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A CLASS OF INTEGRAL FAVARD-SZASZ TYPE OPERATORS

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REZUMAT. - O clasă de operatori integrali de tip Favard-Szasz. În această lucrare se consideră un operator de tip integral, în sensul lui Durrmeyer [2] şi Derriennic [1], care se obţine plecând de la un operator de tip Favard-Szasz (4), introdus în 1969 de către Jakimovski şi Leviatan [4]. Autoarea dă unele estimări cantitative, exprimate cu modulele de continuitate de primele două ordine, pentru aproximarea funcţiilor cu ajutorul operatorului L_n, definit la (6).

Abstract. This paper one considers an integral type operator, in the sense of Durrmeyer [2] and Derriennic [1], which is obtained by starting from a Favard-Szasz operator (4), introduced in 1969 by Jakimovski and Leviatan [4]. The author gives some quantitative estimates, in therms of the first and the second order moduli of continuity, for the approximation of functions by means of the operator L_n, defined at (6).

1. This paper is motivated by the works of J.L. Durrmeyer [2], A. Lupaş [6] and M.M. Derriennic [1], which have obtained and studied a modified Bernstein operator

$$(B_n^* f)(x) = (n+1)\sum_{k=0}^{n} d_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)\,dt,$$

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1}$$

where f is Lebesque integrable on [0,1].

S.M. Mazhar and V. Totik [7], similarly modified the Favard-Szasz operator and they

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have defined another class of positive linear operators

$$(S_n^* f)(x) = n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}f(t)\,dt \tag{2}$$

for functions $f \in L_1[0,\infty)$ ∘

By using a similar way we will modify an operator introduced by A. Jakimovski and D. Leviatan [4]. Let us remind this operators. One considers $g(z) = \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk $|z| < R$, $R > 1$, where $g(1) \neq 0$. It is known that the Appell polynomials $p_k(x)$, $k \geq 0$ can be defined by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{3}$$

To a function f: $[0,\infty) \to R$ one associates the Jakimovski-Leviatan operator

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{4}$$

The case $g(z) = 1$ yields the classical operator of Favard-Szasz

$$(S_n f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

B. Wood [9] has proved that the operator P_n is positive if and only if $\frac{a_n}{g(1)} \geq 0$, $n = 0, 1, \ldots$

Now we will modify the operator P_n, as follows: for a function f, Lebesque integrable in $[0,\infty)$, we replace $f\left(\frac{k}{n}\right)$ into P_n, by a positive linear functional

$$A_k(f) = \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}f(t)\,dt,\ \lambda \geq 0 \tag{5}$$

and so we obtain the operator

$$(L_n f)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}f(t)\,dt. \tag{6}$$

For $g(z)=1$ and $\lambda=0$ the operator defined at (6) becomes the operator S_n^*.

We suppose that this operator is positive, therefore $\frac{a_n}{g(1)} \geq 0$, $n = 0, 1, \ldots$ We denote by E the class of functions of exponential type, which have the property that $|f(t)| \leq e^{At}$, for each $t \geq 0$ and some finite number A.

The following lemma is essential to study the convergence of the sequence $(L_n f)$ to the function f.

LEMMA 1.1. *For all $x \geq 0$, we have*:

$$(L_n e_0)(x) = 1$$

$$(L_n e_1)(x) = x + \frac{1}{n}\left(\lambda + 1 + \frac{g'(1)}{g(1)}\right) \tag{7}$$

$$(L_n e_2)(x) = x^2 + \frac{2x}{n}\left(\lambda + 2 + \frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right],$$

where $e_i(x) = x^i$, $i \in \{0, 1, 2\}$.

Proof. We will use the properties of the gamma function and the values of the operator P_n defined at (4) for the monomials e_0, e_1, e_2:

$$(P_n e_0)(x) = 1$$

$$(P_n e_1)(x) = x + \frac{1}{n}\frac{g'(1)}{g(1)} \tag{8}$$

$$(P_n e_2)(x) = x^2 + \frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1)+g'(1)}{g(1)}.$$

For instance, let us calculate $(L_n e_1)(x)$. We have:

$$A_k(e_1) = \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^\infty e^{-nt}t^{\lambda+k+1}\,dt = \frac{1}{n}(\lambda+k+1)$$

and so we obtain

$$(L_n e_1)(x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^\infty p_k(nx)\frac{1}{n}(\lambda+k+1) = \frac{1}{n}(\lambda+1) + (P_n e_1)(x) =$$

$$= x + \frac{1}{n}\left(\lambda + 1 + \frac{g'(1)}{g(1)}\right)$$

THEOREM 1.2. *If $f \in \mathbf{C}[0,\infty) \cap E$, then $\lim_{n\to\infty}(L_n f)(x) = f(x)$, the convergence being uniform in each compact* $[0,a]$.

Proof. According to Lemma 1.1, we have $\lim_{n\to\infty}(L_n e_i)(x) = e_i(x)$, $i \in \{0,1,2\}$

uniformly on the compact $[0,a]$, so if we invoke the Bohman-Korovkin theorem, we obtain the desired result.

2. Estimate of the order of approximation. In this section we are concened with the estimate of the order of approximation of a function $f \in L_1[0,\infty)$ by means of the linear positive operator L_n. We will use the modulus of continuity defined by $\omega(f;\delta) = \sup |f(x'') - f(x')|$, where x' and x'' are points from $[0,a]$ so that $|x'' - x'| < \delta$, δ being a positive number. By using a standard method we prove

THEOREM 2.1. *If $f \in L_1[0,a]$, then*

$$|(L_n f)(x) - f(x)| \le \left(1 + \sqrt{2x + \frac{1}{n}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right)}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right)$$

Proof. Because $L_n e_0 = e_0$ and L_n is positive, we can write

$$|(L_n f)(x) - f(x)| \le \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\,|A_k(f) - f(x)A_k(e_0)| =$$

$$= \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}\,|f(t) - f(x)|\,dt \le$$

$$\le \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\left(1 + \frac{1}{\delta}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}\,|t-x|\,dt\right)\omega(f;\delta).$$

By making use of the Cauchy inequality, we obtain

$$\int_0^{\infty} e^{-nt}t^{\lambda+k}\,|t-x|\,dt \le \sqrt{\int_0^{\infty} e^{-nt}t^{\lambda+k}\,dt}\,\sqrt{\int_0^{\infty} e^{-nt}t^{\lambda+k}(t-x)^2\,dt} =$$

$$= \frac{\Gamma(\lambda+k+1)}{n^{\lambda+k+1}}\sqrt{x^2 - 2x\frac{k+\lambda+1}{n} + \frac{(k+\lambda+1)(k+\lambda+2)}{n^2}}\,.$$

It results that

$$|(L_n f)(x) - f(x)| \le$$

$$\left(1 + \frac{1}{\delta}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)\sqrt{\frac{(\lambda+1)(\lambda+2)}{n^2} + \frac{k(2\lambda+3)}{n^2} + \frac{k^2}{n^2} - 2x\frac{k+\lambda+1}{n} + x^2}\right)\omega(f;\delta).$$

We use again the Cauchy inequality and we get

$$|(L_n f)(x) - f(x)| \le \left(1 + \frac{1}{\delta}\sqrt{2\frac{x}{n} + \frac{1}{n^2}\left((\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right)}\right)\omega(f;\delta)$$

By inserting into it $\delta = \dfrac{1}{\sqrt{n}}$, we obtain the desired result. ∎

Next we will give some approximation theorems in different normed linear spaces. In order to establish the next results, we need some definitions.

The second order modulus of continuity of $f \in C_B[0,\infty)$ is

$$\omega_2(f;t) = \sup_{|h|\le t} \|f(\circ + 2h) - 2f(\circ + h) + f(\circ)\|_{C_B},\ t \ge 0$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

The Peetre K-functional of function $f \in C_B$ is defined as

$$K(f;t) = \inf_{g\in C_B^2}\left\{\|f - g\|_{C_B} + t\|g\|_{C_B^2}\right\}$$

where $C_B^2 = \left\{f \in C_B \mid f', f'' \in C_B\right\}$, with the norm $\|f\|_{C_B^2} = \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}$ It is known the following inequality;

$$K(f;t) \le A\left\{\omega_2(f;\sqrt{t}) + \min(1,t)\|f\|_{C_B}\right\} \tag{9}$$

for all $t \in [0,\infty)$, the constant A being independent of t and f. We will also use

LEMMA 2.2. *If $z \in C^2[0,\infty)$ and (P_n) is a sequence of linear positive operators with the property $P_n e_0 = e_0$, then*

$$|(P_n z)(x) - z(x)| \le \|z'\|\sqrt{\left(P_n(t-x)^2\right)(x)} + \frac{1}{2}\|z''\|\left(P_n(t-x)^2\right)(x)$$

The proof is analogous to the proof of theorem 2 from [3].

THEOREM 2.3. *If $f \in C[0,a]$, then for any $x \in [0,a]$ we have*

$$|(L_n f)(x) - f(x)| \leq \frac{2h}{a}\|f\| + \frac{3}{4}\left(3 + \frac{a}{h}\right)\omega_2(f;h),$$

where $h = \sqrt{2\frac{x}{n} + \frac{1}{n^2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]}$.

Proof. Let f_h be the Steklov function attached to the function f. We will use the following result of V.V. Juk [5]: if $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$, then $\|f - f_h\| \leq \frac{3}{4}\omega_2(f;h)$ and $\|f_h''\| \leq \frac{3}{2}\frac{1}{h^2}\omega_2(f;h)$. Since $L_n e_0 = e_0$, we can write

$$|(L_n f)(x) - f(x)| \leq |(L_n(f - f_h)(x)| + |(L_n f_h)(x) - f_h(x)| + |f_h(x) - f(x)| \leq$$

$$\leq 2\|f - f_h\| + |(L_n f_h)(x) - f_h(x)|$$

For the function $f_h \in C^2[0,a]$ we use lemma 2.2:

$$|(L_n f_h)(x) - f_h(x)| \leq \|f_h'\| \sqrt{(L_n(t-x)^2)(x)} + \frac{1}{2}\|f_h''\|(L_n(t-x)^2)(x)$$

According to result from [3] and [5], we have

$$\|f_h'\| \leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \leq \frac{2}{a}\|f\| + \frac{a}{2}\|f_h''\| \leq \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h).$$

By making use of this inequality and choosing $h = \sqrt{(L_n(t-x)^2(x)}$ we obtain

$$|(L_n f_h)(x) - f_h(x)| \leq \frac{2}{a}\|f\|h + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h)$$

and therefore we get

$$|(L_n f)(x) - f(x)| \leq 2\|f - f_h\| + \frac{2}{a}\|f\|h + \frac{3}{4}\left(\frac{a}{h} + 1\right)\omega_2(f;h)$$

Here we use the inequality $\|f - f_h\| \leq \frac{3}{4}\omega_2(f;h)$ and we obtain the desired result.

Remark. If we consider $g(z) \equiv 1$ and $\lambda = 0$, we obtain, for the operator due to S.M. Mazhar and V. Totik [7], the estimation

$$|(S_n^* f)(x) - f(x)| \leq \frac{2h}{a}\|f\| + \frac{3}{4}\left(3 + \frac{a}{h}\right)\omega_2(f;h),$$

where $h = \sqrt{2\frac{x}{n} + \frac{2}{n^2}}$.

THEOREM 2.4. *For every function* $f \in C_B^2[0,\infty)$, *we have*

$$|(L_nf)(x) - f(x)| \le \frac{1}{n}\left\{x + \frac{1}{2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\}\|f\|_{C_B^2}$$

Proof. Applying the Taylor expansion to the function $f \in C_B^2$, we have

$$(L_nf)(x) - f(x) = f'(x)(L_n(t-x))(x) + \frac{1}{2}f''(\xi)(L_n(t-x)^2)(x), \text{ where } \xi \in (t,x)$$

By using lemma 1.1, we can write successively

$$|(L_nf)(x) - f(x)| \le \frac{1}{n}\left(\lambda + 1 + \frac{g'(1)}{g(1)}\right)\|f'\|_{C_B} +$$

$$+\frac{1}{2n}\left\{2x + \frac{1}{n}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\}\|f''\|_{C_B} \le \frac{1}{n}\left(\lambda+1+\frac{g'(1)}{g(1)}\right)\|f'\|_{C_B} +$$

$$+\frac{1}{n}\left\{x + \frac{1}{2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\}\|f''\|_{C_B} \le$$

$$\le \frac{1}{n}\left\{x + \frac{1}{2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\}\left(\|f'\|_{C_B} + \|f''\|_{C_B}\right)$$

Remark. If we take into it $g(z) = 1$ and $\lambda = 0$, we obtain

$$|(S_n^*f)(x) - f(x)| \le \frac{1}{n}(x+1)\|f\|_{C_B^2}$$

result obtained by S.P. Singh and M.K. Tiwari [8].

THEOREM 2.5. *If* $f \in C_B[0,\infty)$, *then we have*

$$|(L_nf)(x) - f(x)| \le 2A\left(\omega_2(f;h) + \lambda_n(x)\|f\|_{C_B}\right),$$

where $h = \sqrt{\frac{1}{2n}\left\{x + \frac{1}{2}\left[(\lambda+1)(\lambda+2) + (2\lambda+3)\frac{g'(1)}{g(1)} + \frac{g''(1)+g'(1)}{g(1)}\right]\right\}}$,

$\lambda_n(x) = \min(1, h^2)$ *and A is a constant independent of h and f.*

Proof. We will use the theorem 2.4 and the K-functional. For $f \in C_B[0,\infty)$ and

$z \in C_B^2[0,\infty)$, we have

$$|(L_nf)(x)-f(x)| \le |(L_nf)(x)-(L_nz)(x)| + |(L_nz)(x)-z(x)| + |z(x)-f(x)| \le$$

$$\le 2\|f-z\|_{C_B} + \frac{1}{n}\left\{x+\frac{1}{2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)+g'(1)}{g(1)}\right]\right\}\|z\|_{C_B^2},$$

Because the left side of this inequality does not depend of the function $z \in C_B^2$, it result that

$$|(L_nf)(x)-f(x)| \le 2K(f;A(x,n)),$$

where

$$A(x,n) = \frac{1}{2n}\left\{x+\frac{1}{2}\left[(\lambda+1)(\lambda+2)+(2\lambda+3)\frac{g'(1)}{g(1)}+\frac{g''(1)+g'(1)}{g(1)}\right]\right\}$$

By making use (9), we obtain

$$|(L_nf)(x)-f(x)| \le 2A\left\{\omega_2(f;\sqrt{A(x,n)}\,)+\min(1,A(x,n))\|f\|_{C_B}\right\} =$$

$$= 2A\left(\omega_2(f;h)+\min(1,h^2)\|f\|_{C_B}\right)$$

Remark. For $g(z) = 1$ and $\lambda = 0$, we have $A(x,n) = \dfrac{x+1}{2n}$ and we obtain a result due to S.P. Singh and M.K. Tiwari [8]:

$$|(S_n^*f)(x)-f(x)| \le 2A\left\{\omega_2\left(f;\sqrt{\frac{x+1}{2n}}\right)+\min\left(1,\frac{x+1}{2n}\right)\|f\|_{C_B}\right\}$$

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