c+d > 1$ such that*

$$d^{k}(fx,fy) \geq a\cdot d^{q}(x,fx)\cdot d^{k-q}(x,y) + b\cdot d^{m}(y,fy)\cdot d^{k-m}(x,y) +$$
$$c\cdot d^{p}(x,fx)\cdot d^{k-p}(y,fy) + d\cdot d^{k}(y), \tag{4}$$

where $k \geq 1$, $q \geq 0$, $m \geq 0$, $p \geq 0$ and $q \leq k$, $m \leq k$, $p \leq k$ for each x,y in X with $x \neq y$, and *if* $d > 1$, *then f has a unique fixed point.*

CORROLARY 3 ([5]). *Let* (X,d) *be a complete metric space and* $f: (X,d) \rightarrow (X,d)$ *a surjective mapping. If there exist non-negative* a,b,c *with* $a < 1$ *and* $c > 1$, *then f has a unique fixed point.*

THEOREM 3. *Let* S, T *and* $\{f_i\}_{i\in N}$ *be mappings from a complete metric space* (X,d) *into itself satisfying the conditions:*

(1°) $\{f_i\}_{i\in N}$ *are surjective,*

(2°) S *or* T *or* f_1 *is continuous,*

(3°) S *and* $\{f_i\}_{i\in N}$ *are compatible of type* (A) *and* T *and* $\{f_i\}_{i\in N}$ *are compatible of type* (A).

(4°) *The inequality*

$$d(f_i x, f_{i+1} y) \geq \psi(d(Sx, Ty), d(f_i x, Sx), d(f_{i+1} y, Ty)) \tag{5}$$

hold for all x and y in X, $\forall\, i \in N$, *where* ψ *is continuous, satisfies property* (h) *with* $h > 1$ *and property* (u), *then* $\{f_i\}_{i\in N}$, *A and B have a unique common fixed point.*

Proof. It is similar to the proof of [7, Theorem 4].

COROLLARY 4. *Let* S, T *and* $\{f_i\}_{i\in N}$ *be mappings from a complete metric space* (X,d) *into itself satisfying the conditions* (1°), (2°), (3°) *of Theorem* 3 *and*

$$d^k(f_i x, f_{i+1} y) \geq a \cdot d^k(Sx, Ty) + b \cdot d^k(f_i x, Sx) + c \cdot d^k(f_{i+1} y, Ty), \tag{6}$$

where $k \geq 1$, $0 \leq b$, $c < 1$, $a > 1$ *hold for all x and y in X,* $\forall\, i \in N$, *then* S, T *and* $\{f_i\}_{i\in N}$ *have a unique common fixed point.*

We conclude this paper with the following example, which shows that "surjectivity of A and B" is a necessary condition in Theorem 1.

Example 1. Let $X = [0,\infty)$. Define A,S,B and T: $X \rightarrow X$ given by $Ax = kx + 1$, Sx

$x+1$, $Bx = Tx = 1$ for x in X and $2 \geq k > 1$. Note that the following mapping satisfies properties (h) and (u):

$$\psi(t_1, t_2, t_3) = k \cdot \max\{t_1, t_2, t_3\}, \text{ where } k > 1.$$

Now, $d(Ax, By) = kx = k \cdot \max\{x, (k-1)x, 0\} = k \cdot \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\} = \psi(d(Sx, Ty), d(Ax, Sx), d(By, Ty))$, for all x,y in X, where $2 \geq k > 1$. Consider a sequence $\{x_n\} \subset X$ such that $x_n \to 0$. Then it is to see, by routine calculation, that A,S and B,T are compatible of type (A). Moreover, A,B,S and T are all continuous. Therefore, we see that all the hypothesis of Theorem 1 are satisfied except surjectivity of A and B, but the mappings A,B,S and T have no fixed point in X.

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