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HYPERBOLIC MEAN VALUE THEOREMS OF NON-DIFFERENTIAL FORM

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REZUMAT. - Teoreme de medie de tip hiperbolic în formă nediferențială. În lucrare sunt stabilite mai multe teoreme de medie pentru funcții definite pe un dreptunghi.

1. Introduction. Let I and J be nonempty intervals of the real axis (R). Denote for simplicity $K = I \times J$. By a (standard) *rectangle* of K we mean any subset $\Delta = [a,b] \times [c,d]$ of K, where $[a,b]$, $[c,d]$ are closed sub-intervals of I and J, respectively. In this case, the points

$$A = (a,c),\ B = (b,c),\ C = (b,d),\ D = (a,d)$$

are called the *vertices* of Δ; correspondingly, the rectangle in question may be represented as [ABCD].

Let $(X, \| \ \|)$ be a normed space and $f: K \to X$, a mapping. For each rectangle Δ of K taken as above, denote

$$m_f(\Delta) = f(A) - f(B) + f(C) - f(D). \tag{1.1}$$

This will be referred to as the *hyperbolic* (Lebesque-Stieltjes) *measure* of Δ generated by this function. Note that, when $X = R$, and

$$f(t,s) = ts,\ t,s \in R$$

then, this hyperbolic measure reduces to

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$$m(\Delta) = ac - bc + bd - ad = (b-a)(d-c) \tag{1.2}$$

(the *usual* Lebesgue *measure* of Δ). Finally, denote

$$R_f(\Delta) = \frac{m_f(\Delta)}{m(\Delta)},\ S_f(\Delta) = \|R_f(\Delta)\|. \tag{1.3}$$

These will be referred to as the *variational quotients* of f with respect to Δ.

Now, by a mean value theorem/property for f over Δ we mean an evaluation of $R_f(\Delta)$ of $S_f(\Delta)$ with the aid of some expressions depending on the objective to be attained. More precisely, we may distinguish between

i) mean value theorems of *non-differential* (relative) *form;*

ii) mean value theorems of *differential form.*

The second class of such properties was investigated-in the bi-dimensional setting we dealt with - by Nicolescu [12, ch.19, §2], under the lines in Bögel [6,7]; see also Dobrescu [8]. The first class of such results was only tangentially discussed until now in the paper by Nicolescu [10]. It is our main aim in the present exposition to fill this gap, in a manner suggested by the one-dimensional developments in this area due to the authors [4,5]; see also Aziz and Diaz [1,2,3]. The imposed assumptions upon f are intended to be the largest possible ones; details will be given in Section 3. All preliminary facts were collected in Section 2. And, in Section 4, some aspects involving the real case ($X = R$) will be considered. Finally, it is worth noting these developments are an essential tool to get mean value theorems under differential form. A detailed account of these will be made in a future paper.

2. Preliminaries. Let again I, J be real intervals and $K = I \times J$. We also give a normed space $(X, \|\ \|)$ and take a mapping $f: K \to X$. It is our aim in the following to investigate this function by means of the associated map $\Delta \mapsto m_f(\Delta)$.

We start with an invariance property. By a *hyperbolic constant* over K we mean any map $h : K \to X$ of the form

$$h(t,s) = \varphi(t) + \psi(s),\ (t,s) \in K$$

where $\varphi : I \to X$, $\psi : J \to X$ are given functions. This term is justified by the statement below. (The proof being evident, we do not give details.)

PROPOSITION 1. *For each rectangle Δ of K and each hyperbolic constant h over K, one has*

$$m_{f+h}(\Delta) = m_f(\Delta). \tag{2.1}$$

As an immediate consequence,

$$R_{f+h}(\Delta) = R_f(\Delta) \text{ (hence } S_{f+h}(\Delta) = S_f(\Delta)).$$

In other words, any property of $R_f(\Delta)$ (or $S_f(\Delta)$) may be also tranferred to the function $f + h$ which, in principle, is no longer endowed with the properties of f. Some concrete examples in this direction will be given in Sections 3 and 4.

We are now passing to an additivity property. For any rectangle $\Delta = [a,b] \times [c,d]$ of K, denote

$$\text{int}\,(\Delta) =]a,b[\times]c,d[\text{ (the } \textit{interior} \text{ of } \Delta)$$

This is of course related to the topological structure of the plane given, e.g., by the maximum norm. By a *division* of the rectangle Δ we mean any finite decomposition $\Delta = \bigcup\limits_r \Delta_r$ of Δ into (standard) rectangles of K with the family $\{\text{int}\,(\Delta_r)\}$ being mutually disjoint. Among these, we distinguish the divisions of Δ generated by corresponding divisions of the real intervals generating Δ. Precisely, given finite decompositions

$$[a,b] = \bigcup_i\, [t_i, t_{i+1}],\ [c,d] = \bigcup_j [s_j, s_{j+1}]$$

of these intervals, the considered division may be written as $\Delta = \bigcup\limits_{i,j} \Delta_{ij}$, where

$\Delta_{ij} = [t_i, t_{i+1}] \times [s_j, s_{j+1}]$, for all possible (i,j). These will be referred to in the sequel as *normal divisions* of the underlying rectangle.

PROPOSITION 2. *Let* $\{\Delta_r\}$ *be a division of the rectangle* Δ. *Then*

$$m_f(\Delta) = \sum_r m_f(\Delta_r). \tag{2.2}$$

Proof. Take any vertex, P, of an arbitrary rectangle in this decomposition, distinct from the vertices of Δ. A simple analysis shows that P belongs to either two or four rectangles in this family. (The proof being almost evident, we do not give details.) Let $\leq$ be the ordering in R^2 introduced in the usual way

$$(t_1, s_1) \leq (t_2, s_2) \ \text{ iff } \ t_1 \leq t_2,\ s_1 \leq s_2 .$$

In the first case, the point in question is extremal in one rectangle and non-extremal in another. In the second case, the considered point is two times extremal and two times non-extremal in the rectangles to which it belongs. Consequently, the contribution of $f(P)$ in $\sum_r m_f(\Delta_r)$ is zero, by the definition of these expressions. In other words, only the vertices of Δ are to be retained in this sum, and conclusion follows. ■

Remark. A different proof of this may be given along the following lines (cf. Tolstov [15, ch.2, §6]). Let $\mathcal{V}$ be the set of all vertices for the rectangles in $\{\Delta_r\}$. The projection of $\mathcal{V}$ over $[a,b]$, respectively $[c,d]$ gives finite decompositions of such intervals. Let

$$\Delta = \cup\{\Delta_{ij}; (i,j) \in \Gamma\}$$

be the normal division of Δ induced by these. It clearly follows by the described construction that a partition $\Gamma = \bigcup_r \Gamma_r$ of the index set Γ may be found so that, for each r,

$$\{\Delta_{ij};\ (i,j) \in \Gamma_r\} \text{ is a normal division of } \Delta_r.$$

This, plus (2.2) being valid for normal divisions imply

$$m_f(\Delta) = \sum \{m_f(\Delta_{ij}); (i,j) \in \Gamma\} = \sum_r \sum \{m_f(\Delta_{ij}); (i,j) \in \Gamma_r\} = \sum_r m_f(\Delta_r)$$

and the assertion is proved.

Remark. Of course, the conclusion of this statement remains valid (via Proposition 1) in case f is to be replaced by $f+h$, where h is any hyperbolic constant (over K).

Now, a useful semi-continuity result will be proved. For any pair of points P,Q in the plane, we denote by $PQ = \{\lambda P + (1-\lambda)Q;\ 0 \le \lambda \le 1\}$ the *segment* between these points and by $(PQ) = \{\lambda P + (1-\lambda)Q;\ \lambda \in R\}$ the *line* passing through P and Q. Let $\Delta = [ABCD]$ be a rectangle in K, given by its vertices. Denote

$$fr(\Delta) = AB \cup BC \cup CD \cup DA \text{ (the } \textit{boundary} \text{ of } \Delta).$$

Let P be a point of $fr(\Delta)$, distinct from the vertices of Δ. There exists a unique line passing through P, which is orthogonal to the segment of $fr(\Delta)$, which contains P. This will be referred to as the *normal* to Δ at the considered point, and denoted $\nu_\Delta(P)$. (That P must be distinct from the vertices of Δ in this construction is a consequence of the fact that, otherwise, the normal in question would be not uniquely determined.) Now, call the underlying function $f: K \to X$, *normally continuous* at the point $P \in fr(\Delta)$ (distinct from A,B,C,D) when its restriction to $\nu_\Delta(P) \cap \Delta$ is continuous at P. We also term f, normally continuous on $fr(\Delta)$ when it is normally continuous at any point $P \in fr(\Delta)$ (distinct from the vertices of Δ).

With these conventions, let Δ be a rectangle in K. We also take a sub-rectangle Δ' of Δ in such a way that $fr(\Delta')$ has at least a segment in common with $fr(\Delta)$.

PROPOSITION 3. *Suppose that*

(H.1) *f is continuous at the vertices of Δ*

(H.2) *f is normally continuous at each vertex of Δ' (if any) lying in $fr(\Delta)$, distinct from the vertices of Δ.*

Then, for each $\eta > 0$, there exists a sub-rectangle Δ'' of Δ', interior to Δ, with

$$S_f(\Delta'') \geq (1-\eta) S_f(\Delta'). \tag{2.3}$$

Proof. Without loss, one may assume $m_f(\Delta) \neq 0$ (hence $S_f(\Delta) \neq 0$). We have several situations to discuss.

Case 1. $fr(\Delta')$ has a single segment in common with $fr(\Delta)$ This, e.g., corresponds to the choice $\Delta' = [A'B'C'D']$ where $A'B' \subset AB$ and $C', D' \in \mathrm{int}(\Delta)$; or, in other words (by the adopted notations for the rectangle Δ)

$$A' = (a', c),\ B' = (b', c),\ C' = (b', r),\ D' = (a', r)$$

with $a < a' < b' < b,\ c < r < d$. We now consider the sub-rectangle Δ_λ of K given by the vertices $A'_\lambda, B'_\lambda, C', D'$, where

$$A'_\lambda = (a', c+\lambda),\ B' = (b', c+\lambda),\ \lambda > 0 \text{ small enough.}$$

Clearly, Δ'_λ is in $\Delta' \cap \mathrm{int}(\Delta)$ for all such λ. Moreover, by (H.2),

$$f(A'_\lambda) \to f(A'),\ f(B'_\lambda) \to f(B) \text{ as } \lambda \to 0_+.$$

This, combined with

$$m(\Delta'_\lambda) \to m(\Delta') \text{ as } \lambda \to 0_+ \tag{2.4}$$

shows

$$R_f(\Delta'_\lambda) \to R_f(\Delta') \text{ (hence } S_f(\Delta'_\lambda) \to S_f(\Delta')) \text{ as } \lambda \to 0_+. \tag{2.5}$$

As a consequence, any Δ'_λ, where $\lambda > 0$ is sufficiently small, may be taken as the sub-rectangle Δ'' in the statement.

Case 2. $fr(\Delta')$ has two segments in common with $fr(\Delta)$. This, for example, may be understood as the rectangle in question being represented in the form $\Delta' = [AB'C'D']$, where

$$B' = (p, c),\ C' = (p, q),\ D' = (a, q),\ a < p < b,\ c < q < d.$$

Let us now construct a sub-rectangle Δ'_λ of Δ by the vertices $A_\lambda, B'_\lambda, C', D'_\lambda$, where

$$A_\lambda = (a+\lambda, c+\lambda),\ B'_\lambda = (p, c+\lambda),\ D'_\lambda = (a+\lambda, q).$$

(As before, $\lambda > 0$ is small enough). That $\Delta' \cap \mathrm{int}(\Delta)$ includes Δ'_λ is clear.

We also have, by (H.1) + (H.2),

$$f(A_\lambda) \to f(A),\ f(B'_\lambda) \to f(B'),\ f(D'_\lambda) \to f(D') \text{ as } \lambda \to 0_+.$$

This, in combination with (2.4) being valid in this context gives again (2.5). Hence, any Δ'_λ like before - where $\lambda > 0$ is sufficiently small - is a candidate for sub-rectangle Δ'' in the statement.

Cases 3-4. $fr(\Delta')$ has more than two segments in common with $fr(\Delta)$. (That is, either, $fr(\Delta')$ has three segments in common with $fr(\Delta)$ or else $\Delta' = \Delta$). The argument we just developed may be correspondingly modified to get a family of sub-rectangles $\{\Delta'_\lambda\}$ of Δ', interior to Δ, which in addition has the property (2.5). So, as before, it will suffice taking one of these as Δ'', to get (2.3). Having explored all possible situations, the conclusion follows. ∎

Remark. The working conditions (H.1) + (H.2) must be taken in a relative sense only. Because as results from Proposition 1, the statement above remains valid whenever f-h fulfils (H.1) + (H.2) for some hyperbolic constant h: $K \to X$ (which, in particular, may be discontinuous at any point of the rectangle Δ).

As an immediate consequence of this, we have

COROLLARY 1. *Suppose that the underlying function f satisfies* (H.1) *plus*

(H.3) *f is normally continuous over $fr(\Delta)$.*

Then, conclusion of Proposition 2 *is retainable.*

In particular, a sufficient condition for (H.1) + (H.3) is

(H.3)′ *f is continuous over $fr(\Delta)$.*

Of course, as already precised, these conditions may be put in an even more general framework, via Proposition 1; further details are not given.

Finally, a specific continuity property will be introduced for such functions. Given a pair $P_1 = (t_1, s_1)$, $P_2 = (t_2, s_2)$ of points (in K), denote by $[P_1;P_2]$ the rectangle $[a,b] \times [c,d]$, where

$$a = \min(t_1, t_2),\ b = \max(t_1, t_2);\ c = \min(s_1, s_2),\ d = \max(s_1, s_2).$$

Of course, the order of these points is not essential, here; i.e., $[P_1;P_2]$ is identical to $[P_2;P_1]$. Let P be an interior point of K. We say the function f: $K \to X$ is *hyperbolic continuous* at P whenever

$$m_f([P;Q]) \to 0 \text{ as } Q \to P;$$

or, in other words, for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\|m_f([P;Q])\| < \varepsilon \text{ provided } \|P - Q\| < \delta(\varepsilon).$$

Likewise, the considered function is called hyperbolic continuous over a subset of K when it is hyperbolic continuous at each point of that subset.

The relationships between this notion and the standard continuity one are precised in

PROPOSITION 4. *The following are valid*:

A) *If the function* f: $K \to X$ *is continuous at the point* $P \in \text{int}(K)$ *then it is hyperbolic continuous at this point.*

B) *Suppose the function* f: $K \to X$ *is hyperbolic continuous at* $P \in \text{int}(K)$. *Then, a continuous at* P *function* $g = g_P$: $K \to X$ *and a hyperbolic constant* $h = h_P$: $K \to X$ *may be found so that* f *be represented as the sum* $g+h$.

Proof. The first part is evident. For the second one note that the hyperbolic continuity of f at $P = (t_0, s_0)$ may be also written as

$$f(t_0,s_0) - f(t_0,s) - f(t,s_0) + f(t,s) \to 0, \text{ as } t \to t_0,\ s \to s_0.$$

Denote in this case

$$g(t,s) = f(t,s) - f(t_0,s) - f(t,s_0),\quad (t,s) \in K.$$

$$h(t,s) = f(t_0,s) + f(t,s_0),\quad (t,s) \in K.$$

That g,h satisfy the above requirements is clear. Hence the conclusion. ■

Remark. This result does not admit, in general, a global counterpart. In other words, if $f\colon K \to X$ is hyperbolic continuous over a part H of K then, a representation like $f = g+h$ where $g\colon K \to X$ is continuous over H and $h\colon K \to X$ is a hyperbolic constant (over K) is not obtainable, in general. For an example in this direction we refer to Nicolescu [12, ch.19, §2].

3. Main results (inequality form). Let the notations above be maintained. Letting I,J be real intervals, for each rectangle $\Delta = [a,b] \times [c,d]$ in $K = I \times J$, denote

$$\operatorname{diam}(\Delta) = \max(b-a,\ d-c) \text{ (the } diameter \text{ of } \Delta).$$

This notion is related to the normed structure of the plane (given by the maximum norm). Let also $(x, \|\ \|)$, a normed space and $f : K \to X$, a mapping. As a consequence of the developments above, the first main result of the present paper is

THEOREM 1. *Let Δ be a (standard) rectangle in K. Then, for each $\varepsilon > 0$, there is a sub-rectangle Δ_ε of Δ with*

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon,\quad S_f(\Delta) \le S_f(\Delta_\varepsilon). \tag{3.1}$$

Proof. Construct a (normal) division of Δ by

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b,\ c = s_0 < s_1 < \dots < s_{m-1} < s_m = d$$

with

$$\max(t_{i+1} - t_i, s_{j+1} - s_j) < \varepsilon,\ 0 \le i \le n-1,\ 0 \le j \le m-1.$$

Here, $n,m \geq 3$ are positive integers. Precisely, if we put

$$\Delta_{ij} = [t_i, t_{i+1}] \times [s_j, s_{j+1}],\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1,$$

the normal division in question is $\{\Delta_{ij}\}$. In addition, we have the supplementary property

$$\operatorname{diam}(\Delta_{ij}) < \varepsilon, \text{ for all possible } (i,j).$$

It is clear, via Proposition 2, that

$$R_f(\Delta) = \sum_{i,j} \lambda_{ij} R_f(\Delta_{ij})$$

where, by convention,

$$\lambda_{ij} = \frac{m(\Delta_{ij})}{m(\Delta)},\ \ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1.$$

Therefore, by the triangle inequality,

$$S_f(\Delta) \leq \sum_{i,j} \lambda_{ij} S_f(\Delta_{ij}).$$

The second of this relation is a convex combination of $\{S_f(\Delta_{ij})\}$. Hence The conclusion. ■

Now, by simply adding to this the remark in Section 2 concerning the alternative proof of Proposition 2, one gets

COROLLARY 2. *Let* Δ *be a rectangle in* K *and* $\{\Delta_r\}$ *be a division of* Δ. *Then, for each* $\varepsilon > 0$, *there exists an index* $r = r(\varepsilon)$ *and a sub-rectangle* Δ_ε *of* Δ_r *so that* (3.1) *be valid.*

Note at this moment that no property is required for the function f to get the conclusion in the statement. Nevertheless, the obtained assertion is not very sharp because the possibility that $fr(\Delta_\varepsilon)$ should have a nonempty intersection with $fr(\Delta)$ cannot be avoided in general. It is natural to ask of whether is this removable. The answer is affirmative (via Proposition 3). To state it, we need a new convention. Let Δ be a rectangle in K. Take eight points systems $\{E_1, ..., E_8\}$ on the boundary of Δ, distincts from the vertices of Δ, according to the condition:

there exists a sub-rectangle Δ', interior to Δ such that $\{E_1, ..., E_8\}$ appears as the projection over $fr(\Delta)$ of the vertices of Δ'. (3.2)

Such systems will be termed *admissible* in what follows. Now, let again $(X, \| \ \|)$ be a normed space and $f: K \to X$ a mapping. As a completion of Theorem 1, the second main result of this paper is

THEOREM 2. *Suppose that f satisfies* (H.1) *plus*

(H.4) *f is normally continuous over at least one admissible eight points system* $fr(\Delta)$.

Then, for each $\varepsilon > 0$, *there is a sub-rectangle* Δ_ε *interior to* Δ, *with the property* (3.1).

Proof. Let the ambient rectangle Δ be represented as [ABCD]. Take also an admissible eight points system $\{E_1, ..., E_8\}$ in $fr(\Delta)$ (given by (H.4)). So, there exists a sub-rectangle Δ_0 = [MNPQ] interior to Δ, such that $\{E_1, ..., E_8\}$ appears as the projection of $\mathcal{V} = \{M,N,P,Q\}$ over $fr(\Delta)$. This, e.g., may be understood as

$$MQ \cap (AB \cup CD) = \{E_1, E_5\};\ NP \cap (AB \cup CD) = \{E_2, E_6\}$$
$$MN \cap (AD \cup BC) = \{E_3, E_7\};\ PQ \cap (AD \cup BC) = \{E_4, E_8\}.$$

Now, the admissible system $\{E_1, ..., E_8\}$ generates a normal division $\{\Delta_0, ..., \Delta_8\}$ of Δ. (Here, Δ_0 is the above sub-rectangle and, e.g., $\Delta_1 = [AE_1ME_7]$, $\Delta_2 = [E_1E_2NM]$, etc.) This gives at once

$$R_f(\Delta) = \sum_{i=0}^{8} \mu_i R_f(\Delta_i)$$

where, by convention,

$$\mu_i = \frac{m(\Delta_i)}{m(\Delta)},\ 0 \le i \le 8.$$

So, by the triangle inequality,

$$S_f(\Delta) \le \sum_{i=0}^{8} \mu_i S_f(\Delta_i). \tag{3.3}$$

As an immediate consequence of this,

$$S_f(\Delta) \le \max_{0 \le i \le 8} S_f(\Delta_i). \tag{3.4}$$

We have two cases to discuss.

a) Relation (3.4) is holding with equality. Then, again combining with (3.3),

$$S_f(\Delta) = \sum_{i=0}^{8} \mu_i S_f(\Delta_i),$$

wherefrom

$$\sum_{i=0}^{8} \mu_i (S_f(\Delta) - S_f(\Delta_i)) = 0.$$

But, μ_0, ..., μ_8 are strictly positive. Therefore

$$S_f(\Delta) = S_f(\Delta_i),\ 0 \le i \le 8;$$

and from this, conclusion is clear.

b) Relation (3.4) is holding strictly (with $<$ in place of $\le$). If one happens that $S_f(\Delta) < S_f(\Delta_0)$, then we are done (by applying Theorem 1 to the same function f and the rectangle Δ_0). Otherwise,

$$S_f(\Delta) < S_f(\Delta_i), \text{ for some } i \in \{1, ..., 8\}.$$

By (H.4) plus Proposition 3, we must have that for each $\eta > 0$ (small enough) there exists a sub-rectangle $\Delta_i^{(\eta)}$ of Δ_i, interior to Δ, with

$$S_f(\Delta_i^{(\eta)}) \ge (1 - \eta) S_f(\Delta_i).$$

Choose $\eta > 0$ in such a way that $(1 - \eta) S_f(\Delta_i) \ge S_f(\Delta)$. (This is possible, by the strict inequality above.) Combining these, yields

$$S_f(\Delta) \le S_f(\Delta_i^{(\eta)});$$

and this, again with Theorem 1 gives conclusion in the statement.

Now, a) + b) are the only possible situations in this discussion. Hence the result. ■

As a direct consequence of this, we have

COROLLARY 3. *Let the rectangle Δ in K and the function f: $K \to X$ be such that* *conditions* (H.1) + (H.3) *are accepted. Then, conclusion of Theorem* 2 *is retainable.*

In particular, a sufficient condition for (H.1) + (H.3) is $(\mathrm{H.3})'$. A natural question appearing in this context is that of determining to what extent are these statements valid when $(\mathrm{H.3})'$ is to be substituted by its weaker counterpart

$(\mathrm{H.3})^{*}$ *f is hyperbolic continuous over* $fr(\Delta)$.

To give a partial answer, we note that, by Proposition 4, one has at each point P in $fr(\Delta)$, the representation $f = g_P + h_P$ where $g_P\colon K \to X$ is continuous at P and $h_P\colon K \to X$ is a hyperbolic constant. Hence the functions in this representation are depending on the points in $fr(\Delta)$. But, if this dependence would be removed (i.e., the underlying functions remain unchanged when P describes $fr(\Delta)$) it follows by Proposition 1 that, in fact, $(\mathrm{H.3})'$ is necessarily fulfilled under $(\mathrm{H.3})^{*}$; and so, conclusion of Theorem 2 is retainable, in view of Corollary 3. Summing up, hyperbolic continuity conditions (over $fr(\Delta)$ or, even, the all of Δ) imposed upon f are - generally - insufficient for the truth of such results. This, in particular, applied to the statement of Lemma 1 in Nicolescu [10], shows we must delete the word "hyperbolic" (as a weaker form of continuity for f) to retain its conclusion. But then, the result in question reduced to Corollary 3 above.

Remark. From a methodological viewpoint, the developments above may be viewed as a bi-dimensional counterpart of the contributions in this area due to Bantaş and Turinici [4]; see also Aziz and Diaz [1].

Now, it would be of interest to determine of whether or not is (H.4) removable; or, in other words, to what extent can we diminish the cardinality of an admissible system (of points in $fr(\Delta)$). The answer is affirmative: it is based on a few remarks about the associated sub-rectangles in the division of Δ. Let $\{E_1, \ldots, E_8\}$ be an admissible eight points system in $fr(\Delta)$

generated by a sub-rectangle $\Delta' = [MNPQ]$ of Δ (and interior to Δ). We associate to each vertex of Δ' its closest projections over $fr(\Delta)$. This generates a decomposition of our system into four groups of such projections. For example, under the notations encountered in the proof of Theorem 2, these groups may be depicted as

$$U_1 = \{E_1, E_7\},\ U_2 = \{E_2, E_3\},\ U_3 = \{E_4, E_6\},\ U_4 = \{E_5, E_8\}.$$

Now, let us call a four points system $\{G_1, G_2, G_3, G_4\}$ in $\{E_1, \ldots, E_8\}$, *admissible* provided

$$G_i \in U_i,\ 1 \le i \le 4.$$

There are $2^4 = 16$ such admissible four points systems generated by an admissible eight points system. However, for symmetry reasons only 4 systems from these are essential. For example, taking AB as a basis, the systems in question are

$$\{E_7, E_2, E_4, E_5\},\ \{E_7, E_3, E_6, E_5\},\ \{E_7, E_3, E_4, E_6\},\ \{E_7, E_3, E_4, E_8\}.$$

Now, given any admissible four points system $G = \{G_1, G_2, G_3, G_4\}$, there exists a division

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$$

of the rectangle Δ, where Δ_0 is the above one and (for $1 \le i \le 4$) the vertices of the sub-rectangle Δ_i lyung in $fr(\Delta)$ and distinct from those of Δ are necessarily in G. (For example, to verify this for $G = \{E_7, E_2, E_4, E_5\}$, it will suffice putting

$$\Delta_1 = [AE_2NE_7],\ \Delta_2 = [E_2BE_4P],\ \Delta_3 = [E_4CE_5Q],\ \Delta_4 = [E_7ME_5D];$$

the remaining situations are treatable in a similar way.) As a direct consequence, the argument used in Theorem 2 is also applicable to this larger setting. We thus proved

COROLLARY 4. *Suppose that f satisfies conditions* (H.1) *plus*

(H.4)′ *f is normally continuous over at least one admissible four points system of $fr(\Delta)$.*

Then, for each $\varepsilon > 0$, there is a sub-rectangle Δ_ε interior to Δ, with the property (3.1).

Concerning the further reduction of this number, call the two points system $\{E_1,E_2\}$ of $fr(\Delta)$ (distinct from the vertices of Δ), *admissible*, when E_1,E_2 are an opposite segments of $fr(\Delta)$ and the normals to E_1 and E_2 are identical (e.g., $E_1 \in AB$, $E_2 \in CD$ and E_1,E_2 is parallel to AD or BC). Suppose now (H.4)' is to be replaced by

(H.4)" *f is normally continuous over an admissible two points system of $fr(\Delta)$.*

Let Δ_1 and Δ_2 be the division of Δ generated by E_1E_2 (in the usual way) and assume

(H.5) $S_f(\Delta) < S_f(\Delta_i)$, *for some* $i \in \{1,2\}$.

By Proposition 3, there must be a sub-rectangle Δ_i' of Δ_i, interior to Δ, with $S_f(\Delta) < S_f(\Delta_i')$; this, plus Theorem 1 give us immediately conclusion of Theorem 2. Therefore, condition (H.4) - or its variants - has a relative character (from a cardinality viewpoint). This forces us to ask of whether or not is this condition effective in such statements. We conjecture that the answer is negative.

4. The real case. In the following, the choice $X = R$ will be considered, from an equality perspective. Precisely, let I, J be real intervals and put $K = I \times J$. Let $f: K \to R$ be a function and Δ, be a (standard) rectangle in K. As a counterpart of Theorem 2, the third main result of the paper is

THEOREM 3. *Suppose that*

(H.6) *f is continuous over Δ.*

Then, for each $\varepsilon > 0$, there is a sub-rectangle Δ_ε interior to Δ, with the properties

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon,\ R_f(\Delta) = R_f(\Delta_\varepsilon). \tag{4.1}$$

Proof. Let us construct an equi-distant division of Δ by

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b,\ \ \rho = t_{i+1} - t_i < \varepsilon,\ 0 \le i \le n-1$$

$$c = s_0 < s_1 < \ldots < s_{m-1} < s_m = d,\ \ \sigma = s_{j+1} - s_j < \varepsilon,\ 0 \le j \le m-1.$$

(Here, $n,m \ge 3$ are fixed positive integers.) Denote for simplicity

$$\Delta(t,s) = [t, t+\rho] \times [s, s+\sigma],\ \ a \le t \le t_{n-1},\ c \le s \le s_{m-1}.$$

Of course, $\Delta(t_i, s_j)$ is, for $0 \le i \le n-1$, $0 \le j \le m-1$, nothing but Δ_{ij} alluded to in Theorem 1. Denote also

$$\phi(t,s) = R_f(\Delta(t,s)),\ a \le t \le t_{n-1},\ c \le s \le s_{m-1}.$$

It is clear that

$$R_f(\Delta) = \sum_{i,j} \lambda_{ij}\phi(t_i, s_j) \tag{4.2}$$

where (λ_{ij}) are again as in Theorem 1. Two situations are now open before us.

Case 1. The set $\{\phi(t_i, s_j);\ 0 \le i \le n-1,\ 0 \le j \le m-1\}$ consists of exactly one element. As a consequence,

$$R_f(\Delta) = R_f(\Delta(t_1, s_1))$$

and conclusion is clear (because $\Delta(t_1, s_1)$ is interior to Δ and its diameter is inferior to ε).

Case 2. The set $\{\phi(t_i, s_j);\ 0 \le i \le n-1,\ 0 \le j \le m-1\}$ has at least two distinct elements. Hence

$$\min_{i,j} \{\phi(t_i, s_j)\} < \max_{i,j} \{\phi(t_i, s_j)\}. \tag{4.3}$$

On the other hand, by convexity arguments,

$$\min_{i,j} \{\phi(t_i, s_j)\} \le R_f(\Delta) \le \max_{i,j} \{\phi(t_i, s_j)\}. \tag{4.4}$$

Suppose one of these relations holds with equality; e.q., the second. We have, by (4.2)

$$\sum_{i,j} \lambda_{ij}(R_f(\Delta) - \phi(t_i, s_j)) = 0.$$

As $\{\lambda_{ij};\ 0 \le i \le n-1,\ 0 \le j \le m-1\}$ are stricly positive,

$$R_f(\Delta) = \phi(t_i, s_j),\ 0 \le i \le n-1,\ 0 \le j \le m-1,$$

absurd by (4.3). Hence, both inequalities in (4.4) are strict. Suppose

$$\min_{i,j} \{\phi(t_i, s_j)\} = \phi(t_p, s_q),\ \max_{i,j} \{\phi(t_i, s_j)\} = \phi(t_u, s_v)$$

for some $p, u \in \{0, \dots, n-1\}$, $q, v \in \{0, \dots, m-1\}$. Denote for simplicity $\Delta' = [a, t_{n-1}] \times [c, s_{m-1}]$, and let $x = x(\tau)$, $y = y(\tau)$, $0 \leq \tau \leq 1$ be a continuous path luing in Δ' with

(i) $(x(\tau), y(\tau)) \in \operatorname{int}(\Delta') \subset \operatorname{int}(\Delta)$, $0 < \tau < 1$

(ii) $(x(0), y(0)) = (t_p, s_q)$, $(x(1), y(1)) = (t_u, s_v)$.

The composed function (from $[0,1]$ to R)

$$\psi(\tau) = \phi(x(\tau), y(\tau)),\quad 0 \leq \tau \leq 1$$

is continuous, by (H.6); and, in view of the assumptions we just made,

$$\psi(0) < R_f(\Delta) < \psi(1).$$

Hence, by the Cauchy intersection theorem, there must be some point τ_0 in $]0,1[$, with $\psi(\tau_0) = R_f(\Delta)$; or in other words,

$$R_f(\Delta) = R_f(\Delta(x(\tau_0), y(\tau_0))).$$

It is now clear that $\Delta_\varepsilon = \Delta(x(\tau_0), y(\tau_0))$ has all the properties we need. This ends the argument.■

As an immediate application, the following "weak" counterpart of Theorem 2 is available. Let $(X, \| \ \|)$ be a normed space and $f: K \to X$, a mapping. Let also Δ be a rectangle in K.

COROLLARY 5. *Suppose that*

(H.6)* *f is weakly continuous over Δ.*

Then, for each $\varepsilon > 0$, there is a sub-rectangle Δ_ε interior to Δ, such that (3.1) *be fulfilled.*

Proof. By the Hahn-Banach theorem, we may find a linear continuous functional x^* over X, with

$$\|x^*\| = 1, \; x^*(R_f(\Delta)) = S_f(\Delta).$$

The function g: $K \to R$ given by

$$g(t,s) = x^*(f(t,s)), \; (t,s) \in K$$

fulfils, by (H.6)*, conditions of Theorem 3. So, for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε interior to Δ, with

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon, \; R_g(\Delta) = R_g(\Delta_\varepsilon).$$

But, evidently,

$$R_g(\Delta) = x^*(R_f(\Delta)) = S_f(\Delta);$$

and, moreover,

$$R_g(\Delta_\varepsilon) = |x^*(R_f(\Delta_\varepsilon))| \leq S_f(\Delta_\varepsilon).$$

Combining these facts yields the desired conclusion.■

Remark. As already precised in Section 2, the continuity condition (H.6) is relative in nature. Because, as results from Proposition 1, conclusion of the above theorem is retainable whenever (H.6) is to be admitted for some function f-h where h: $K \to R$ is a hyperbolic constant (which, in principle may be discontinuous over Δ).

Remark. These results are methodologically comparable with the statements in this direction due to Nicolescu [11]. And from a dimensional viewpoint, these may be deemed as direct extensions of the ones obtained in Bantaş and Turinici [4]; see also Aziz and Diaz [2,3]. The idea of the argument goes back to Bögel [6] and, respectively, Pompeiu [13,14]. Further aspects of the problem may be found in the survey paper by Nashed [9].

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