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NOTE ABOUT A METHOD FOR SOLVING NONLINEAR SYSTEM OF EQUATIONS IN FINIT DIMENSIONAL SPACES

Béla FINTA[*]

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REZUMAT. - O metodă de rezolvare a sistemelor de ecuaţii neliniare în spaţii finit dimensionale. În această lucrare se aplică ideea lui Seidel şi metoda SOR pentru forma iterativă a unui sistem de ecuaţii neliniare şi se dau condiţii suficiente, care asigură convergenţa şirului iterativ.

It is well known to obtain solutions for a linear and nonlinear system of equations one kind are the iterative methods (see [1] pages 177-188, [2] pages 40-49 and 127-166, or [3] pages 82-106 and 322-363). For the linear case the simplest method of such type is known as Jacobi's method. For the nonlinear case this method appears, too as Jacobi's theorem.

Let us consider the function $f: D \subset R^n \to R^n$, where $D \neq \emptyset$, and let us transform the system of equations $f(x) = \theta_{R^n}$ in the iterative form $x = \phi(x)$ (see [4] pages 21-22).

THEOREM (Jacobi). *Let us suppose that D' is a domain, and $\phi : D' \subset R^n \to R^n$ a Fréchet differentiable map. If $A \subset D'$ is a closed convex subset such that $\phi(A) \subset A$, and there exists $\alpha \in (0,1)$ with property: $\sum_{j=1}^{n} \left| \frac{\partial \phi_i}{\partial x_j}(x) \right| \leq \alpha$ for every $x \in A$ and for $i = \overline{1,n}$, then the system $\phi(x) = x$ has a unique solution in A* (see [5] page 81).

For the linear system of equations transformed in the iterative form there exist other methods, which increase the rapidity of convergence for the iterative sequence obtained by the Jacobi's iterative method, like the Gauss-Seidel, and more, the successive overrelaxation

[*] *Technical University, Str. N. Iorga, 1, Târgu-Mureş, Romania*

methods.

In the case of nonlinear systems of the form $f(x) = \theta_{R^n}$, it is known the so called Seidel-SOR method, which combines Seidel's idea with the successive overrelaxation. The existing result in this direction is the following:

Let us consider the function $f: D \subset R^n \to R^n, f = (f_1, \dots, f_n)$. We suppose it is known the k-th term x^k of the iterative sequence, and we want to find x^{k+1}. If we suppose that the first i-1 components of x^{k+1} are determined, than let's consider x_i' to be the solution of the equation: $f_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \dots, x_n^k) = 0$. We can calculate this approximative value x_i' by one of the methods of solving nonlinear equations in one variable.

Then we obtain the i-th component like: $x_i^{k+1} = x_i^k + \omega \cdot (x_i' - x_k')$, where $\omega \in R^*$ is a factor of relaxation. We consider the decomposition $f(x) = D(x) - L(x) - U(x)$ for the Jacobian of f, where $D(x)$ is the diagonal matrix formed by the diagonal elements, $L(x)$ is the lower triangular matrix, and $U(x)$ is the upper triangular matrix. We note:

$$B(x) = \omega^{-1} \cdot [D(x) - \omega \cdot L(x)],$$

$$C(x) = \omega^{-1} \cdot [(1 - \omega) \cdot D(x) + \omega \cdot U(x)], \; H(x) = B^{-1} C.$$

Now we are ready to announce the theorem obtained by the local linearization around the solution x^* of the nonlinear system of equations:

THEOREM (Seidel-SOR). *Let us consider the function* $f: D \subset R^n \to R^n$ *and let us suppose that* x^* *is a solution of the system* $f(x) = \theta_{R^n}$. *If we suppose the following conditions hold:* i) *f is continuously differentiable in a neighborhood* $V(x^*) \subset D$ *of* x^*, ii) $D(x^*)$ *is not singular,* iii) *the spectral radius* $\rho(H(x^*)) < 1$, *then there exists a sphere* $S(x^*, r) = \{x \in R^n \mid \|x - x^*\| \le r\} \subset V(x^*)$ *such that for every* $x \in S(x^*, r)$ *the iterative sequence* $\{x^k\}_{k \in N}$ *generated by the Seidel-SOR method is unique defined and converges to* x^*, *i.e.* x^* *is an attractive point for* f (see [6] pages 89-95).

The purpose of this work is to apply the Siedel's idea and the SOR method for the iterative form $\phi(x) = x$ of the nonlinear system of equations and to find sufficient conditions which assure us the convergence of the iterative sequence. First we rewrite the system $f(x) = \theta_{R^n}$ in the nonlinear relaxation form:

We suppose that we can form the function $\phi^*: D' \subset R^n \to R^n$ in the following way:

$$\phi_1^*(x) = x_1 + \omega\cdot(\phi_1(x) - x_1),$$

$$\phi_2^*(x) = x_2 + \omega\cdot(\phi_2(\phi_1^*(x), x_2, \ldots, x_n) - x_2), \ldots,$$

$$\phi_i^*(x) = x_i + \omega\cdot(\phi_i(\phi_1^*(x), \ldots, \phi_{i-1}^*(x), x_i, \ldots, x_n) - x_i), \ldots,$$

$$\phi_n^*(x) = x_n + \omega\cdot(\phi_n(\phi_1^*(x), \ldots, \phi_{n-1}^*(x), x_n) - x_n)$$

for every $x \in D'$, where $\omega \in R^*$ is the factor of relaxation. For $A \neq \emptyset$, $A \subset D'$ closed, convex set, let us consider the numbers:

for $i = \overline{1,n}$:

$$a_{ii} = \sup\left\{\left|1 - \omega + \omega\cdot\frac{\partial\phi_i}{\partial x_i}(x)\right| \;\middle|\; x \in A\right\},$$

for $i,j = \overline{1,n}$ and $i \neq j$

$$a_{ij} = \sup\left\{\left|\omega\cdot\frac{\partial\phi_i}{\partial x_j}(x)\right| \;\middle|\; x \in A\right\}.$$

For $i = \overline{1,n}$ we generate the numbers:

$$M_i = \sum_{j=1}^{i-1} a_{ij}\cdot M_j + \sum_{j=i}^{n} a_{ij},$$

with convention: $\sum_{j=1}^{0} a_{1j}\cdot M_j = 0$.

THEOREM 1. *If we suppose that:* i) $\phi: D' \subset R^n \to R^n$ *is a Fréchet differentiable function,* ii) *there exists a closed convex subset* $A \neq \emptyset$, $A \subset D'$ *such that* $\phi^*(A) \subset A$, iii) *for this set* A, $\max\{M_i | i = \overline{1,n}\} < 1$, *then for every* $x^0 \in A$ *the iterative sequence* $x^{k+1} = \phi^*(x^k)$ *exists and converges to the unique fixed point of the function* ϕ.

Proof. We consider the norm:

$$\|x\|_\infty = \max\{|x_i| \mid i = \overline{1,n}\}$$

on R^n, and we apply the Banach fixed point theorem to the function ϕ^*. For every $i = \overline{1,n}$ we obtain:

$$|\phi_i^*(y) - \phi_i^*(x)| =$$

$$= |y_i - x_i + \omega\cdot\{\phi_i(\phi_1^*(y),\dots,\phi_{i-1}^*(y),y_i,\dots,y_n) - \phi_i(\phi_1^*(x),\dots,\phi_{i-1}^*(x),x_i,\dots,x_n) - (y_i - x_i)\}| =$$

$$= |(y_i - x_i)(1-\omega) + \omega\cdot\{\phi_i(\phi_1^*(y),\dots,\phi_{i-1}^*(y),y_i,\dots,y_n) - \phi_i(\phi_1^*(x),\dots,\phi_{i-1}^*(x),x_i,\dots,x_n)\}| =$$

$$= |(y_i - x_i)(1-\omega) + \omega\cdot d\phi_i(u)|,$$

where

$$u = (\phi_1^*(x),\dots,\phi_{i-1}^*(x),x_i,\dots,x_n) + \xi\cdot(\phi_1^*(y) - \phi_1^*(x),\dots,\phi_{i-1}^*(y) - \phi_{i-1}^*(x), y_i - x_i,\dots,y_n - x_n)),$$

with $\xi \in (0,1)$, and

$$|(y_i - x_i)(1-\omega) + \omega\sum_{j=1}^{i-1}\frac{\partial\phi_i}{\partial x_j}(u)(\phi_j^*(y) - \phi_j^*(x)) + \omega\sum_{j=i}^{n}\frac{\partial\phi_i}{\partial x_j}(u)(y_j - x_j)| \le$$

$$\le \sum_{j=1}^{i-1}|\omega\cdot\frac{\partial\phi_i}{\partial x_j}(u)|\cdot|\phi_j^*(y) - \phi_j^*(x)| + |(1-\omega) + \omega\frac{\partial\phi_i}{\partial x_j}(u)|\cdot|y_i - x_i| +$$

$$+ \sum_{j=i+1}^{n}|\omega\frac{\partial\phi_i}{\partial x_j}(u)|\;|(y_i - x_j)| \le$$

$$\le (\sum_{j=1}^{i-1}a_{ij}\cdot M_j + a_{ii} + \sum_{j=i+1}^{n}a_{ij})\|y - x\|_\infty$$

Consequently

$$\|\phi^*(y) - \phi^*(x)\|_\infty = \max\{|\phi_i^*(y) - \phi_i^*(x)\| \; i = \overline{1,n}\} \le$$

$$\le \max\{M_i \mid i = \overline{1,n}\}\cdot\|y - x\|_\infty,$$

with $\max\{M_i \mid i = \overline{1,n}\} < 1$. So ϕ^* is a contraction and we can easily see that the fixed point of ϕ^* will be a fixed point for ϕ, too.

For $\omega = 1$ we obtain the Seidel's method for the system of nonlinear equations in the iterative form. In this case we define the function $\phi^*: D' \subset R^n \to R^n$ in the following way:

$$\phi_1^*(x) = \phi_1(x),$$

$$\phi_2^*(x) = \phi_2(\phi_1^*(x), x_2, \dots, x_n), \dots,$$

$$\phi_i^*(x) = \phi_i(\phi_1^*(x), \dots, \phi_{i-1}^*(x), x_i, \dots, x_n), \dots,$$

$$\phi_n^*(x) = \phi_n(\phi_1^*(x), \dots, \phi_{n-1}^*(x), x_n),$$

and for $A \neq \varnothing$, $A \subset D'$ closed, convex set, we consider the numbers:

$$a_{ij} = \sup\left\{ \left| \frac{\partial \phi_i}{\partial x_j}(x) \right| \,\middle|\, x \in A \right\}, \text{ for } i,j = \overline{1,n}$$

and we generate the numbers:

$$M_i = \sum_{j=1}^{i-1} a_{ij} \cdot M_j + \sum_{j=i}^{n} a_{ij}, \text{ for } i = \overline{1,n}$$

with convention: $\sum_{j=1}^{0} a_{1j} \cdot M_j = 0$.

THEOREM 2. *If we suppose that:* i) $\phi: D' \subset R^n \to R^n$ *is a Fréchet differentiable function,* ii) *there exists a closed convex subset* $A \neq \varnothing, A \subset D'$ *such that* $\phi^*(A) \subset A$, iii) *for this set* A, $\max\{M_i | i = \overline{1,n}\} < 1$, *then for every* $x^0 \in A$ *the iterative sequence* $x^{k+1} = \phi^*(x^k)$ *exists and converges to the unique fixed point of the function* ϕ.

Example. We solve the following nonlinear system of equations:

$$7\sin x = x^2 + yz + \cos z$$

$$9\sin y = xz^2 + y\cos(xyz) + 1$$

$$8\sin z = x\sin z^2 + y^2\cos(xy)$$

for $x,y,z \in [-1,1]$. We transform the system in the following iterative form:

$$x = \arcsin[(x^2 + yz + \cos z)|7]$$

$$y = \arcsin[(xz^2 + y\cos(xyz) + 1)|9]$$

$$z = \arcsin[(x\sin z^2 + y^2\cos(xy))|8]$$

and we solve it using Jacobi's theorem, Theorem 2 and Theorem 1 with $\omega = 1.1$. If we consider the initial point $x^0 = (0.5, 0.5, 0.5)$ then we obtain the solution with accurate to two, three, four, five decimal places by making 4, 3, 3; and 5, 4, 4; and 6, 5, 5; and 7, 6, 5 iterations, respectively.

Remark. We can obtain theorems like Jacobi's theorem, Theorem 1 and Theorem 2 by using other norms on R^n. One problem is to find such a norm, for that the conditions on the system are larger.

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