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ON A CERTAIN INEQUALITY USED IN THE THEORY OF DIFFERENCE EQUATIONS

B. G. PACHPATTE*

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REZUMAT. - Asupra unei inegalităţi folosite în teoria ecuaţiilor cu diferenţe. Sunt stabilite câteva noi inegalităţi cu diferenţe finite legate de o inegalitate folosită în teoria ecuaţiilor cu diferenţe.

Abstract. In the present paper we establish some new finite difference inequalities related to a certain inequality used in the theory of difference equations. The inequalities established here can be used as tools in the qualitative analysis of certain new classes of difference and sum-difference equations.

Introduction. In a recent paper [4, p.250] Mate and Navai used the following inequality while extending the well known results established by H. Poincaré in [9].

LEMMA. *Let* $u(n) \geq 0$, $p(n) \geq 0$ *be real-valued functions defined on integers and let* $c \geq 0$ *be a real constant. If*

$$u(n) \leq c + \sum_{s=n+1}^{\infty} p(s)\, u(s),$$

then

$$u(n) \leq c \exp\left(\sum_{s=n+1}^{\infty} p(s)\right).$$

Finite difference inequalities of this type are most useful in the qualitative analysis of various classes of difference equations. In the past few years, many papers on finite difference

* *Marathwada University, Department of Mathematics, Aurangabad 431004 (Maharashtra) India*

inequalities of the above type and their applications have appeared in the literature, see [1-8, 10] and the references given therein. In view of the important role played by such inequalities in the study of difference equations, it is natural to expect that some new finite difference inequalities of the type given in Lemma, would also be equally important in certain new applications. The main purpose of the present paper is to establish some new finite difference inequalities of the type given in Lemma, which can be used as tools in the analysis of certain new classes of difference and sum-difference equations for which earlier inequalities fail to apply directly. An application to obtain a bound on the solution of a certain sum-difference equation is also given.

2. Statement of results. In what follows we let $N_0 = \{0, 1, 2, \ldots\}$ and use the notations m, n, p, q to denote the elements of N_0. Let R denote the set of real numbers and $R_+ = [0,\infty)$. For $n > m$, $n,m \in N_0$ and any function $h: N_0 \to R$, we use the usual conventions $\sum_{s=n}^{m} h(s) = 0$ and $\prod_{s=n}^{m} h(s) = 1$. Throughout, without further mention, we assume that all the sums and products converge on the respective domain of their definitions.

Our main results are given in the following theorems.

THEOREM 1. *Let $u(n), f(n), g(n), h(n)$ be functions defined on N_0 into R_+ and $c \geq 0$ be a real constant.*

(i) *If*

$$u^2(n) \leq c^2 + 2\sum_{s=n+1}^{\infty} [f(s)u^2(s) + h(s)u(s)],\ n \in N_0, \tag{1}$$

then

$$u(n) \leq c\prod_{t=n+1}^{\infty} [1 + f(t)] + \sum_{s=n+1}^{\infty} h(s) \prod_{t=n+1}^{s-1} [1 + f(s)],\ n \in N_0 \tag{2}$$

(ii) *If*

$$u^2(n) \le c^2 + 2\sum_{s=n+1}^{\infty}\left[f(s)u(s)\left(u(s) + \sum_{t=s+1}^{\infty} g(t)u(t)\right) + h(s)u(s)\right],\ n \in N_0, \tag{3}$$

then

$$u(n) \le c\sum_{t=n+1}^{\infty}[1 + f(t) + g(t)] + \sum_{s=n+1}^{\infty} h(s)\cdot\prod_{t=n+1}^{s-1}[1 + f(t) + g(t)],\ n \in N_0. \tag{4}$$

(iii) *If*

$$u^2(n) \le c^2 + 2\sum_{s=n+1}^{\infty}\left[f(s)u(s)\left(\sum_{t=s+1}^{\infty} g(t)u(t)\right) + h(s)u(s)\right],\ n \in N_0, \tag{5}$$

then

$$u(n) \le c\sum_{t=n+1}^{\infty}\left[1 + f(t) + \sum_{\tau=t+1}^{\infty} g(\tau)\right] + \sum_{s=n+1}^{\infty} h(s)\prod_{t=n+1}^{s-1}\left[1 + f(t) + \sum_{\tau=t+1}^{\infty} g(\tau)\right],\ n \in N_0. \tag{6}$$

THEOREM 2. *Let $u(m,n)$, $f(m,n)$, $g(m,n)$, $h(m,n)$ be functions defined for $m,n \in N_0$ into R_+ and $c \ge 0$ is a real constant.*

(iv) *If*

$$u^2(m,n) \le c^2 + 2\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[f(s,t)u^2(s,t) + h(s,t)u(s,t)],\ m,n \in N_0, \tag{7}$$

then

$$u(m,n) \le \phi(m,n)\prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\right],\ m,n \in N_0, \tag{8}$$

where

$$\phi(m,n) = c + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} h(s,t). \tag{9}$$

(v) *If*

$$u^2(m,n) \le c^2 + 2\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[f(s,t)u(s,t)(u(s,t) + \sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)u(x,y)) + h(s,t)u(s,t)],\ m,n \in N_0, \tag{10}$$

then

$$u(m,n) \le \phi(m,n)\prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty}[f(s,t) + g(s,t)]\right],\ m,n \in N_0, \tag{11}$$

where $\phi(m,n)$ *is defined as in* (9).

(vi) *If*

$$u^2(m,n) \le c^2 + 2\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)u(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\,u(x,y)\right) + h(s,t)\,u(s,t)\right],\ m,n \in N_0, \tag{12}$$

then

$$u(m,n) \le \phi(m,n)\prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right)\right],\ m,n \in N_0, \tag{13}$$

where $\phi(m,n)$ *is defined as in* (9).

3. Proofs of theorems 1 and 2. Since the proofs of (i)-(vi) resemble one another, we give the details for (ii) and (vi) only, the proofs of the remaining inequalities can be completed by following the proofs of (ii) and (vi).

(ii) Define a function $z(n)$ by

$$z(n) = (c+\varepsilon)^2 + 2\sum_{s=n+1}^{\infty}\left[f(s)u(s)\left(u(s) + \sum_{t=s+1}^{\infty} g(t)\,u(t)\right) + h(s)\,u(s)\right], \tag{14}$$

where $\varepsilon > 0$ is an arbitrary small constant. From (14) and using the fact that $u(n+1) \le \sqrt{z(n+1)}$, $n \in N_0$, we observe that

$$z(n) - z(n+1) \le 2\sqrt{z(n+1)}\left[f(n+1)\left(\sqrt{z(n+1)} + \sum_{t=n+2}^{\infty} g(t)\sqrt{z(t)}\right) + h(n+1)\right]. \tag{15}$$

Using the facts that $\sqrt{z(n+1)} > 0$, $\sqrt{z(n+1)} \le \sqrt{z(n)}$ for $n \in N_0$ and (15) we observe that

$$\sqrt{z(n)} - \sqrt{z(n+1)} = \frac{z(n) - z(n+1)}{\sqrt{z(n)} + \sqrt{z(n+1)}} \le \frac{z(n) - z(n+1)}{2\sqrt{z(n+1)}}$$

$$\le f(n+1)\left(\sqrt{z(n+1)} + \sum_{t=n+2}^{\infty} g(t)\sqrt{z(t)}\right) + h(n+1). \tag{16}$$

Define a function $v(n)$ by

$$v(n) = \sqrt{z(n)} + \sum_{t=n+1}^{\infty} g(t)\sqrt{z(t)} . \tag{17}$$

From (17) and (16) it is easy to observe that

$$v(n) - [1 + f(n+1) + g(n+1)]\, v(n+1) \le h(n+1). \tag{18}$$

Now multiplying (18) by $\prod_{t=n+1}^{m} [1 + f(t) + g(t)]^{-1}$, for an arbitrary $m \in N_0$, then setting $n = s$ and taking the sum over $s = n, n+1, ..., m-1$ we obtain

$$v(n)\prod_{t=n+1}^{m} [1 + f(t) + g(t)]^{-1} \le v(m) + \sum_{s=n+1}^{m} h(s)\prod_{t=s}^{m} [1 + f(t) + g(t)]^{-1}. \tag{19}$$

From (19) we have

$$v(n) \le v(m)\prod_{t=n+1}^{m} [1 + f(t) + g(t)] + \sum_{s=n+1}^{m} h(s)\prod_{t=n+1}^{s-1} [1 + f(t) + g(t)]. \tag{20}$$

Noting that $\lim_{m\to\infty} v(m) = \lim_{m\to\infty} \sqrt{z(m)} = c + \varepsilon$ and letting $m \to \infty$ in (20) we get

$$v(n) \le (c + \varepsilon)\prod_{t=n+1}^{\infty} [1 + f(t) + g(t)] + \sum_{s=n+1}^{\infty} h(s)\sum_{t=n+1}^{s-1} [1 + f(t) + g(t)]. \tag{21}$$

The required inequality in (4) now follows from (21) and using the facts that $u(n) \le \sqrt{z(n)}$ and $\sqrt{z(n)} \le v(n)$ and by taking $\varepsilon \to 0$.

(vi) Define a function $z(m,n)$ by

$$z(m,n) = (c+\varepsilon)^2 + 2\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f(s,t)\,u(s,t)\cdot\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\,u(x,y)\right) + h(s,t)\,u(s,t)\right], \tag{22}$$

where $\varepsilon > 0$ is an arbitrary small constant. From (22) and using the facts that $u(m,n) \le \sqrt{z(m,n)}$ for $m,n \in N_0$, we observe that

$$[z(m,n) - z(m+1,n)] - [z(m,n+1) - z(m+1,n+1)]$$

$$\le 2\sqrt{z(m+1,n+1)}\left[f(m+1,n+1)\left(\sum_{x=m+2}^{\infty}\sum_{y=n+2}^{\infty} g(x,y)\sqrt{z(x,y)}\right) + h(m+1,n+1)\right]. \tag{23}$$

Using the facts that $\sqrt{z(m,n)} > 0$, $\sqrt{z(m,n+1)} \le \sqrt{z(m,n)}$, $\sqrt{z(m+1,n+1)} \le \sqrt{z(m+1,n)}$,

$\sqrt{z(m+1,n+1)} \le \sqrt{z(m,n+1)}$ for $m,n \in N_0$, we observe that (see, [7, p.379])

$$\left[\sqrt{z(m,n)} - \sqrt{z(m+1,n)}\right] = \frac{[z(m,n) - z(m+1,n)]}{\left[\sqrt{z(m,n)} + \sqrt{z(m+1,n)}\right]},$$

and

$$\left[\sqrt{z(m,n)} - \sqrt{z(m+1,n)}\right] - \left[\sqrt{z(m,n+1)} - \sqrt{z(m+1,n+1)}\right]$$

$$\le \frac{[z(m,n) - z(m+1,n)] - [z(m,n+1) - z(m+1,n+1)]}{\left[\sqrt{z(m+1,n+1)} + \sqrt{z(m+1,n+1)}\right]}. \tag{24}$$

From (24) and (23) we observe that

$$\left[\sqrt{z(m,n)} - \sqrt{z(m+1,n)}\right] - \left[\sqrt{z(m,n+1)} - \sqrt{z(m+1,n+1)}\right]$$

$$\le f(m+1,n+1)\left(\sum_{x=m+2}^{\infty}\sum_{y=n+2}^{\infty} g(x,y)\sqrt{z(x,y)}\right) + h(m+1,n+1). \tag{25}$$

Now keeping m fixed in (25), set $n = t$ and sum over $t = n, n+1, \ldots, q-1$ to obtain

$$\left[\sqrt{z(m,n)} - \sqrt{z(m+1,n)}\right] - \left[\sqrt{z(m,q)} - \sqrt{z(m+1,q)}\right]$$

$$\le \sum_{t=n+1}^{q}\left[f(m+1,t)\left(\sum_{x=m+2}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\sqrt{z(x,y)}\right) + h(m+1,t)\right]. \tag{26}$$

Noting that $\lim_{q\to\infty}\sqrt{z(m,q)} = \lim_{q\to\infty}\sqrt{z(m+1,q)} = c + \varepsilon$, and by letting $q \to \infty$ in (26) we get

$$\left[\sqrt{z(m,n)} - \sqrt{z(m+1,n)}\right]$$

$$\le \sum_{t=n+1}^{\infty}\left[f(m+1,t)\left(\sum_{x=m+2}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\sqrt{z(x,y)}\right) + h(m+1,t)\right]. \tag{27}$$

Keeping n fixed in (27), set $m = s$ and sum over $s = m, m+1, \ldots, p-1$ to obtain

$$\sqrt{z(m,n)} - \sqrt{z(p,n)} \le \sum_{s=m+1}^{p}\sum_{t=n+1}^{\infty}\left[f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\cdot\sqrt{z(x,y)}\right) + h(s,t)\right]. \tag{28}$$

Noting that $\lim_{p\to\infty}\sqrt{z(p,n)} = c+\varepsilon$, and by letting $p \to \infty$ in (28) we get

$$\sqrt{z(m,n)} \le \phi_\varepsilon(m,n) + \sum_{s=n+1}^{\infty}\sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\sqrt{z(x,y)}\right), \tag{29}$$

where $\phi_\varepsilon(m,n) = c+\varepsilon+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} h(s,t)$. From (29) it is easy to observe that

$$\frac{\sqrt{z(m,n)}}{\phi_\varepsilon(m,n)} \le 1 + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\frac{\sqrt{z(x,y)}}{\phi_\varepsilon(x,y)}\right). \tag{30}$$

Define $v(m,n)$ by

$$v(m,n) = 1 + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\frac{\sqrt{z(x,y)}}{\phi_\varepsilon(x,y)}\right). \tag{31}$$

From (31) and (30) it is easy to observe that

$$[v(m,n)-v(m+1,n)]-[v(m,n+1)-v(m+1,n+1)]$$

$$\le f(m+1,n+1)\left(\sum_{x=m+2}^{\infty}\sum_{y=n+2}^{\infty} g(x,y)\right)v(m+1,n+1). \tag{32}$$

From the definition of $v(m,n)$ given in (31) we observe that $v(m+1,n+1) \le v(m+1,n)$ for $m,n \in N_0$. Using this in (32) we observe that

$$\frac{[v(m,n)-v(m+1,n)]}{v(m+1,n)} - \frac{[v(m,n+1)-v(m+1,n+1)]}{v(m+1,n+1)}$$

$$\le f(m+1,n+1)\sum_{x=m+2}^{\infty}\sum_{y=n+2}^{\infty} g(x,y). \tag{33}$$

Now keeping m fixed in (33), set $n = t$ and sum over $t = n, n+1, \ldots, q-1$ to obtain

$$\frac{[v(m,n)-v(m+1,n)]}{v(m+1,n)} - \frac{[v(m,q)-v(m+1,q)]}{v(m+1,q)}$$

$$\le \sum_{t=n+1}^{q} f(m+1,t)\left(\sum_{x=m+2}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right). \tag{34}$$

Noting that $\lim_{q\to\infty} v(m,q) = \lim_{q\to\infty} v(m+1,q) = 1$, and by letting $q \to \infty$ in (34) we get

$$\frac{v(m,n) - v(m+1,n)}{v(m+1,n)} \le \sum_{t=n+1}^{\infty} f(m+1,t)\left(\sum_{x=m+2}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right). \tag{35}$$

From (35) we have

$$v(m,n) \le v(m+1,n)\left[1 + \sum_{t=n+1}^{\infty} f(m+1,t)\left(\sum_{x=m+2}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right)\right]. \tag{36}$$

Now keeping n fixed in (36), set $m = s$ and sum over $s = m, m+1, \ldots, p-1$ succesively to obtain

$$v(m,n) \le v(p,n)\prod_{s=m+1}^{p}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right)\right]. \tag{37}$$

Noting that as $p \to \infty$, $v(p,n) = 1$, and letting $p \to \infty$ in (37) we have

$$v(m,n) \le \prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\left(\sum_{x=s+1}^{\infty}\sum_{y=t+1}^{\infty} g(x,y)\right)\right]. \tag{38}$$

The desired inequality in (13) now follows by using (38) in (30), the fact that $u(m,n) \le \sqrt{z(m,n)}$ and by taking $\varepsilon \to 0$. This completes the proof of (vi).

4. An application. In this section we present an application of our inequality given in Theorem 1 part (i) to obtain bound on the solution of the following sum-difference equation

$$y^2(n) = p(n) + \sum_{s=n+1}^{\infty} k(n,s)\,y(s)\,F(s,y(s)),\ n \in N_0, \tag{39}$$

where $p : N_0 \to R$, $k : N_0 \times N_0 \to R$, $F : N_0 \times R \to R$. We assume that

$$|p(n)| \le c^2,\ |k(n,s)F(s,y(s))| \le 2\,[f(s)|y(s)| + h(s)], \tag{40}$$

where f, h and c are as defined in Theorem 1. From (39) and (40) we obtain

$$|y(n)|^2 \le c^2 + 2\sum_{s=n+1}^{\infty}[f(s)\,|y(s)|^2 + h(s)|y(s)|\,]. \tag{41}$$

Now an application of the inequality given in Theorem 1 part (i) to (41) yields

$$|y(n)| \le c\prod_{t=n+1}^{\infty}[1 + f(t)] + \sum_{s=n+1}^{\infty} h(s)\prod_{t=n+1}^{s-1}[1 + f(t)],\ n \in N_0. \tag{42}$$

The inequality (42) gives the bound on the solution $y(n)$ of equation (39) in terms of the known functions.

Finally, we note that the inequalities established in Theorem 2 can be extended very easily to functions of several independent variables. We also note that there are many possible applications of the inequalities established in Theorems 1 and 2 to certain new classes of difference and sum-difference equations. However, the discussion of such applications is left to another place.

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