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PROBABILISTIC POSITIVE LINEAR OPERATORS

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REZUMAT. - Operatori liniari pozitivi probabilistici. Pentru un şir de operatori probabilistici se indică un algoritm de tip Casteljau. Se prezintă apoi câteva aplicaţii.

1. Introduction. For every x in an interval I of the real axis let us consider a sequence of independent and identically distributed random variables $(Y_n^x)_{n\geq 1}$. Let $p_{ni} \geq 0$, $i = 1, \ldots, n$, such that $p_{n1} + \ldots + p_{nn} = 1$ for each $n \geq 1$.

For a continuous function f on the real line let us denote

$$L_n f(x) = Ef\left(\sum_{i=1}^{n} p_{ni} Y_i^x\right) \tag{1}$$

provided that the expectation is finite.

Many classical positive linear operators (in particular Bernstein, Szász, Gamma, Weierstrass and Baskakov operators) are of the form (1). The probabilistic positive linear operators have been extensively studied; see [1], [3], [7], [8] and the references therein.

Our approach is based on a recursive algorithm related to Casteljau's algorithm. It allows us to deduce some properties of L_n from those of L_1. Finally we shall generalize a result from [7] concerning monotonic convergence under convexity. Other results of this type are to be found in [4] and [13].

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2. The algorithm. Let f be a given continuous function on R. For $x \in I$ and $t_1, \dots, t_n \in R$ denote

$$f_0^x(t_1, \dots, t_n) = f(p_{n1}t_1 + \dots + p_{nn}t_n)$$

$$f_k^x(t_1, \dots, t_{n-k}) = Ef_{k-1}^x(t_1, \dots, t_{n-k}, Y_{n-k+1}^x),\ k = 1, \dots, n-1.$$

Then we have

$$L_n f(x) = Ef_0^x(Y_1^x, \dots, Y_n^x) = Ef_1^x(Y_1^x, \dots, Y_{n-1}^x) = \dots =$$

$$= Ef_{n-1}^x(Y_1^x) = L_1 f_{n-1}^x(x) \tag{2}$$

Examples. (a) Let $p_{ni} = 1/n$, $n \geq 1$, $i = 1, \dots, n$. Let $(X_k)_{k \geq 1}$ be a sequence of independent and on [0,1] uniformly distributed random variables. Let $Y_n^x = I_{(X_n \leq x)}$, $0 \leq x \leq 1$, where I_C denotes the indicator function of C. Then $L_n f(x)$ coincides with the Bernstein operator $B_n f(x)$; see [1].

For $x \in [0,1]$, $f \in C[0,1]$, $k = 1, \dots, n-1$, $t_1, \dots, t_n \in \{0,1\}$ we have

$$f_0^x(t_1, \dots, t_n) = f((t_1 + \dots + t_n)/n)$$

$$f_k^x(t_1, \dots, t_{n-k}) = (1-x)f_{k-1}^x(t_1, \dots, t_{n-k}, 0) + xf_{k-1}^x(t_1, \dots, t_{n-k}, 1)$$

$$L_n f(x) = (1-x)f_{n-1}^x(0) + xf_{n-1}^x(1)$$

It follows that the computation of $L_n f(x)$ by means of (2) is equivalent to the computation of $B_n f(x)$ by means of the Casteljau algorithm [9] (see also [11] and [14]).

(b) In the case of the Szász operator (see [7]) we have for $x \geq 0$, $k = 1, \dots, n-1$, $t_i = 0, 1, \dots,$

$$f_0^x(t_1, \dots, t_n) = f((t_1 + \dots + t_n)/n)$$

$$f_k^x(t_1, \dots, t_{n-k}) = e^{-x} \sum_{j=0}^{\infty} f_{k-1}^x(t_1, \dots, t_{n-k}, j)x^j/j!$$

$$S_n f(x) = e^{-x} \sum_{j=0}^{\infty} f_{n-1}^x(j)x^j/j!$$

(c) Let $p_{ni} = 1/n$ and let Y_n^x be uniformly distributed on $[x-1, x+1]$. Then $L_n f(x)$ is the operator of Pečarić and Zwick [12]. We have for $k = 1, \ldots, n-1$,

$$f_0^x(t_1, \ldots, t_n) = f((t_1 + \ldots + t_n)/n)$$

$$f_k^x(t_1, \ldots, t_{n-k}) = (1/2) \int_{x-1}^{x+1} f_{k-1}^x(t_1, \ldots, t_{n-k}, t)\, dt$$

$$L_n f(x) = (1/2) \int_{x-1}^{x+1} f_{n-1}^x(t)\, dt$$

Remark 1. Let $p_{ni} = 1/n$. Denote $g_0^x = f$ and

$$g_k^x(u) = Ef((n-k)\, u/n + (Y_{n-k+1}^x + \ldots + Y_n^x)/n), \; k = 1, \ldots, n-1.$$

Then $f_k^x(t_1, \ldots, t_{n-k}) = g_k^x((t_1 + \ldots + t_{n-k})/(n-k))$.

Consider again the above example (c) and express $L_n f(x)$ by means of a divided difference (see [12]); we deduce

$$L_n f(x) = \int_{\mathbb{R}} g_{n-1}^x(u)\, B_0^x(u)\, du = \int_{\mathbb{R}} g_{n-2}^x(u)\, B_1^x(u)\, du = \ldots =$$
$$= \int_{\mathbb{R}} g_0^x(u)\, B_{n-1}^x(u)\, du$$

where B_{j-1}^x is the B-spline function [9] of degree j-1 corresponding to the equidistant points $x-1 = t_0 < t_1 < \ldots < t_j = x+1$, $j = 1, \ldots, n$.

In particular, $L_n f(0) = \int_{\mathbb{R}} f(u)\, B_{n-1}^0(u)\, du$. This means that the probability density of $(Y_1^0 + \ldots + Y_n^0)/n$ is the spline function B_{n-1}^0. The characteristic function of the same variable is

$$\varphi(t) = ((n/t) \sin(t/n))^n$$

It follows that the Fourier transform of B_{n-1}^0 is φ (see also [5]).

3. Applications. For $M > 0$ denote

$$\mathrm{Lip}\,(M; I) = \{ f \in C(I) : |f(x) - f(y)| \leq M|x-y|,\; x, y \in I \}.$$

The following lemma can be proved by induction and we omit the details.

LEMMA 1. (i) *If* $f \in \mathrm{Lip}\,(M;R)$ *then*

$$f_k^x(t_1, \ldots, t_{n-k-1}, \cdot) \in \mathrm{Lip}\,(Mp_{n,n-k}; R),\ k = 0, \ldots, n-1.$$

(ii) *If* f *is increasing, then* $f_k^x(t_1, \ldots, t_{n-k-1}, \cdot)$ *is increasing,* $k = 0, \ldots, n-1$.

THEOREM 1. *Let* $M, N > 0$. *If* L_1 *transforms the functions from* $\mathrm{Lip}(M;R)$ [*the increasing functions*] *into functions from* $\mathrm{Lip}(N;I)$ [*increasing functions*], *then the same is true for each* L_n, $n > 1$.

Proof. Let $x,y \in I$, $f \in \mathrm{Lip}(M;R)$, $n > 1$ and $q = |x-y|$. Then, by (i), $f_k^y(t_1, \ldots, t_{n-k-1}, \cdot)$ is in $\mathrm{Lip}(Mp_{n,n-k}; R)$, hence $L_1 f_k^y(t_1, \ldots, t_{n-k-1}, \cdot)$ is in $\mathrm{Lip}(Np_{n,n-k}; I)$. This means that the function $t \to Ef_k^y(t_1, \ldots, t_{n-k-1}, Y_{n-k}^t)$ is in $\mathrm{Lip}(Np_{n,n-k}; I)$ for each $k = 0, \ldots, n-1$.

Let F_x be the distribution function of Y_1^x. Since $f_0^x = f_0^y$, we have

$$L_n f(x) = Ef_0^x(Y_1^x, \ldots, Y_n^x) = Ef_0^y(Y_1^x, \ldots, Y_n^x) =$$

$$= \int_{R^{n-1}} Ef_0^y(t_1, \ldots, t_{n-1}, Y_n^x)\, dF_x(t_1) \ldots dF_x(t_{n-1}) \leq$$

$$\leq \int_{R^{n-1}} Ef_0^y(t_1, \ldots, t_{n-1}, Y_n^y)\, dF_x(t_1) \ldots dF_x(t_{n-1}) + Nqp_{nn} =$$

$$\int_{R^{n-1}} f_1^y(t_1, \ldots, t_{n-1})\, dF_x(t_1) \ldots dF_x(t_{n-1}) + Nqp_{nn} =$$

$$= Ef_1^y(Y_1^x, \ldots, Y_{n-1}^x) + Nqp_{nn}.$$

By repeating this argument we obtain finally

$$L_n f(x) \leq Ef_{n-1}^y(Y_1^x) + Nq(p_{nn} + \ldots + p_{n2}) \leq Ef_{n-1}^y(Y_1^y) + Nq.$$

By virtue of (2) we have $L_n f(x) \leq L_n f(y) + Nq$. It follows immediately that $|L_n f(x) - L_n f(y)| \leq N|x-y|$, hence $L_n f \in \mathrm{Lip}(N; I)$.

The assertion concerning increasing functions can be proved similarly.

4. Monotonic convergence. In what follows we put $p_{n,n+1} = 0$, $n \geq 1$ and we shall suppose that

$$(p_{n,1}, \dots, p_{n,n+1}) \text{ majorizes } (p_{n+1,1}, \dots, p_{n+1,n+1}) \tag{3}$$

(Concerning majorization, see [10]).

THEOREM 2. *Under the above hypothesis we have $L_n f \geq L_{n+1} f$ iff f is convex.*

Proof. Let $x \in I$. If f is convex then the function

$$(q_1, \dots, q_{n+1}) \to Ef\left(\sum_{i=1}^{n+1} q_i Y_i^x\right)$$

is convex and symmetric, hence it is Schur-convex [10; 3.C.2]. Now from (3) it follows that

$$Ef\left(\sum_{i=1}^{n+1} p_{ni} Y_i^x\right) \geq Ef\left(\sum_{i=1}^{n+1} p_{n+1,i} Y_i^x\right)$$

This means that $L_n f(x) \geq L_{n+1} f(x)$ and the proof is finished.

Remark 2. The above proof is suggested by Theorems 3.7 and 3.8 of [6]. From Theorem 2 with $p_{ni} = 1/n$ we obtain the inequality contained in [7; Theorem 3] (see also [2]) and proved there by means of a martingale-type property and the conditional version of Jensen's inequality.

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