Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 4, 749–771
DOI: 10.24193/subbmath.2022.4.07

Existence for stochastic sweeping process with fractional Brownian motion

Tayeb Blouhi, Mohamed Ferhat and Safia Benmansour

Abstract. This paper is devoted to the study of a convex stochastic sweeping process with fractional Brownian by time delay. The approach is based on discretizing stochastic functional differential inclusions.

Mathematics Subject Classification (2010): 34A37, 34K45, 60H99, 60H05, 65C30, 47H10.

Keywords: Sweeping process, evolution inclusion, perturbation, normal cone, fixed point.

1. Introduction

The so-called sweeping process is a particular differential inclusion of the general form

$$-x'(t) \in N_{C(t)}(x(t))\, a, e.\, t \in [0,T] \tag{1.1}$$

$$x(0) \in C(0) \tag{1.2}$$

where $C(t)$ is a convex time dependance set,and $N_C(t)(x(t))$ is the normal cone to $C(t)$ at $x(t)$.The sweeping process, introduced by Moreau in the early 1970s, and extensively studied by himself and other authors (see, e.g., [2, 7, 8, 5]).These models prove to be quite useful in elastoplasticity, non smooth mechanics, convex optimization, mathematical economics, queuing theory, etc. In this paper, we propose a simple extension of the sweeping process. More precisely, We consider the problem formally

Received 28 November 2019; Accepted 17 January 2020.

expressed by

$$\begin{cases} -dx(t) & \in N_{C_1(t)}(x(t))dt + G^1(t, x_t, y_t)dB^{H_1} \ a, e. t \in J := [0, T] \\ -dy(t) & \in N_{C_2(t)}(y(t))dt + G^2(t, x_t, y_t)dB^{H_2} \ a, e. t \in J := [0, T] \\ x(t) & = \phi(t), t \in [-r, 0], \ x(0) \in C_1(0) \\ y(t) & = \overline{\phi}(t), t \in [-r, 0], \ y(0) \in C_2(0) \end{cases} \tag{1.3}$$

where $C_1(t), C_2(t)$ is convex for all t, X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\|\cdot\|$, $G^j : M_2([-r, 0], X) \times M_2([-r, 0], X) \to L^0_{Q_{H_j}}(Y, X)$ are given functions. Here, $L^0_{Q_{H_j}}(Y, X)$ denotes the space of all Q_{H_j}-Hilbert-Schmidt operators from Y into X,B^{H_j} is sequence of mutually independent fractional Brownian motions with $H_1 \neq H_2$ i.e $(B^{H_1} \neq B^{H_2})$ for each $j = 1, 2$, with Hurst parameter $H_j > \frac{1}{2}$. Here $y(\cdot, \cdot) : [-r, T] \times \Omega \to X$, then for any $t \geq 0$, $y_t(\cdot, \cdot) : [-r, 0] \times \Omega \to X$ is given by:

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in [-r, 0], \ \omega \in \Omega.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. Let $M^2([-r, 0], X)$ be the following space defined by

$$M^2([-r, 0], X) = \{\phi, \overline{\phi} \colon \ [-r, 0] \times \Omega \to X, \quad \phi, \overline{\phi} \in C([-r, 0], L^2(\Omega, X))\},$$

endowed with the norm

$$||\phi(t)||_{M^2_{\mathcal{F}_0}} = \int_{-r}^{0} |\phi(t)|^2 dt$$

Now, for a given $T > 0$, we define

$$\begin{cases} M^2([-r, T], X) = y \colon \ [-r, T] \times \Omega \to X, \quad \phi, \overline{\phi} \in C([-r, T], L^2(\Omega, X)) \text{ and} \\ \qquad \qquad \sup\limits_{t \in [0, T]} E(|y(t)|^2) < \infty, \ \int_{-r}^{0} |\phi(t)|^2 dt < \infty. \end{cases}$$

Endowed with the norm

$$\|y\|_{M^2_{\mathcal{F}_b}} = \sup_{-r \leq s \leq T} (\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}}.$$

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monographs by Da Prato and Zabczyk [3], Gard [4],Sobzyk [10] and Tsokos and Padgett [11]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [11] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs by Bharucha-Reid [1], Mao[6], Øksendal[9], Tsokos and Padgett [11].

This paper is organized as follows. In Section 2 and 3, we recall some definitions and results that will be used in all the sequel. Section 4 is devoted to the study of the existence problem of (1.3).In Section 5, we restrict our attention to the case when the perturbation with F.

2. Basic definitions of stochastic calculus

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.Actually we will borrow them from [**?**].Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For a stochastic process $x(\cdot, \cdot) : [0, T] \times \Omega \to X$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t, \omega)$.

Definition 2.1. Given $H_1, H_2 \in (0, 1)$,$H_1 \neq H_2$ a continuous centered Gaussian process B^H is said to be a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H_j,$j = 1, 2$ if its covariance function $R_{H_j}(t, s) = \mathbb{E}[B^{H_j}(t))B^{H_j}(s)]$ satisfies

$$R_{H_j}(t, s) = \frac{1}{2}(|t|^{2H_j} + |s|^{2H_j} - |t - s|^{2H_j}) \quad t, s \in [0, T].$$

It is known that $B^H(t)$ with $H_j > \frac{1}{2}$ admits the following Volterra representation

$$B^{H_j}(t) = \int_0^t K_{H_j}(t, s)dW(s) \tag{2.1}$$

where W is a standard Brownian motion given by

$$W(t) = B^{H_j}((K^*_{H_j})^{-1}\xi_{[0,t]}),$$

and the Volterra kernel the kernel $K(t, s)$ is given by

$$K_{H_j}(t, s) = c_{H_j} s^{1/2-H_j} \int_s^t (u - s)^{H_j - \frac{3}{2}} \left(\frac{u}{s}\right)^{H_j - \frac{1}{2}} du, \quad t \geq s,$$

where $c_{H_j} = \sqrt{\frac{H_j(2H_j-1)}{\beta(2H_j-2, H_j-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ denotes the Beta function, $K(t, s) = 0$ if $t \leq s$, and it holds

$$\frac{\partial K_{H_j}}{\partial t}(t, s) = c_H \left(\frac{t}{s}\right)^{H_j - \frac{1}{2}} (t - s)^{H_j - \frac{3}{2}},$$

and the kernel $K^*_{H_j}$ is defined as follows. Denote by $\mathcal{E}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_{H_j}(t, s),$$

and consider the linear operator $K^*_{H_j}$ from $\mathcal{E}$ to $L^2([0, T])$ defined by,

$$(K^*_{H_j}\phi^j)(t) = \int_s^T \phi^j(t) \frac{\partial K_{H_j}}{\partial t}(t, s)dt.$$

Notice that,

$$(K^*_{H_j}\chi_{[0,t]})(s) = K_{H_j}(t, s)\chi_{[0,t]}(s).$$

The operator $K^*_{H_j}$ is an isometry between $\mathcal{E}$ and $L^2([0,T])$ which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\langle K^*_{H_j}\chi_{[0,t]}, K^*_{H_j}\chi_{[0,t]}\rangle_{L^2([0,T])} = \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}} = R_{H_j}(t,s).$$

In addition, for any $\phi^j \in \mathcal{H}$,

$$\int_0^T \phi^j(s)dB^{H_j}(s) = \int_0^T (K^*_{H_j}\phi^j)(s)dW(s),$$

if and only if $K^*_{H_j}\phi \in L^2([0,T])$. Next we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Definition 2.2. An $\mathcal{F}_t$-adapted process ϕ^j on $[0,T]\times\Omega \to X$ is an elementary or simple process if for a partition $\psi = \{\bar{t}_0 = 0 < \bar{t}_1 < \ldots < \bar{t}_n = T\}$ and $(\mathcal{F}_{\bar{t}_i})$-measurable X-valued random variables $(\phi^j_{\bar{t}_i})_{1\le i\le n}$, ϕ_t satisfies

$$\phi^j_t(\omega) = \sum_{i=1}^n \phi^j_i(\omega)\chi_{(\bar{t}_{i-1},\bar{t}_i]}(t), \quad \text{for } \ 0 \le t \le T, \quad \omega \in \Omega.$$

The Itô integral of the simple process ϕ^j is defined as

$$I_{H_j}(\phi^j) = \int_0^T \phi^j(s)dB^{H_j}(s) = \sum_{i=1}^n \phi^j(\bar{t}_i)(B^{H_j}_l(\bar{t}_i) - B^{H_j}_l(\bar{t}_{i-1})), \tag{2.2}$$

whenever $\phi^j_{\bar{t}_i} \in L^2(\Omega, \mathcal{F}_{\bar{t}_i}, \mathbb{P}, X)$ for all $i \le n$.

Let $(X, \langle\cdot,\cdot\rangle, |\cdot|_X)$, $(Y, \langle\cdot,\cdot\rangle, |\cdot|_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y,X)$ denote the space of all linear bounded operators from Y into X. Let $e_n, n = 1,2,\ldots$ be a complete orthonormal basis in Y and $Q_{H_j} \in \mathcal{L}(Y,X)$ be an operator defined by $Q_{H_j}e_n = \lambda^j_n e_n$ with finite trace $trQ_{H_j} = \sum_{n=1}^{\infty}\lambda^j_n < \infty$ where λ^j_n, $n = 1,2,\ldots$, are non-negative real numbers. Let $(\beta^{H_j}_n)_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. If we define the infinite dimensional fBm on Y with covariance Q_{H_j} as

$$B^{H_j}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta^{H_j}_n(t)e_n, \tag{2.3}$$

then it is well defined as an Y-valued Q_{H_j}-cylindrical fractional Brownian motion (see [?]) and we have

$$\mathbb{E}\langle \beta^{H_j}_l(t), x\rangle\langle\beta^H_k(s), y\rangle = R_{H_{lk}}(t,s)\langle Q_{H_j}(x), y\rangle, \quad x,y \in Y \quad and\, s,t \in [0,T]$$

such that

$$R_{H^j_{lk}} = \frac{1}{2}\{|\, t\,|^{2H_j} + |\, s\,|^{2H_j} + |\, t-s\,|^{2H_j}\}\delta_{lk} \quad t,s \in [0,T],$$

where

$$\delta_{lj} = \begin{cases} 1 & k = l, \\ 0, & k \ne l. \end{cases}$$

In order to define Wiener integrals with respect to a $Q_{H_j} - fBm$, we introduce the space $L^0_{Q_{Hj}} := L^0_{Q_{Hj}}(Y, X)$ of all Q_{H_j}−Hilbert-Schmidt operators $\varphi^j : Y \longrightarrow X$. We recall that $\varphi^j \in L(Y, X)$ is called a Q_{H^j}−Hilbert-Schmidt operator, if

$$\|\varphi^j\|^2_{L^0_{Q_{H_j}}} = \|\varphi Q^{1/2}_{H_j}\|^2_{HS} = tr(\varphi_j Q \varphi^*_j) < \infty.$$

Definition 2.3. Let $\phi^j(s), s \in [0, T]$, be a function with values in $L^0_{Q_{Hj}}(Y, X)$. The Wiener integral of ϕ^j with respect to fBm given by (2.3) is defined by

$$\int_0^T \phi^j(s) dB^{H_j}(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi^j(s) e_n d\beta^{H_j}_n$$
$$= \sum_{n=1}^{\infty} \int_0^T \sqrt{\lambda_n} K^*_{H_j}(\phi^j e_n)(s) d\beta_n. \quad (2.4)$$

Notice that if

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\|_{L^{1/H_j}([0,T];X)} < \infty, \quad (2.5)$$

the next result ensures the convergence of the series in the previous definition. It can be proved by similar arguments to those used to prove Lemma 2.4 in Caraballo *et al.* [?].

Lemma 2.4. *For any* $\phi^j : [0, T] \to L^0_{Q_{Hj}}(Y, X)$ *such that* (2.5) *holds, and for any* $\alpha, \beta \in [0, T]$ *with* $\alpha > \beta$, *for each* $j = 1, 2$

$$\mathbb{E}\left|\int_\alpha^\beta \phi^j(s) dB^{H_j}(s)\right|^2_X \le c_2(H_j) H_j (2H_j - 1)(\alpha - \beta)^{2H_j - 1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left|\phi^j(s) Q^{1/2} e_n\right|^2_X ds. \quad (2.6)$$

where $c_2(H_j)$ *is a constant depending on* H_j. *If, in addition,*

$$\sum_{n=1}^{\infty} |\phi^j Q^{1/2} e_n|_X \text{ is uniformly convergent for } t \in [0, T],$$

then,

$$\mathbb{E}\left|\int_\alpha^\beta \phi^j(s) dB^{H_j}(s)\right|^2_X \le c_2(H_j) H_j (2H_j - 1)(\alpha - \beta)^{2H_j - 1} \int_\alpha^\beta \left\|\phi^j(s)\right\|^2_{L^0_{Q_{Hj}}} ds. \quad (2.7)$$

3. Nonsmooth analysis

Let $x, y \in X$; the projection of x,y into $C_j \subset X$ is the set

$$Proj(y, C_j) = \{z \in C_j \ : d(z, C_j) = ||z - y||\}.$$

This set is nonempty if, for example, C_j is weakly closed.Let C_j be a closed subset of space X;and let $x, y \in C_i$: We say that a vector $v \in X$ is a proximal normal to C_j at z if $v = y - z$ for some $y \in X$ with $z \in Proj(y, C_j)$. We denote by $N^p(z, C_j)$.

the normal cone. One can show that $\eta \in N^p(y, C_j)$ if and only if there exists M such that the following proximal normal inequality holds,

$$\langle \eta, z - y \rangle \leq M||z - y||,$$

for all $z \in C_j$. (In general, M will depend on x). On the other hand

$$N^p(z, C_j) = \bigcup_{n=1}^{\infty} \left\{ v \in X \ : \ d(y + \frac{v}{n}) = \frac{||v||}{n} \right\}.$$

This cone is convex, but in general not closed . An useful characterization of the proximal normal cone is the following (see,e.g., [**?**], Proposition 1.1.5(a)):

$$N^p(z, C_j) = \cup_{\mu > 0}\{v \in X \ : \ \langle v, a - z \rangle \leq \mu ||z - y||^2, \, a \in C_j\}.$$

If C_j is closed and convex then we have

$$z \in N^p(z, C_j) \Longleftrightarrow y \in C_j \text{ and } \langle z, y \rangle = \sigma(z, C_i) \Longleftrightarrow y \in C_j, x \in \partial\varphi_{C_j}(y)$$

where σ is the support function of a subset C_j of X, $\partial\varphi_{C_j}$ is the subdifferential in the sense of convex analysis and C_i is the indicator function of a subset C_j of X

$$\partial\varphi_{C_j}(y) = \begin{cases} 0, & \text{if} \quad y \in C_j, \\ \\ \emptyset, & \text{if}\, y \in C_j. \end{cases}$$

We define the Bouligand cone by

$$T_{C_j}(x) = \left\{ v \in X \ : \ \lim_{h \to 0} \inf \frac{d(z + hv, C_j)}{h} \right\} = \bigcap_{\epsilon > 0} \bigcap_{\delta > 0} \bigcup_{0 < h < \delta} \Big(\frac{C_j - z}{h} + \epsilon \overline{B}(0, 1) \Big).$$

For more informations about nonsmooth analysis we see the monographs of Clarke and Ledyaev et al [**?**] and Clarke [**?**].

3.1. Multi-valued analysis

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X) : y \text{ closed } \},$$
$$\mathcal{P}_{b}(X) = \{y \in \mathcal{P}(X) : y \text{ bounded } \},$$
$$\mathcal{P}_{c}(X) = \{y \in \mathcal{P}(X) : y \text{ convex } \},$$
$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X) : y \text{ compact } \}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) \quad := \quad \begin{pmatrix} H_{d_1}(A, B) \\ ... \\ H_{d_n}(A, B) \end{pmatrix}$$

Let (X, d) be a generalized metric space with

$$d(x, y) \quad := \quad \begin{pmatrix} d_1(x, y) \\ ... \\ d_n(x, y) \end{pmatrix}$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1,..,n$ are metrics on X,

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b), d(a,B) = \inf_{b\in B} d(a,b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y\in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.

If the multivalued map F is completely continuous with nonempty compact valued, then F is u.s.c. if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \longrightarrow \mathcal{P}_{cp,c}$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 3.1. The set-valued map $F : J \times X \times X \to \mathcal{P}(X \times X)$ is said to be L^2-Carathéodory if

- **(i).** $t \mapsto F(t,v)$ is measurable for each $v \in X \times X$;
- **(ii).** $v \mapsto F(t,v)$ is u.s.c. for almost all $t \in J$;
- **(iii).** for each $q > 0$, there exists $h_q \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,v)\|^2 := \sup_{f\in F(t,v)} \|f\|^2 \leq h_q(t), \text{for all } \|v\|^2 \leq q \text{ and for a.e. } t \in J.$$

We denote the graph of G to be the set $gr(G) = \{(x,y) \in X \times Y, \quad y \in G(x)\}$.

Lemma 3.2. [?] *If $G : X \to P_{cl}(Y)$ is u.s.c., then $gr(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 3.3. [?] *If $G : X \to P_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

Definition 3.4. A set-valued operator $G : J \longrightarrow \mathcal{P}_{cl}(X)$ is said to be a contraction if there exists $0 \leq \gamma < 1$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x,y), \text{ for all } x,y \in X,$$

The following two results are easily deduced from the limit properties.

Lemma 3.5. (See e.g. [?], Theorem 1.4.13) *If $G : X \to \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x\to x_0} G(x) = G(x_0).$$

Lemma 3.6. (See e.g. [?], Lemma 1.1.9) *If Let* $(K_n)_{n\in N} \subset K \subset X$ *be a sequence of subsets where* K *is compact in the separable Banach space* X*. Then*

$$\overline{co}(\limsup_{n\to\infty} K_n) = \cap_{N>0}\overline{co}(\cup_{n\geq N} K_n)$$

where $\overline{co}A$ *refers to the closure of the convex hull of* A*.*

The second one is due to Mazur, 1933:

Lemma 3.7. (Mazur's Lemma, ([?] [Theorem 21.4])) *Let* X *be a normed space and* $\{x_k\}_{k\in N} \subset X$ *be a sequence weakly converging to a limit* $x \in X$*. Then there exists a sequence of convex combinations* $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ *with* $\alpha_{mk} > 0$ *for* $k = 1, 2, .., m$ *and* $\sum_{k=1}^{m} \alpha_{mk} = 1$*, which converges strongly to* x*.*

Lemma 3.8. [?] $C : [0, T] \to \mathcal{P}_{cl}(X)$ *such that*

(i). C *is Hausdorff lower semicontinuous at* $t = 0$*;*
(ii). ∂C *is Hausdorff upper semicontinuous at* $t = 0$*;*
(iii). *there exist* $x \in X$ *and* $r_0 > 0$ *such that* $B(x, r_0) \subseteq C(0)$

Then for every $r \in (0, r_0)$ *there exists* $\delta > 0$ *such that* $B(x, r) \subset C(r)$ *for all* $t \in [0, \delta]$*.*

4. Statement of the main results

Definition 4.1. A function $x, y \in M^2([-r, T], X)$, is said to be a solution of (1.3) if x, y satisfies the equation

$$\begin{cases} dx(t) & \in N^p(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1} \ a, e. t \in [0, T] \\ \\ dy(t) & \in N^p(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2} \ a, e. t \in [0, T] \end{cases}$$

and the conditions $(x(t), y(t)) \in (C_1(t), C_2(t))$, for all $t \in [0, T]$.

First, we will list the following hypotheses which will be imposed in our main theorem. In this section,

(H_1) $C_j(t)$ is convex for every $t \in [0, T]$ and there exists $\lambda > 0$ such that

$$H_{d_j}(C_j(t), C_j(s)) \leq \lambda|t - s|,$$

for all $t, s \in [0, T]$,

(H_2) there exists a positive constant α_j, β_j for each $j = 1, 2$ such that

$$\mathbb{E}|G^j(t, x, y) - G^j(t, \overline{x}, \overline{y})| \leq \alpha_j||x - \overline{x}||_{M^2_{\mathcal{F}_0}} + \beta_j||y - \overline{y}||_{M^2_{\mathcal{F}_0}},$$

for all $t \in [0, T]$ and $x, y, \overline{x}, \overline{y} \in M^2([-r, 0], X)$

Theorem 4.2. *Assume that* (H_1) *and* (H_2) *hold. Then, problem* (1.3) *possesses a unique solution on* $[0, T]$*.*

Proof. The existence part. Therefore, we pass immediately to uniqueness. We shall obtain the solution by a well-establish discretization procedure.
The following discretization scheme lies at the heart of many proofs for sweeping processes. Consider for every $n \in \mathbb{N}$, the following partition of $[0, T]$,

$$t_{n,i} := \frac{iT}{2^n},\ 0 \le i \le 2^n \text{ and } I_{n,i} = (t_{n,i}, t_{n,i+1}],\ \text{if } 0 \le i \le 2^n - 1,\ n \ge 0.$$

$$x_{n,0} = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0), & t \in [0, t_{n,0}], \end{cases}$$

for any $I_{n,0} = (t_{n,0}, t_{n,1}]$, we have

$$x_{n,1} = \begin{cases} x_{n,0}(t),\ t \in [-r, t_{n,0}], \\ proj\Big(\phi(0) + G^1(t_{n,0}, x(_{n,0})_{t_{n,0}}, y(_{n,0})_{t_{n,0}})(B^{H_1}(t_{n,1}) - B^{H_1}(t_{n,0}), C_1(t_{n,1})\Big), \\ t \in [t_{n,0}, t_{n,1}] \end{cases}$$

for any $I_{n,1} = (t_{n,1}, t_{n,2}]$, we have

$$x_{n,2} = \begin{cases} x_{n,1}(t), t \in [-r, t_{n,1}], \\ proj\Big(x_{n,1}(t_{n,1}) + G^1(t_{n,1}, x(_{n,1})_{t_{n,1}}, y(_{n,1})_{t_{n,1}})(B^{H_1}(t_{n,2}) \\ -B^{H_1}(t_{n,1}), C_1(t_{n,2})\Big), \\ t \in [t_{n,1}, t_{n,2}]. \end{cases}$$

With the same argument we can define recursively

$$x_{n,i+1} = \begin{cases} x_{n,i}(t), t \in [-r, t_{n,i}], \\ proj\Big(x_{n,i}(t_{n,i}) \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,1}}, y(_{n,i})_{t_{n,1}})(B^{H_1}(t_{n,i+1}) \\ -B^{H_1}(t_{n,i}), C_1(t_{n,i+1})\Big),\ t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

Estimate (x_n, y_n) by norm $M^2([-r, T], X) \times M^2([-r, T], X)$, since (x_n, y_n) is piecewise affine, by direct calculations,

$$\sup\{\sqrt{E|x_{n,i+1}(t) - x_{n,i}(t)|^2} \quad : \quad t \in [-r, T]\} \le \lambda \frac{T}{2^n}. \tag{4.1}$$

Observe that $(x_{n,i}(t), y_{n,i}(t)) \in (C_1(t_{n,i}), C_2(t_{n,i}))$,and

$$\mathbb{E}|x_{n,i+1}(t) - x_{n,i}(t)| \le \mathbb{E}H_{d_1}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n} \tag{4.2}$$

and

$$\mathbb{E}|y_{n,i+1}(t) - y_{n,i}(t)| \le \mathbb{E}H_{d_2}(C_2(t_{n,i}), C_2(t_{n,i+1})) \le \lambda \frac{T}{2^n}, \tag{4.3}$$

for all $t \in (t_{n,i-1}, t_{n,i}]$, for every $0 \le i \le 2^n$.

By affine interpolation we define a corresponding sequence of approximate solutions $x_n, y_n \in M^2([-r,T],X)$; for $t \in I_{n,i}$ the explicit formula is

$$x_n(t) = \begin{cases} x_{n,i}(t), & t \in [-r, t_{n,i}] \\ x_{n,i}(t_{n,i}) + \frac{t-t_{n,i}}{\epsilon_n}(x_{n,i+1}(t) - x_{n,i}(t)) & \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

and

$$y_n(t) = \begin{cases} y_{n,i}(t), & t \in [-r, t_{n,i}] \\ y_{n,i}(t_{n,i}) + \frac{t-t_{n,i}}{\epsilon_n}(y_{n,i+1}(t) - y_{n,i}(t)) & \\ +G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

where $\epsilon_n = \frac{T}{2^n}$ and for every $0 \le i \le 2^n - 1$.

From the definition of normal proximal cone, we have

$$\begin{aligned} dx_n(t) &\in -N(x_{n,i+1}, C_1(t_{n,i+1}))dt \\ &\quad + G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})). \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} dy_n(t) &\in -N(y_{n,i+1}, C_2(t_{n,i+1}))dt \\ &\quad + G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})). \end{aligned} \tag{4.5}$$

Now we prove that $\{x_n, y_n,\ n \in \mathbb{N}\}$ is compact in $M^2([-r,T],X)$, for each $z_n = (x_n, y_n)$ in $M^2([-r,T],X) \times M^2([-r,T],X)$.

Step 1. $\{(x_n, y_n)\ n \in \mathbb{N}\}$ are bounded sets in $M^2([-r,T],X) \times M^2([-r,T],X)$.

We obtain

$$\begin{aligned} |x_n(t)| &\le |x_{n,i}(t)| + |x_{n,i+1}(t) - x_{n,i}(t)| \\ &+b|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\le |x_{n,0}(t)| + \sum_{k=1}^{i+1} |x_{n,k-1}(t) - x_{n,k}(t)| \\ &+T|G^1(t_{n,i}, x(_{n,i}, y(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\le ||\phi|| + 2T + T\Big(|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) \\ &-G^1(t_{n,i}, 0, 0)| + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\le ||\phi|| + 2T + T\Big(\alpha_1||(x_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} \\ &+\beta_1||(y_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|. \end{aligned}$$

By definition $(x_{n,i}, y_{n,i})$ we can prove that there exist $M, \overline{M} > 0$ such that

$$\sup\{\mathbb{E}|x_{n,i}(t)| \ :\ t \in [-r,T]\} \le M$$

and

$$\sup\{\mathbb{E}|y_{n,i}(t)| \ : \ t \in [-r,T]\} \leq \overline{M}.$$

Hence, by using (4.2) and (4.3), we have

$$\begin{aligned}
\mathbb{E}|x_n(t)|^2 \leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 E||(x_{n,i})_{t_{n,i}}||^2 + \beta_1 E||(y_{n,i})_{t_{n,i}}||^2 \\
& + \sup_{t\in[0,b]} |G^1(t,0,0)|^2\Big)\mathbb{E}|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|^2 \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 \mathbb{E}||(x_{n,i})_{t_{n,i}}||^2 + \beta_1 E||(y_{n,i})_{t_{n,i}}||^2 \\
& + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|t - t_{n,1}|^{2H_1} \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|t - t_{n,1}|^{2H_1} \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)T^{2H_1} = l_1.
\end{aligned}$$

Similarly, we have

$$\mathbb{E}|y_n(t)|^2 \leq 2\mathbb{E}||\overline{\phi}||^2 + 4T^2 + 2T^2\Big(\alpha_2 \overline{M} + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)T^{2H_2} = l_2.$$

which implies that

$$\begin{pmatrix} \mathbb{E}|x_n(t)|^2 \\ \mathbb{E}|y_n(t)|^2 \end{pmatrix} \leq \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

Step 2. $\{(x_n, y_n) \ n \in \mathbb{N}\}$ are equicontinuous sets in $M^2([-r,T],X) \times M^2([-r,T],X)$. Let $\tau_1, \tau_2 \in [t_{n,i}, t_{n,i+1}], \tau_1 < \tau_2$. Thus

$$\begin{aligned}
& \mathbb{E}|x_n(\tau_2) - x_n(\tau_1)|^2 \\
= \ & \mathbb{E}\Big|\frac{\tau_2 - \tau_1}{\epsilon_n}(x_{n,i+1} - x_{n,i}) + G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(\tau_2) - B^{H_1}(\tau_1))\Big|^2 \\
\leq \ & 2|\tau_2 - \tau_1|^2 + 2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}|y_n(\tau_2) - y_n(\tau_1)|^2 \leq & \ 2|\tau_2 - \tau_1|^2 \\
& + 2\Big(\alpha_2 M + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_2}.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small. From Steps 1, 2. By the Arzela-Ascoli theorem, we conclude that there is a subsequence of (x_n, y_n), again denoted (x_n, y_n) which converges to $(x, y) \in M^2([-r,T],X)$.

Now, we prove that $(x(t), y(t)) \in (C_1(t), C_2(t))$. Let $\rho_n(t)$,$\mu_n(t)$ be two functions from $[0,T]$ into $[0,T]$ defined by

$$\begin{aligned}
\rho_n(t) = t_{n,i}, \qquad & \text{if} \quad t \in [t_{n,i}, t_{n,i+1}), \qquad \rho_n(0) = 0 \\
\mu_n(t) = t_{n,i+1} \qquad & \text{if} \quad t \in [t_{n,i}, t_{n,i+1}), \qquad \mu_n(0) = 0,
\end{aligned}$$

for all $t \in [0,T]$. From (4.4) and (4.5) we have

$$dx_n(t) \in -N(x_n(\mu_n(t)), C_1(\mu_n(t)))dt$$
$$+G^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_1}(\rho_n(t)), \text{ a.e. } t \in [0,T] \tag{4.6}$$

and

$$dy_n(t) \in -N(x_n(\mu_n(t)), C_2(\mu_n(t)))dt$$
$$+G^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_2}(\rho_n(t)), \text{ a.e. } t \in [0,T]. \tag{4.7}$$

Moreover, for all n large enough,we have

$$\rho_n(t) \to t, \qquad \mu_n(t) \to t \qquad \text{uniformly on} \quad [0,b]$$

Since $|\rho_n(t) - t| \leq \frac{T}{2^n}$ and $|\mu_n(t) - t| \leq \frac{T}{2^n}$. Thus

$$|y_n(\rho_n(t)) - y_n(t)| \leq H_{d_1}(C_1(\rho_n(t)), C_1(t)) \leq \lambda|\rho_n(t) - t|,$$

which immediately yields

$$\sup\{\sqrt{\mathbb{E}|y_n(\rho_n(t)) - y_n(t)|^2} \ : \ t \in [0,T]\} \leq \lambda\sqrt{E|\rho_n(t) - t|^2} \to 0 \,\text{as}\, n \to \infty.$$

Let $t \in [0,T]$.From (4.1) for each $n \in \mathbb{N}$,$t_{n,i} \in I_{n,i}$ for some i,

$$\begin{aligned} |x_n(t) - C_1(t)| &\leq |x_n(t) - x_n(t_{n,i})| + d(x_n(t_{n,i}), C_1(t)) \\ &\leq \lambda\frac{T}{2^n} + H_{d_1}(C_1(t_{n,i}), C_1(t)). \end{aligned}$$

Thus

$$|x_n(t) - C_1(t)| \leq \lambda\frac{T}{2^{n-1}}. \tag{4.8}$$

Since (x_n, y_n) is defined by linear interpolation, we obtain

$$|x'_n(t)| \leq \frac{1}{\epsilon_n}\sup_i |x_{n,i+1}(t) - x_{n,i}(t)|,$$

and

$$|y'_n(t)| \leq \frac{1}{\epsilon_n}\sup_i |y_{n,i+1}(t) - y_{n,i}(t)|.$$

By letting $n \to \infty$ in(4.8) for all $t \in [0,T]$,we obtain that

$$(x(t), y(t)) \in (C_1, C_2).$$

Now, we prove that the sequences of composition mappings $(x_n \circ \mu_n, y \circ \mu_n)$ and $(x_n \circ \rho_n, y \circ \rho_n)$ converge uniforms to (x_t, y_t) in $M^2([-r,0], X)$

$$\begin{aligned} \mathbb{E}|x_n(\rho_n(t) + \tau) - x(t+\tau)|^2 &\leq 3\mathbb{E}|x_n(\rho_n(t) + \tau) - x_n(t+\tau)|^2 \\ &+ 3\mathbb{E}|x_n(\rho_n(t) + \tau) - x_n(\mu_n(t) + \tau)|^2 \\ &+ 3\mathbb{E}|x_n(\mu_n(t) + \tau) - x_n(t+\tau)|^2. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{\tau\in[-r,0]} \mathbb{E}|(x_n)_{\rho_n(t)} - x_t|^2 &\leq 3\lambda^2\mathbb{E}|\rho_n(t) - t|^2 + 3\mathbb{E}|\rho_n(t) - \mu_n(t)|^2 \\ &+ 3\sup_{\tau\in[-r,T]} \mathbb{E}|x_n(\mu_n(t)) - x(t)|^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $|(\rho_n(t)-\tau)-(t-\tau)| \leq \frac{T}{2^n}$ and $|\mu_n(t)-\rho_n(t)| \leq \frac{T}{2^{n-1}}$. We can pass to the limit when $n \to \infty$, we deduce from

$$(x_{\rho_n(t)}, y_{\rho_n(t)}) \to (x_t, y_t) \in M^2([-r,0], X)$$

and,the fact that $G^i(.,.,.)$ is a continuous function then we have

$$G^i(\rho_n(t), x_{\rho_n(t)}, y_{\rho_n(t)}) \to G^i(t, x_t, y_t).$$

Now, we show that

$$dx(t) \in -N(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T]. \tag{4.9}$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T]. \tag{4.10}$$

Since (x_n, y_n) is bounded in $X \times X$,there exists a subsequence of (x_n, y_n)converge to (x, y). Then

$$\begin{aligned}\int_0^T \sigma\Big(-x_n'(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt \\ \leq \int_0^T \Big(-x_n'(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), x(\mu_n(t))\Big)dt.\end{aligned} \tag{4.11}$$

Using the fact that $\sigma(., C_j(t))$ is lower semicontinuous [**?**],then

$$\begin{aligned}\liminf_{n\to\infty} \int_0^T \sigma\Big(-x_n'(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt \\ \geq \int_0^T \Big(-x'(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt.\end{aligned} \tag{4.12}$$

By (5.16) and (5.18),we obtain

$$\begin{aligned}\int_0^T \Big(-x'(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt \\ \geq \int_0^T \sigma\Big(-x'(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt.\end{aligned} \tag{4.13}$$

Thus,

$$dx(t) \in -N(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T].$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T].$$

Finally, we prove the uniqueness of solutions of the problem (1.3).Let us assume that (x, y) and $(\overline{x}, \overline{y})$ are two solutions of (1.3).

$$d\overline{x}(t) \in -N(\overline{x}(t), C_1(t))dt + G^1(t, \overline{x}_t, \overline{y}_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T],$$

and

$$d\overline{y}(t) \in -N(\overline{y}(t), C_2(t))dt + G^2(t, \overline{x}_t, \overline{y}_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T].$$

Since $C(t) = (C_1(t), C_2(t))$ is a convex set, then

$$T_{C_j}(z) = \cup_{h>0} \overline{\frac{C_j(t) - z}{h}},$$

for all $t \in [0,T]$,

$$T_{C_j}(z) \subset \{v \in X : \langle v, \xi\rangle \leq 0 \text{ for all } \xi \in N^p(z,\xi)\},$$

which immediately yields

$$\Big\langle x'(t) - \overline{x}'(t) + \Big(G^1(t,x_t,y_t) - G^1(t,\overline{x}_t,\overline{y}_t)\Big)dB^{H_1}(t), x(t) - \overline{x}(t)\Big\rangle \leq 0.$$

Thus, we deduce

$$\Big\langle x'(t) - \overline{x}'(t), x(t) - \overline{x}(t)\Big\rangle + \Big\langle \Big(G^1(t,x_t,y_t) - G^1(t,\overline{x}_t,\overline{y}_t)\Big)dB^{H_1}(t), x(t) - \overline{x}(t)\Big\rangle \leq 0.$$

By assumptions (H_1), (H_2) imply

$$\begin{aligned} \frac{1}{2}.\frac{d}{dt}\Big|x(t) - \overline{x}(t)\Big|^2 &\leq \alpha_1 ||x_t - \overline{x}_t||_{M^2_{\mathcal{F}_0}}\Big|x(t) - \overline{x}(t)\Big|dB^{H_1}(t) \\ &+ \beta_1 ||y_t - \overline{y}_t||_{M^2_{\mathcal{F}_0}}\Big|x(t) - \overline{x}(t)\Big|dB^{H_1}(t) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \frac{1}{2}.\frac{d}{dt}\Big|y(t) - \overline{y}(t)\Big|^2 &\leq \alpha_2 ||x_t - \overline{x}_t||_{M^2_{\mathcal{F}_0}}\Big|y(t) - \overline{y}(t)\Big|dB^{H_1}(t) \\ &+ \beta_2 ||y_t - \overline{y}_t||_{M^2_{\mathcal{F}_0}}\Big|y(t) - \overline{y}(t)\Big|dB^{H_1}(t). \end{aligned} \tag{4.15}$$

Integrating (4.14)and (4.15) over $(0,t)$ we arrive at

$$\begin{aligned} \Big|x(t) - \overline{x}(t)\Big|^2 &\leq \alpha_1 \int_0^t ||x_s - \overline{x}_s||_{M^2_{\mathcal{F}_0}}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &+ \beta_1 \int_0^t ||y_s - \overline{y}_s||_{M^2_{\mathcal{F}_0}}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &\leq \alpha_1 \int_0^t \sup_{s\in[0,t]} \sqrt{E|x(s) - \overline{x}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &+ \beta_1 \int_0^t \sup_{s\in[0,t]} \sqrt{E|y(s) - \overline{y}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s). \end{aligned}$$

Then, for each $t \in [0,T]$ and thanks to Lemma 2.4,

$$\begin{aligned} \mathbb{E}\Big|x(t) - \overline{x}(t)\Big|^4 &\leq 2\alpha_1 \mathbb{E}\Big|\int_0^t \sup_{s\in[0,t]} \sqrt{\mathbb{E}|x(s) - \overline{x}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s)\Big|^2 \\ &+ 2\beta_1 \mathbb{E}\Big|\int_0^t \sup_{s\in[0,t]} \sqrt{\mathbb{E}|y(s) - \overline{y}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s)\Big|^2 \\ &\leq 2c_2(H_1)H_1(2H_1 - 1)T^{2H_1 - 1}\alpha_1 \int_0^t \sup_{s\in[0,t]} E|x(s) - \overline{x}(s)|^4 ds \\ &+ 2c_2(H_1)H_1(2H_1 - 1)T^{2H_1 - 1}\beta_1 \\ &\quad \int_0^t \sup_{s\in[0,t]} \mathbb{E}|x(s) - \overline{x}(s)|^2 E|y(s) - \overline{y}(s)|^2 ds. \end{aligned}$$

Thus

$$\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4 \leq A_1\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds + B_1\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds,$$

where

$$A_1 = 2c_2(H_1)H_1(2H_1-1)T^{2H_1-1}(2\alpha_1+\beta_1)$$

and

$$B_1 = c_2(H_1)H_1(2H_1-1)T^{2H_1-1}\beta_1.$$

In the same way, we also have

$$\begin{aligned}\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq 2c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\alpha_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds \\ &+ 2c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\beta_2 \\ &\quad \int_0^t \sup_{s\in[0,t]} E|x(s)-\overline{x}(s)|^2\mathbb{E}|y(s)-\overline{y}(s)|^2ds,\end{aligned}$$

and, consequently,

$$\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 \leq A_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds + B_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds,$$

where

$$A_3 = c_2(H_2)H_2(2H_2-1)T^{2H_2-1}(2\alpha_2+\beta_2),$$

and

$$A_4 = c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\beta_2.$$

Adding these we obtain

$$\begin{aligned}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq A_*\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds \\ &+ B_*\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds,\end{aligned}$$

where $A_* = A_1+B_2,\ \ B_* = A_2+B_1$. Then

$$\begin{aligned}\sup_{s\in[0,t]}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq A_{**}\int_0^t \sup_{s\in[0,t]}\Big(\mathbb{E}|x(s)-\overline{x}(s)|^4 \\ &+ \mathbb{E}|y(s)-\overline{y}(s)|^4\Big)ds,\end{aligned}$$

where $A_{**} = \max\{A_*, B_*\}$.

By a generalization of Gronwall inequality, we have

$$\sup_{s\in[0,t]}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 = 0 \Longrightarrow (x(t),y(t)) = (\overline{x}(t),\overline{y}(t)),\ \text{a.e. } t\in[0,T].$$

The proof is therefore complete. □

5. Perturbation Problem (1.3)

To prove the main result we will need the following auxiliary inclusion:

$$\begin{cases} -dx(t) & \in N_{C_1(t)}(x(t))dt + F^1(t, x_t, y_t)dt \\ & +G^1(t, x_t, y_t)dB^{H_1}, \text{ a.e. } t \in [0, T] \\ -dy(t) & \in N_{C_2(t)}(y(t))dt + F^2(t, x_t, y_t)dt \\ & +G^2(t, x_t, y_t)dB^{H_2}, \text{ a.e. } t \in [0, T] \\ x(t) & = \phi(t), t \in [-r, 0], \; x(0) \in C_1(0) \\ y(t) & = \overline{\phi}(t), t \in [-r, 0], \; y(0) \in C_2(0) \end{cases} \tag{5.1}$$

Very recently in the case where $G^i = 0$ the perturbation problem was studied by Castaing et al . [**?**]. The aim in those works, is to study the existence of a solution of the problem (5.1) and investigated the topological structure of the solution set. The goal of this section is to study the existence result of the problem (5.1).

Theorem 5.1. *Assume that* (H_1) *and* (H_2) *and the conditions .*

(H_3) $F^j : [0, T] \times M^2([-r, 0], X) \times M^2([-r, 0], X) \to \mathcal{P}_{cp,cv}(X)$ *be a u.s.c. Caratheodory multimap, and for each* $t \in [0, T]$, *scalarly* $\mathcal{L}([0, T]) \otimes \mathcal{B}(M^2([-r, 0], X), X)$ *measurable, where* $\mathcal{L}([0, T])$ *is the* $\sigma-$ *algebra of Lebesgue measurable sets of* $[0, T]$ *and* $\mathcal{B}(M^2)$ *is the Borel tribe of* M^2 *and* $|F^j(t, x, y)| \leq k_j$ *for all* $(t, x, y) \in [0, T] \times M^2([-r, 0], X) \times M^2([-r, 0], X)$ *or some constant* $k_j > 0$.

Then, problem (5.1) *has at least one solution on* $[0, T]$.

Proof. Consider for every $n \in \mathbb{N}$, the following partition of $[0, T]$,

$$t_{n,i} := \frac{iT}{2^n}, \; 0 \leq i \leq 2^n \text{ and } I_{n,i} = (t_{n,i}, t_{n,i+1}], \text{ if } 0 \leq i \leq 2^n - 1, \; n \geq 0.$$

$$x_{n,0} = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0), & t \in [0, t_{n,0}], \end{cases}$$

for any $I_{n,0} = (t_{n,0}, t_{n,1}]$, we have

$$x_{n,1} = \begin{cases} x_{n,0}(t), \; t \in [-r, t_{n,0}], \\ proj\Big(\phi(0) + g_0^1(t_{n,0}) \\ +G^1(t_{n,0}, x(_{n,0})_{t_{n,0}}, y(_{n,0})_{t_{n,0}})(B^{H_1}(t_{n,1}) \\ -B^{H_1}(t_{n,0}), C(t_{n,1})\Big), \; t \in [t_{n,0}, t_{n,1}]. \end{cases}$$

Similarly,for any $I_{n,1} = (t_{n,1}, t_{n,2}]$, we have

$$x_{n,2} = \begin{cases} x_{n,1}(t), \ t \in [-r, t_{n,1}], \\ proj\Big(x_{n,1}(t_{n,1}) + g_0^1(t_{n,1}) \\ +G^1(t_{n,1}, x(_{n,1})_{t_{n,1}}, y(_{n,1})_{t_{n,1}})(B^{H_1}(t_{n,2}) \\ -B^{H_1}(t_{n,1}), C(t_{n,2})\Big), t \in [t_{n,1}, t_{n,2}]. \end{cases}$$

With the same argument we can define recursively, for any $I_{n,i} = (t_{n,i}, t_{n,i+1}]$,

$$x_{n,i+1} = \begin{cases} x_{n,i}(t), t \in [-r, t_{n,i}], \\ proj\Big(x_{n,i}(t_{n,i}) + g_0^1(t_{n,i}) \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,1}}, y(_{n,i})_{t_{n,1}})(B^{H_1}(t_{n,i+1}) \\ -B^{H_1}(t_{n,i}), C(t_{n,i+1})\Big), t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

where

$$g_0^j(t,u) = \min\{|x| \ : x \in F^j(t,u)\}.$$

By construction, we have $(x_{n,i}, y_{n,i}) \in (C_1, C_2)$, for all $t \in [t_{n,i-1}, t_{n,i}]$.
Then for every $0 \le i \le 2^n$,

$$|x_{n,i+1}(t) - x_{n,i}(t)| \le H_{d_1}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n}$$

and

$$|y_{n,i+1}(t) - y_{n,i}(t)| \le H_{d_2}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n}$$

and, consequently,

$$\sup\Big\{\sqrt{\mathbb{E}|x_{n,i+1}(t) - x_{n,i}(t)|^2} \ : t \in [-r, T]\Big\} \le \lambda \frac{T}{2^n} \tag{5.2}$$

and

$$\sup\Big\{\sqrt{\mathbb{E}|y_{n,i+1}(t) - y_{n,i}(t)|^2} \ : t \in [-r, T]\Big\} \le \lambda \frac{T}{2^n} \tag{5.3}$$

Put

$$x_n(t) = \begin{cases} x_{n,i}(t), & t \in [-r, t_{n,i}] \\ x_{n,i}(t_{n,i}) + \frac{t - t_{n,i}}{\epsilon_n}(x_{n,i+1}(t) - x_{n,i}(t)) + (t - t_{n,i}) g_0^1(t_{n,i}) & \\ +G^1(t_{n,i}, x_{t_{n,i}}, y_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

and

$$y_n(t) = \begin{cases} y_{n,i}(t), & t \in [-r, t_{n,i}] \\ y_{n,i}(t_{n,i}) + \frac{t - t_{n,i}}{\epsilon_n}(y_{n,i+1}(t) - y_{n,i}(t)) + (t - t_{n,i}) g_0^2(t_{n,i}) & \\ +G^2(t_{n,i}, x_{t_{n,i}}, y_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

Since (x_n, y_n) is defined by linear interpolation, we have

$$|x_n'(t)| \le \frac{1}{\epsilon_n} \sup_i |x_{_{n,i+1}}(t) - x_{_{n,i}}(t)|$$

and

$$|y_n'(t)| \leq \frac{1}{\epsilon_n} \sup_i |y_{n,i+1}(t) - y_{n,i}(t)|.$$

Using the fast that the projections are non-expansive, thus

$$|x_{n,i+1}(t) - proj(x_{n,i}(t), C_1(t_{n,i+1}))| \leq \epsilon_n |g_0^1(t_{n,i})| \leq \epsilon_n k_1.$$

and

$$|y_{n,i+1}(t) - proj(y_{n,i}(t), C_2(t_{n,i+1}))| \leq \epsilon_n |g_0^2(t_{n,i})| \leq \epsilon_n k_2.$$

Hence

$$|x_{n,i+1}(t) - x_{n,i}(t)| \leq \epsilon_n (k_1 + \lambda). \tag{5.4}$$

Thus

$$|x_n'(t)| \leq k_1 + \lambda \quad \text{ and } \sup_{t\in J} |x_n'(t)|^2 \leq (k_1 + \lambda)^2. \tag{5.5}$$

From the definition of normal proximal cone, we have

$$dx_n(t) \in -N(x_{n,i+1}, C_1(t_{n,i+1}))dt + g_0^1(t_{n,i})dt$$

$$+G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), \text{ a.e. } t \in [0,T] \tag{5.6}$$

and

$$dy_n(t) \in -N(y_{n,i+1}, C_2(t_{n,i+1}))dt + g_0^2(t_{n,i})dt$$

$$+G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), \text{ a.e. } t \in [0,T]. \tag{5.7}$$

Now we prove that $\{(x_n, y_n) \quad , n \in \mathbb{N}\}$ is compact in $M^2([-r,T],X) \times M^2([-r,T],X)$.
Step 1. $\{(x_n, y_n)\ n \in \mathbb{N}\}$ are bounded sets in $M^2([-r,T],X) \times M^2([-r,T],X)$.
We have

$$\begin{aligned}
|x_n(t)| \leq & |x_{n,i}(t)| + |x_{n,i+1}(t) - x_{n,i}(t)| + T|g_0^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})| \\
& + |G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & |x_{n,0}(t)| + 2\sum_{k=1}^{i+1} |x_{n,k-1}(t) - x_{n,k}(t)| + Tk_1 \\
& + |G^1(t_{n,i}, x(_{n,i}, y(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & ||\phi|| + 2T + \Big(|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) - G^1(t_{n,i}, 0, 0)| \\
& + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & ||\phi|| + 2T + Tk_1 \\
& + T\Big(\alpha_1 ||(x_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} + \beta_1 ||(y_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} \\
& + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|.
\end{aligned}$$

Then,

$$
\begin{aligned}
\mathbb{E}|x_n(t)|^2 \leq & 2(||\phi||^2 + 2T + Tk_1)^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} \\
& + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)\mathbb{E}|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|^2 \\
\leq & 2(||\phi||^2 + 2T + Tk_1)^2 \\
& + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)T^{2H_1} := \bar{l}_1.
\end{aligned}
$$

Hence

$$\sup\{\sqrt{\mathbb{E}|x_n(t)|^2} \ : \ t \in [-r,T]\} \leq \bar{l}_1.$$

and

$$\sup\{\sqrt{\mathbb{E}|y_n(t)|^2} \ : \ t \in [-r,T]\} \leq \bar{l}_2.$$

Which implies that

$$\begin{pmatrix} \mathbb{E}|x_n(t)|^2 \\ \mathbb{E}|y_n(t)|^2 \end{pmatrix} \leq \begin{pmatrix} \bar{l}_1 \\ \bar{l}_2 \end{pmatrix}$$

Step 2. $\{(x_n, y_n), \ n \in \mathbb{N}\}$ are equicontinuous sets in $M^2([-r,T],X)$.
Let $\tau_1, \tau_2 \in [t_{n,i}, t_{n,i+1}], \tau_1 < \tau_2$. Thus

$$
\begin{aligned}
& \mathbb{E}|x_n(\tau_2) - x_n(\tau_1)|^2 \\
= & \ \mathbb{E}\Big|\frac{\tau_2 - \tau_1}{\epsilon_n}(x_{n,i+1} - x_{n,i}) + (\tau_2 - \tau_1)g_0^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) \\
+ & \ G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(\tau_2) - B^{H_1}(\tau_1))\Big|^2 \\
\leq & \ 3|\tau_2 - \tau_1|^2 + 3\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_1} \\
+ & \ 3k_1^2|\tau_2 - \tau_1|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}|y_n(\tau_2) - y_n(\tau_1)|^2 \leq & \ 3|\tau_2 - \tau_1|^2 + 3\Big(\alpha_2 M + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_2} \\
& + 3k_2^2|\tau_2 - \tau_1|^2.
\end{aligned}
$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small. From Steps 1, 2, by the Arzela-Ascoli theorem, we conclude that there is a subsequence of (x_n, y_n), again denoted (x_n, y_n) which converges to (x, y) in $M^2([-r,T],X) \times M^2([-r,T],X)$. It remains to prove that $(x(t), y(t)) \in (C_1(t), C_2(t))$. Let $t \in [0,T]$,from (5.5) ,we obtain

$$\begin{array}{rcl} 0 \leq |x_n(t) - C_1(t)| & = & d(x_n(t), C_1(t)) \\ & \leq & \Big| x_n(t) - x_n(t_{n,i}) \Big| + d(x_n(t_{n,i}), C_1(t)) \\ & \leq & (k_1 + \lambda)|t - t_{n,i}| + H_{d_1}(C_1(t_{n,i}), C_1(t)) \\ & \leq & \dfrac{(k_1+\lambda)b}{2^{n-1}}. \end{array}$$

Then

$$|x_n(t) - C_1(t)| \leq \frac{(k_1+\lambda)T}{2^{n-1}}. \tag{5.8}$$

and

$$|y_n(t) - C_2(t)| \leq \frac{(k_2+\lambda)T}{2^{n-1}}. \tag{5.9}$$

By letting $n \to \infty$ in (5.8) and (5.9) ,we obtain that

$$(x(t), y(t)) \in (C_1, C_2) \tag{5.10}$$

Now, we define, for $t \in [0, T]$

$$\rho_n(t) = t_{n,i}, \qquad \mu_n(t) = t_{n,i+1} \quad \text{if} \quad t \in [t_{n,i}, t_{n,i+1}).$$

Hence, by using (4.4) and (4.5) we have

$$\begin{array}{l} dx_n(t) \in -N(x_n(\mu_n(t)), C_1(\mu_n(t)))dt + g_0^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)}) \\ \qquad +G^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_1}(\rho_n(t)) \text{ a,e. } t \in [0, T]. \end{array} \tag{5.11}$$

and

$$\begin{array}{l} dy_n(t) \in -N(x_n(\mu_n(t)), C_2(\mu_n(t)))dt + g_0^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)}) \\ \qquad +G^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_2}(\rho_n(t)) \ t \in \text{ a,e. } t \in [0, T]. \end{array} \tag{5.12}$$

Hence

$$\rho_n(t) \to t, \qquad \mu_n(t) \to t \qquad \text{uniformly on} \quad [0, b]$$

Since $|\rho_n(t) - t| \leq \frac{T}{2^n}$ and $|\mu_n(t) - t| \leq \frac{T}{2^n}$. Moreover,

$$|x_n(\rho_n(t)) - x_n(t)| \leq H_{d_1}(C_1(\rho_n(t)), C_1(t)) \leq \lambda|\rho_n(t) - t|.$$

Similarly,

$$|y_n(\rho_n(t)) - y_n(t)| \leq H_{d_2}(C_2(\rho_n(t)), C_2(t)) \leq \lambda|\rho_n(t) - t|.$$

Therefore,

$$\sup\{\sqrt{\mathbb{E}|x_n(\rho_n(t)) - x_n(t)|^2} \ : \ t \in [0, T]\} \leq \lambda\sqrt{\mathbb{E}|\rho_n(t) - t|^2} \to 0 \text{ as } n \to \infty.$$

and

$$\sup\{\sqrt{\mathbb{E}|y_n(\rho_n(t)) - y_n(t)|^2} \ : \ t \in [0, T]\} \leq \lambda\sqrt{\mathbb{E}|\rho_n(t) - t|^2} \to 0 \text{ as } n \to \infty.$$

In Theorem (4.2) was proved that $(x_{\rho_n(t)}, y_{\rho_n(t)})$ converge to (x_t, y_t) in $M^2([-r, T], X)$.

Let $v_n^j(t) = g_0^j(\rho_n(t), (x_n)_{\rho_n(t)}), (y_n)_{\rho_n(t)}))$.From H_3 we have $|v_n^j(t)| \leq k_j$ for $n \in \mathbb{N}$ implies that $v_n^j(t) \in lB(0,1)$, hence $(v_n^j)_{n\in\mathbb{N}}$ which converges weakly to some limit $v^j \in L^2(J, X)$. Since $F(., x, y)$ is u.s.c. with closed and convex values and $F^j(.,.,.)$

is bounded for each $j=1,2$, then exists a sequence $\{F_m\}_{m\in\mathbb{N}}$ of globally u.s.c. set-valued mappings on $J\times M^2([-r,0],X)\times M^2([-r,0],X)$ with convex compact values in $X\times X$ satisfying the following conditions:

$$||F^j_m(t,x,y)||\leq k_j,$$

for all $(t,x,y)\in J\times M^2([-r,0],X)\times M^2([-r,0],X)$ and $j=1,2$,

$$F^j_{m+1}(t,x,y)\subset F^j_m(t,x,y),\quad F(t,x,y)=\cap_{m\geq 1}F^j_m(t,x,y).$$

Now we need to prove that $v^j(t)\in F^j(t,x_t,y_t)$, for a.e. $t\in J$. Lemma 3.7 yields the existence of constants $\alpha^n_i\geq 0$, $l=1,2..,k(n)$ and $j=1,2$ such that $\sum_{l=1}^{k(n)}\alpha^n_l=1$ and the sequence of convex combinations $\psi^j_n(.)=\sum_{l=1}^{k(n)}\alpha^n_l v^j_l(.)$ converges strongly to some limit $v^j\in L^2(J,X)$. Since F^j takes convex values, using Lemma 3.6, we obtain that

$$\begin{aligned} v^j(t) &\in \bigcap_{n\geq 1}\overline{\{\psi^j_n(t)\}},\quad a.e\quad t\in J\\ &\subset \bigcap_{n\geq 1}\overline{co}\{v^j_k(t),\quad k\geq n\}\\ &\subset \bigcap_{n\geq 1}\overline{co}\{\bigcup_{k\geq n}F^j_m(\rho_k(t),(x_k)_{\rho_k(t)},(y_k)_{\mu_k(t)})\}\\ &=\overline{co}\{\limsup_{k\to\infty}F^j_m(\mu_k(t),(x_k)_{\mu_k(t)},(y_k)_{\mu_k(t)})\}. \end{aligned} \tag{5.13}$$

Since F^j_m is u.s.c. and has compact values, then by Lemma 3.5, we have

$$\limsup_{n\to\infty}F^j_m(\rho_n(t),(x_n)_{\rho_n(t)},(y_n)_{\rho_n(t)})=F^j_m(t,x_t,y_t)\quad \text{for a.e}\quad t\in J.$$

This and (5.13) imply that $v^j(t)\in\overline{co}(F^j(t,x_t,y_t)$. Since, for each $j=1,2$, $F^j_m(.,.,.)$ has closed, convex values, we deduce that $v^j(t)\in F^j_m(t,x_t,y_t)$ for a.e. $t\in J$,then $v^j(t)\in F^j(t,x_t,y_t)$.
We can pass to the limit when $n\to\infty$, we deduce from

$$(x_{\rho_n(t)},y_{\rho_n(t)})\to(x_t,y_t)\in M^2([-r,0],X)\text{ as }n\to\ \infty.$$

Using the fact that $G^j(.,.,.)$ is a continuous function then we have

$$G^j(\rho_n(t),x_{\rho_n(t)},y_{\rho_n(t)})\to G^j(t,x_t,y_t)\text{ as }n\to\ \infty.$$

Now, we show that

$$dx(t)\in -N(x(t),C_1(t))dt+v^1(t)dt+G^1(t,x_t,y_t)dB^{H_1}(t)\text{ a.e. }t\in[0,T]. \tag{5.14}$$

and

$$dy(t)\in -N(y(t),C_2(t))dt+v^2(t)dt+G^2(t,x_t,y_t)dB^{H_2}(t)\text{ a.e. }t\in[0,T]. \tag{5.15}$$

Since (x_n, y_n) is bounded in $X \times X$,there exists a subsequence of (x_n, y_n)converge to (x, y).Then

$$\int_0^T \sigma\Big(- x_n'(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t) dB^{H_1}(t), C_1(\mu_n(t))\Big) dt$$

$$\leq \int_0^T \Big(- x_n'(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t) dB^{H_1}(t), x(\mu_n(t))\Big) dt. \tag{5.16}$$

Using the fact that $\sigma(., C_1(t))$ is lower semicontinuous ,then

$$\liminf_{n\to\infty} \int_0^T \sigma\Big(- x_n'(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t) dB^{H_1}(t), C_1(\mu_n(t))\Big) dt$$

$$\geq \int_0^T \Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t) dB^{H_1}(t), C_1(t)\Big) dt. \tag{5.17}$$

By (5.16) and (5.18),we obtain

$$\int_0^T \Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t) dB^{H_1}(t), C_1(t)\Big) dt$$

$$\geq \int_0^T \sigma\Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t) dB^{H_1}(t), C_1(t)\Big) dt. \tag{5.18}$$

Thus,

$$dx(t) \in -N(x(t), C_1(t))dt + F^1(t, x_t, y_t)dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0, T].$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + F^1(t, x_t, y_t)dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0, T].$$

and the proof is finished. □

References

[1] Bharucha-Reid, A.T., *Random Integral Equations*, Academic Press, New York, 1972.

[2] Castaing, C., *Sur un nouvelle classe d'equation d'evolution dans les espaces de Hilbert*, expose no 10, Seminaire d'analyse convexe, University of Montpellier, 24 pages, 1983.

[3] Da Prato, G., Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge Univ Press, Cambridge, 1992.

[4] Gard, T.C., *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[5] Gavioli, A., *Approximation from the exterior of a multifunction and its applications in the "sweeping process"*, J. Diff. Equations, **92**(1992), 121-124.

[6] Mao, X., *Stochastic Differential Equations and Applications*, Harwood, Chichester, 1997.

[7] Moreau, J.J., *Rale par un convexe variable*, (premiere partie), expose no 15, Seminaire d'analyse convexe, University of Montpellier, 43 pages, 1971.

[8] Moreau, J.J., *Probleme d'evolution associé a un convexe mobile d'un espace hilbertien*, C. R. Acad. Sci. Paris, Serie A-B, (1973), 791-794.

[9] Øksendal, B., *Stochastic Differential Equations: An Introduction with Applications*, (Fourth Edition), Springer-Verlag, Berlin, 1995.

[10] Sobczyk, H., *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, London, 1991.

[11] Tsokos, C.P., Padgett, W.J., *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

Tayeb Blouhi
Faculty of Mathematics and Computer Science,
Department of Mathematics,
University of Science and Technology,
Mohamed-Boudiaf El Mnaouar, BP 1505,
Bir El Djir 31000, Oran, Algeria
e-mail: blouhitayeb1984@gmail.com

Mohamed Ferhat
Faculty of Mathematics and Computer Science,
Department of Mathematics,
University of Science and Technology,
Mohamed-Boudiaf El Mnaouar, BP 1505,
Bir El Djir 31000, Oran, Algeria
e-mail: ferhat22@hotmail.fr

Safia Benmansour
01, Rue Barka Ahmed Bouhannak Imama
Prés du Commissariat des 400 Logements de Bouhannak,
Tlemcen 13000 Algeria
e-mail: safiabenmansour@hotmail.fr