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Fekete-Szegő inequalities for certain subclass of analytic functions associated with quasi-subordination

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Abstract. In this present investigation, we introduce a certain subclass $\mathcal{S}_q(\lambda, \gamma, h)$ of analytic functions which is specify in terms of a quasi-subordination. Sharp bounds of the Fekete-Szegő coefficient for functions belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$ are obtained. The results presented give improved versions for the classes involving the quasi-subordination and majorization.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the family of normalized functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.
A function f in $\mathcal{A}$ is said to be univalent in $\mathbb{U}$ if f is one to one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. Let g and f be two analytic functions in $\mathbb{U}$ then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$g(z) = f(w(z)) \quad (z \in \mathbb{U}).$$

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We denote this subordination by $g \prec f$.
In particular, if the f is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$g(0) = f(0) \ \ and \ \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further, [14] function g is said to be quasi-subordinate to f in the unit disk $\mathbb{U}$ if there exist the functions w (with constant coefficient zero) and ϕ which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$g(z) = \phi(z) f(w(z))$$

and this is equivalent to

$$\frac{g(z)}{\phi(z)} \prec f(z) \quad (z \in \mathbb{U}).$$

We denote this quasi-subordination by

$$g(z) \prec_q f(z) \quad (z \in \mathbb{U}).$$

It is observed that if $\phi(z) = 1 \quad (z \in \mathbb{U})$, then the quasi-subordination $\prec_q$ become the usual subordination $\prec$, and for the function $w(z) = z \quad (z \in \mathbb{U})$, the quasi-subordination $\prec_q$ become the majorization '$\ll$'. In this case:

$$g(z) \prec_q f(z) \ \Rightarrow g(z) = \phi(z) f(w(z)) \ \Rightarrow g(z) \ll f(z), \ \ (z \in \mathbb{U}).$$

The concept of majorization is due to MacGregor [8].
In geometric function theory, study a functional made up of combinations of the coefficients of the original function is a typical problem. Initially, a sharp bound of the functional $|a_3 - \nu a_2^2|$ for univalent functions $f \in \mathcal{A}$ of the form with real ν was obtained by Fekete and Szegő [3] in 1933. Since then, the problem of finding the sharp bounds for this functional $|a_3 - \nu a_2^2|$ of any compact family of functions $f \in \mathcal{A}$ with any complex number ν is generally known as the classical Fekete-Szegő problem or inequality. Fekete-Szegő problem for several subclasses of $\mathcal{A}$ have been studied by many authors (see [1], [2], [4], [12], [13], [15], [17], [18]).
Throughout this paper it is assumed that functions ϕ and h are analytic in $\mathbb{U}$.
Also let

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|\phi(z)| \leq 1, \ z \in \mathbb{U}) \tag{1.2}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 \in \mathbb{R}^+). \tag{1.3}$$

Motivated by earlier works in ([5],[6],[11],[16]) on quasi-subordination, we introduce here the following subclass of analytic functions:

Definition 1.1. *For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{S}_q(\lambda, \gamma, h)$, if the following condition are satisfied :*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \prec_q (h(z) - 1), \tag{1.4}$$

where h is given by (1.3) and $z \in \mathbb{U}$.

It follows that a function f is in the class $\mathcal{S}_q(\lambda,\gamma,h)$ if and only if there exists an analytic function ϕ with $|\phi(z)| \leq 1$, in $\mathbb{U}$ such that

$$\frac{\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)}{\phi(z)} \prec (h(z)-1)$$

where h is given by (1.3) and $z \in \mathbb{U}$.
If we set $\phi(z) \equiv 1 \ \ (z \in \mathbb{U})$, then the class $\mathcal{S}_q(\lambda,\gamma,h)$ is denoted by $\mathcal{S}(\lambda,\gamma,h)$ satisfying the condition that

$$1+\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big) \prec h(z) \ \ (z \in \mathbb{U}).$$

In the present paper, we find sharp bounds on the Fekete-Szegő functional for functions belonging in the class $\mathcal{S}_q(\lambda,\gamma,h)$. Several known and new consequences of these results are also pointed out. In order to derive our main results, we have to recall here the following well-known lemma:
Let Ω be class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + ... \tag{1.5}$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)| < 1$.

Lemma 1.1. ([7], p. 10) *If $w(z) \in \Omega$, then for any complex number ν:*

$$|w_1| \leq 1, |w_2 - \nu w_1^2| \leq 1 + (|\nu|-1)|w_1^2| \leq max\{1, |\nu|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2. Main results

Theorem 2.1. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda,\gamma,h)$, then*

$$|a_2| \leq \frac{|\gamma|B_1}{2-\lambda} \tag{2.1}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3-\lambda}\max\{1, |\frac{B_2}{B_1} - KB_1|\}, \tag{2.2}$$

where

$$K = \gamma\Big(\frac{\nu(3-\lambda)}{(2-\lambda)^2} - \frac{\lambda}{2-\lambda}\Big). \tag{2.3}$$

The results are sharp.

Proof. Let $f \in \mathcal{S}_q(\lambda,\gamma,h)$. In view of Definition1.1, there exist then Schwarz functions w and an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big) = \phi(z)(h(w(z))-1) \quad (z \in \mathbb{U}). \tag{2.4}$$

Series expansions for f and its successive derivatives from (1.1) gives us

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\frac{1}{\gamma}\big[(2-\lambda)a_2z+[(3-\lambda)a_3-\lambda(2-\lambda)a_2^2]z^2+\cdots\big]. \quad (2.5)$$

Similarly from (1.2), (1.3) and (1.5), we obtain

$$h(w(z))-1=B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots$$

and

$$\phi(z)\big(h(w(z))-1\big)=A_0B_1w_1z+[A_1B_1w_1+A_0(B_1w_2+B_2w_1^2)]z^2+\cdots. \quad (2.6)$$

Equating (2.5) and (2.6) in view of (2.4) and comparing the coefficients of z and z^2, we get

$$a_2=\frac{\gamma A_0B_1w_1}{2-\lambda} \quad (2.7)$$

and

$$a_3=\frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+A_0\{w_2+\Big(\frac{\gamma\lambda A_0B_1}{2-\lambda}+\frac{B_2}{B_1}\Big)w_1^2\}\Big]. \quad (2.8)$$

Thus, for any $\nu\in\mathbb{C}$, we have

$$\begin{aligned} a_3-\nu a_2^2 &= \frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(\frac{\nu\gamma(3-\lambda)}{(2-\lambda)^2}-\frac{\gamma\lambda}{2-\lambda}\Big)B_1A_0^2w_1^2\Big] \\ &= \frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-KB_1A_0^2w_1^2\Big], \quad (2.9)\end{aligned}$$

where K is given by (2.3).
Since $\phi(z)=A_0+A_1z+A_2z^2+\cdots$ is analytic and bounded by one in $\mathbb{U}$, therefore we have (see[10], p. 172)

$$|A_0|\le 1 \ \ \textit{and} \ \ A_1=(1-A_0^2)y \quad (y\le 1). \quad (2.10)$$

From (2.9) into (2.10), we obtain

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[yw_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(B_1Kw_1^2+yw_1\Big)A_0^2\Big]. \quad (2.11)$$

If A_0=0 in (2.11), we at once get

$$|a_3-\nu a_2^2|\le\frac{|\gamma|B_1}{3-\lambda}. \quad (2.12)$$

But if $A_0\ne 0$, let us then suppose that

$$G(A_0)=yw_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(B_1Kw_1^2+yw_1\Big)A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0|\le 1$ and maximum value of $|G(A_0)|$ is attained at $A_0=e^{\iota\theta}$ $(0\le\theta<2\pi)$, we find that

$$\begin{aligned}\max|G(A_0)| &= \max_{0\le\theta<2\pi}|G(e^{\iota\theta})|=|G(1)| \\ &= \Big|w_2-\Big(KB_1-\frac{B_2}{B_1}\Big)w_1^2\Big|.\end{aligned}$$

Therefore, it follows from (2.11) that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda} \Big| w_2 - \Big(KB_1 - \frac{B_2}{B_1}\Big) w_1^2 \Big|, \tag{2.13}$$

which on using Lemma1.1, shows that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda} \max\{1, |\frac{B_2}{B_1} - KB_1|\},$$

and this last above inequality together with (2.12) establish the results.
The results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z),$$

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z^2)$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = z(h(z) - 1).$$

This completes the proof of Theorem 2.1. □

For $\lambda = 1$ the Theorem 2.1 reduces to following corollary:

Corollary 2.2. *If $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{f(z)} - 1\Big) \prec_q (h(z) - 1) \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}),$$

then

$$|a_2| \leq |\gamma| B_1,$$

and for some $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{2} \max\Big\{1, \Big|\frac{B_2}{B_1} + \gamma(1-2\nu)B_1\Big|\Big\},$$

The results are sharp.

Remark 2.3. For $\phi \equiv 1$, $\gamma = \lambda = 1$, Theorem 2.1 reduces to an improved result of given in [9].

The next theorems gives the result based on majorization.

Theorem 2.4. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \ll (h(z) - 1) \quad (z \in \mathbb{U}), \tag{2.14}$$

then

$$|a_2| \leq \frac{|\gamma| B_1}{2-\lambda}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda} \max\Big\{1, |\frac{B_2}{B_1} - KB_1|\Big\},$$

where K is given by (2.3). The results are sharp.

Proof. Assume that (2.14) holds. From the definition of majorization, there exist an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\phi(z)(h(z)-1) \quad (z\in\mathbb{U}).$$

Following similar steps as in the proof of Theorem 2.1, and by setting $w(z)\equiv z$, so that $w_1=1, w_n=0, n\geq 2$, we obtain

$$a_2=\frac{\gamma A_0 B_1}{2-\lambda}$$

and also we obtain that

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[A_1+\frac{B_2}{B_1}A_0-KB_1A_0^2\Big]. \tag{2.15}$$

On putting the value of A_1 from (2.10) into (2.15), we obtain

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[y+\frac{B_2}{B_1}A_0-\Big(B_1K+y\Big)A_0^2\Big]. \tag{2.16}$$

If $A_0=0$ in (2.16), we at once get

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}. \tag{2.17}$$

But if $A_0\neq 0$, let us then suppose that

$$T(A_0)=y+\frac{B_2}{B_1}A_0-\Big(B_1K+y\Big)A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0|\leq 1$ and maximum value of $|T(A_0)|$ is attained at $A_0=e^{\iota\theta}$ $(0\leq\theta<2\pi)$, we find that

$$\max|T(A_0)| \;=\; \max_{0\leq\theta<2\pi}|T(e^{\iota\theta})|=|T(1)|.$$

Hence, from (2.16), we obtain

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}\Big|KB_1-\frac{B_2}{B_1}\Big|.$$

Thus, the assertion of Theorem 2.4 follows from this last above inequality together with (2.17). The results are sharp for the function given by

$$1+\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=h(z),$$

which completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $0\leq\lambda\leq 1$ and $\gamma\in\mathbb{C}\backslash\{0\}$. If $f\in\mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}(\lambda,\gamma,h)$, then*

$$|a_2|\leq\frac{|\gamma|B_1}{2-\lambda}$$

and for any $\nu\in\mathbb{C}$

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}\max\left\{1,\Big|\frac{B_2}{B_1}-KB_1\Big|\right\},$$

where K is given by (2.3), the results are sharp.

Proof. The proof is similar to Theorem 2.1, Let $f \in \mathcal{S}(\lambda, \gamma, h)$.
If $\phi(z) = 1$, then $A_0 = 1, A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (2.7) and (2.10) and by application of Lemma 1.1, we obtain the desired assertion. The results are sharp for the function $f(z)$ given by

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z),$$

or

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z^2).$$

Thus, the proof of Theorem 2.5 is completed. □

Now, we determine the bounds for the functional $|a_3 - \nu a_2^2|$ for real ν.

Theorem 2.6. *Let $0 \le \lambda \le 1$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| \le \begin{cases} \frac{|\gamma| B_1}{3-\lambda}\big[B_1\big(\frac{\lambda}{2-\lambda} - \frac{3-\lambda}{(2-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\big] & (\nu \le \sigma_1), \\ \frac{|\gamma| B_1}{3-\lambda} & (\sigma_1 \le \nu \le \sigma_1 + 2\rho), \\ -\frac{|\gamma| B_1}{3-\lambda}\big[B_1\big(\frac{\lambda}{2-\lambda} - \frac{3-\lambda}{(2-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\big] & (\nu \ge \sigma_1 + 2\rho), \end{cases} \tag{2.18}$$

where

$$\sigma_1 = \frac{\lambda(2-\lambda)}{(3-\lambda)} - \frac{(2-\lambda)^2}{\gamma(3-\lambda)}\Big(\frac{1}{B_1} - \frac{B_2}{B_1^2}\Big) \tag{2.19}$$

and

$$\rho = \frac{(2-\lambda)^2}{\gamma(3-\lambda)B_1}. \tag{2.20}$$

Each of the estimates in (2.18) are sharp.

Proof. For real values of ν and γ the above bounds can be obtained from (2.2), respectively, under the following cases:

$$B_1 K - \frac{B_2}{B_1} \le -1, \ -1 \le B_1 K - \frac{B_2}{B_1} \le 1 \text{ and } B_1 K - \frac{B_2}{B_1} \ge 1,$$

where K is given by (2.3). We also note the following:
(i) When $\nu < \sigma_1$ or $\nu > \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z$ or one of its rotations.
(ii) When $\sigma_1 < \nu < \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.
(iii) Equality holds for $\nu = \sigma_1$ if and only if $\phi(z) \equiv 1$ and $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations, while for $\nu = \sigma_1 + 2\rho$, the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations. □

The bounds of the functional $a_3 - \nu a_2^2$ for real values of ν and γ for the middle range of the parameter ν can be improved further as follows:

Theorem 2.7. *Let $0 \le \lambda \le 1$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \le \frac{|\gamma|B_1}{3-\lambda} \quad (\sigma_1 \le \nu \le \sigma_1 + \rho) \tag{2.21}$$

and

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \le \frac{|\gamma|B_1}{3-\lambda} \quad (\sigma_1 + \rho \le \nu \le \sigma_1 + 2\rho), \tag{2.22}$$

where σ_1 and ρ are given by (2.19) and (2.20), respectively.

Proof. Let $f \in \mathcal{S}_q(\lambda, \gamma, h)$. For real ν satisfying $\sigma_1 + \rho \le \nu \le \sigma_1 + 2\rho$ and using (2.7) and (2.13) we get

$$\begin{aligned} |a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \le \frac{|\gamma|B_1}{3-\lambda} \Bigg[&|w_2| - \frac{|\gamma|B_1(3-\lambda)}{(2-\lambda)^2}(\nu - \sigma_1 - \rho)|w_1|^2 \\ &+ \frac{|\gamma|B_1(3-\lambda)}{(2-\lambda)^2}(\nu - \sigma_1)|w_1|^2 \Bigg]. \end{aligned}$$

Therefore, by virtue of Lemma 1.1, we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \le \frac{|\gamma|B_1}{3-\lambda}[1 - |w_1|^2 + |w_1|^2],$$

which yields the assertion (2.21).

If $\sigma_1 + \rho \le \nu \le \sigma_1 + 2\rho$, then again from (2.7), (2.13) and the application of Lemma 1.1, we have

$$\begin{aligned} |a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 &\le \frac{|\gamma|B_1}{3-\lambda} \Bigg[|w_2| + \frac{|\gamma|B_1(3-\lambda)}{(2-\lambda)^2}(\nu - \sigma_1 - \rho)|w_1|^2 \\ &\quad + \frac{|\gamma|B_1(3-\lambda)}{(2-\lambda)^2}(\sigma_1 + 2\rho - \nu)|w_1|^2 \Bigg] \\ &\le \frac{|\gamma|B_1}{3-\lambda}[1 - |w_1|^2 + |w_1|^2], \end{aligned}$$

which estimates (2.22). □

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