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A dynamic problem with wear involving electro-elastic-viscoplastic materials with damage

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Abstract. A dynamic contact problem is considered in the paper. The material behavior is described by electro-elastic-viscoplastic law with piezoelectric effects. The body is in contact with damage and an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. The damage of the material caused by elastic deformations. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, variational equation for the electric potential and a parabolic variational inequality for the damage. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

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1. Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and other devoted to boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects(see [2]). For the case of coupled laws of behavior between mechanical and electric effects, general models can be found in (see [5]). Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes

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are just a few examples. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials,

In this paper, we consider a general model for the dynamic process of frictional contact bilateral between a deformable body and an obstacle which results in the wear of the contacting surface. The material obeys an electro-elastic-viscoplastic constitutive law with piezoelectric effects. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence of a unique weak solution of the problem. The idea is to reduce the second order evolution nonlinear inequality of the system to first order evolution inequality. After this,we use classical results on first order evolution nonlinear inequalities and aquation , a parabolic variational inequality and the fixed point arguments.

The paper is structured as follows. In Section 1 we present the electro-elastic-viscoplastic contact model with friction and provide comments on the contact boundary conditions. In Section 2 we list the assumptions on the data and derive the variational formulation. In Section 3 we present our main results on existence and uniqueness which state the unique weak solvability.

2. Problem statement

Problem P: Find a displacement field $u : \Omega\times[0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega\times[0,T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0,T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$, the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}^d$, and the wear $\omega : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ such that

$$\begin{aligned}&\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(\varepsilon(u(t)), \beta(t))\\&+\int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}(\varepsilon(\dot{u}(s))), \varepsilon(u(s))\right) ds - \xi^* E\left(\varphi\right), \text{in } \Omega \text{ a.e. } t \in [0,T],\end{aligned} \tag{2.1}$$

$$\begin{aligned}&D = BE\left(\varphi\right) + \xi\varepsilon\left(u\right)\\&\text{in } \Omega \times [0,T],\end{aligned} \tag{2.2}$$

$$\rho\ddot{u} = Div\ \sigma + f_0, \quad \text{in } \Omega \times [0,T], \tag{2.3}$$

$$\text{div } D = q_0 \quad \text{in } \Omega \times [0,T], \tag{2.4}$$

$$\dot{\beta} - K_1\Delta\beta + \partial\varphi_K\left(\beta\right) \ni S(\varepsilon(u), \beta), \quad \text{in } \Omega \times [0,T], \tag{2.5}$$

$$u = 0, \text{on } \Gamma_1 \times [0,T], \tag{2.6}$$

$$\sigma\nu = h, \text{on } \Gamma_2 \times [0,T], \tag{2.7}$$

$$\begin{cases}\sigma_\nu = -\alpha\left|\dot{u}_\nu\right|, \ \ |\sigma_\tau| = -\mu\sigma_\nu,\\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right), \ \lambda \geq 0, \ \dot{\omega} = -kv^*\sigma_\nu, \ k > 0. \text{ on } \Gamma_3 \times [0,T],\end{cases} \tag{2.8}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \text{ on } \Gamma \ \times [0,T]\,, \tag{2.9}$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0,T]\,, \tag{2.10}$$

$$D\nu = q_2 \text{on } \Gamma_b \times [0,T]\,, \tag{2.11}$$

$$u\,(0) = u_0,\ v\,(0) = v_0,\ \beta\,(0) = \beta_0,\ \omega(0) = \omega_0,\ \text{in } \Omega, \tag{2.12}$$

where (2.1) and (2.2) represent the electro-elastic-viscoplastic constitutive law with damage. we denote $\varepsilon\,(u)$ (respectively; $E\,(\varphi) = -\nabla\varphi$, $\mathcal{A}, \mathcal{G}$, ξ, ξ^*, B) the linearized strain tensor (respectively; electric field, the viscosity nonlinear tensor, the viscoplasticity tensor, the third order piezoelectric tensor and its transpose, the electric permittivity tensor), (2.3) represents the equation of motion where ρ represents the mass density, (2.4) represents the equilibrium equation, we mention that $Div\sigma$, $divD$ are the divergence operators. Inclusion (2.5) describes the evolution of damage field, governed by the source damage function φ, where $\partial\varphi_K\,(\zeta)$ is the subdifferential of indicator function of the set K of admissible damage functions.

Equalities (2.6) and (2.7) are the displacement-traction boundary conditions, respectively. (2.8) describes the frictional bilateral contact with wear described above on the potential contact surface Γ_3. (2.9) represents on Γ, a homogeneous Neumann boundary condition for the damage field. (2.10) , (2.11) represent the electric boundary conditions.The functions u_0, v_0, β_0 and ω_0 in (2.12) are the initial data.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we itroduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e. g: $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations

$$H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\},$$
$$\mathcal{H} = \sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega),$$
$$H_1 = u = (u_i)/\varepsilon(u) \in \mathcal{H} = H^1\,(\Omega)^d$$
$$\mathcal{H}_1 = \sigma \in \mathcal{H}/Div\sigma \in H,$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(u,v)_H = \textstyle\int_\Omega u_i v_i dx, \forall u,v \in H,$$
$$(\sigma,\tau)_{\mathcal{H}} = \textstyle\int_\Omega \sigma_{ij}\tau_{ij} dx, \forall \sigma,\tau \in \mathcal{H},$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \forall u,v \in H_1,$$
$$(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \sigma,\tau \in \mathcal{H}_1,$$

We denote by $|\cdot|_H$ (respectively; $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$and $|\cdot|_{\mathcal{H}1}$) the associated norm on the space H (respectively; $\mathcal{H}$, H_1 and $\mathcal{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, v_\tau = v - v_\nu \nu$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote it trace on Γ. We recall that the summation convention applies to a repeated index.

For the electric displacement field we use two Hilbert spaces

$$\mathcal{W} = \mathbb{L}^2(\Omega)^d, \mathcal{W}_1 = \left\{D \in \mathcal{W}, \operatorname{div} D \in \mathbb{L}^2(\Omega)\right\}$$

endowed with the inner products

$$(D, E)_{\mathcal{W}} = \int_\Omega D_i E_i dx, (D, E)_{\mathcal{W}_1} = (D, E)_{\mathcal{W}} + (\operatorname{div} D, \operatorname{div} E)_{\mathbb{L}^2(\Omega)}$$

and the associated norm $|.|_{\mathcal{W}}$(respectively; $|.|_{\mathcal{W}_1}$). The electric potential field is to be found in

$$W = \left\{\psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_a\right\}.$$

Since $meas\ (\Gamma_a) > 0$, the following Friedrichs-Poincaré's inequality holds, thus

$$|\nabla\psi|_{\mathcal{W}} \geq c_F\, |\psi|_{H^1(\Omega)}\, \forall\psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{\mathcal{W}},$$

and let $|.|_W$ be the associated norm. It follows from (3.1) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace Theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\psi|_{\mathbb{L}^2(\Gamma_3)} \leq \widetilde{c}_0\, |\psi|_W \quad \forall\psi \in W. \tag{3.2}$$

We recall that when $D \in \mathcal{W}_1$ is a sufficiently regular function,the Green's type formula holds

$$(D, \nabla\psi)_{\mathcal{W}} + (\operatorname{div} D, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da. \tag{3.3}$$

When σ is a regular function, the following Green 's type formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \qquad \forall v \in H_1.$$

Next, we define the space

$$V = \{u \in H_1/\ u = 0 \text{ on } \Gamma_1\}.$$

Since $meas\ (\Gamma_1) > 0$, the following Korn's inequality holds

$$|\varepsilon(u)|_{\mathcal{H}} \geq c_K\, |v|_{H_1}\, \forall v \in V, \tag{3.4}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.5}$$

let $|.|_V$ be the associated norm. It follows by (3.4) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leq c_0 |v|_V \, \forall v \in V. \tag{3.6}$$

Finally, for a real Banach space $(X, |.|_X)$ we use the usual notation for the space $\mathbb{L}^p(0,T;X)$ and $W^{k.p}(0,T;X)$, where $1 \leq p \leq \infty, k = 1, 2......$; we also denote by $C(0,T;X)$ and $C^1(0,T;X)$ the spaces of continuous and continuously differentiable function on $[0,T]$with values in X, with the respective norms:

$$|x|_{C(0,T;X)} = \max_{t\in[0,T]} |x(t)|_X ,$$

$$|x|_{C^1(0,T;X)} = \max_{t\in[0,T]} |x(t)|_X + \max_{t\in[0,T]} \left|\dot{x}(t)\right|_X .$$

In what follows, we assume the following assumptions on the problem P.
The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} (a)\, \exists\, M_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq M_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \\ \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (b)\ \exists\, m_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2), \varepsilon_1 - \varepsilon_2| \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \\ \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (c)\ \text{The mapping } x \to \mathcal{A}(x, \varepsilon) \text{ is lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d)\ \text{The mapping } x \to \mathcal{A}(x, 0) \in \mathcal{H}. \end{cases} \tag{3.7}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$satisfies

$$\begin{cases} (a)\ \exists\, L_{\mathcal{B}} > 0 \text{ such that} \\ \ |\mathcal{B}(x, \varepsilon_1, \alpha_1) - \mathcal{B}(x, \varepsilon_2, \alpha_2)| \leq L_{\mathcal{B}} (|\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|) \\ \ \forall\, \varepsilon_1, \varepsilon_2 \in S^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, a.\ e.\ x \in \Omega, \\ (b)\ \text{The mapping } x \to \mathcal{B}(x, \varepsilon, \alpha) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R} \\ (c)\, \text{The mapping } x \to \mathcal{B}(x, 0, 0) \in \mathcal{H}, \end{cases} \tag{3.8}$$

The viscoplasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \exists\, L_{\mathcal{G}} > 0 \text{ such that} \\ \ |\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)| \leq L_{\mathcal{G}} (|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\ \ \forall\, \sigma_1, \sigma_2 \in \mathbb{S}^d, \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.\ e.\ x \in \Omega, \\ (b)\ \text{The mapping } x \to \mathcal{G}(x, \sigma, \varepsilon) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \sigma,\ \varepsilon \in \mathbb{S}^d \\ (c)\ \text{The mapping } x \to \mathcal{G}(x, 0, 0) \in \mathcal{H}, \end{cases} \tag{3.9}$$

The damage source function $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \exists\ M_S > 0 \text{ such that} \\ \ |S(x,\varepsilon_1,\alpha_1) - S(x,\varepsilon_2,\alpha_2)| \le M_S(|\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|) \\ \forall\ \varepsilon_1\ ,\varepsilon_2 \in S^d,\ \forall \alpha_1,\alpha_2 \in \mathbb{R}, a.\ e.\ x \in \Omega, \\ (b)\ \text{The mapping } x \to S(x,\varepsilon,\alpha) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R} \\ (c)\ \text{The mapping } x \to S(x,0,0) \in \mathbb{L}^2(\Omega), \end{cases} \tag{3.10}$$

The piezoelectric tensor $\xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) : \xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d, \\ (b) : \xi(x,\tau) = (e_{ijk}(x)\,\tau_{jk})\ \ \forall \tau = (\tau_{ij}) \in S^d, a.\ e.\ x \in \Omega, \\ (c) : e_{ijk} = e_{ikj} \in \mathbb{L}^\infty(\Omega), \end{cases} \tag{3.11}$$

The electric permittivity tensor $B = (B_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} (a) : B = (B_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ (b) : B(x,E) = (b_{ij}(x)\,E_j)\ \ \forall E = (E_i) \in \mathbb{R}^d,\ \text{a.e. } x \in \Omega, \\ (c) : b_{ij} = b_{ji} \in \mathbb{L}^\infty(\Omega), \\ (d) : \exists\ m_B > 0 \text{ such that} : b_{ij}(x)\,E_iE_j \ge m_B\,|E|^2 \\ \forall E = (E_i) \in \mathbb{R}^d,\ x \in \Omega. \end{cases} \tag{3.12}$$

The mass density ρ satisfy

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \ge \rho^*,\ \text{a.e. } x \in \Omega \tag{3.13}$$

The body forces, surface tractions,the densities of electric charges, and the functions α and μ, satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0,T;H), h \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Gamma_2)^d), \\ q_0 \in L^2\left(0,T;\mathbb{L}^2(\Omega)\right), q_2 \in L^2\left(0,T;\mathbb{L}^2(\Gamma_b)\right). \\ \alpha \in \mathbb{L}^\infty(\Gamma_3)\alpha(x) \ge \alpha^* > 0,\ \text{a.e. on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3), \mu(x) > 0,\ \text{a.e. on } \Gamma_3, \\ K_1 > 0, i = 0,1. \end{cases} \tag{3.14}$$

The set K of admissible damage functions defined by

$$K = \left\{\beta \in H^1(\Omega)/\ 0 \le \beta \le 1\ p.p \text{ in } \Omega\right\} \tag{3.15}$$

The initial data satisfy

$$u_0 \in V, \beta_0 \in K, \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{3.16}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u,v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u.v \in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}, \forall v \in H.$$

We use the notation $(.,.)_{V'\times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u,v)_{V'\times V}=((u,v))\,,\forall u\in H,\forall v\in V.$$

It follows from assumption (3.13) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V,|.|_V)$ into $(H,\|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V\subset H=H'\subset V'$.

We define the function $f(t)\in V$ and $q:[0.T]\to W$ by

$$\begin{aligned}(f(t),v)_V&=\textstyle\int_\Omega f_0(t)\,vdx+\int_{\Gamma_2}h(t)\,vda\forall v\in V,t\in[0,T]\,,\\(q(t),\psi)_W&=-\textstyle\int_\Omega q_0(t)\,\psi dx+\int_{\Gamma_b}q_2(t)\,\psi da\ \forall\psi\in W,t\in[0.T]\,,\end{aligned}$$

for all u, $v\in V$, $\psi\in W$ and $t\in[0.T]$, and note that condition (3.14) imply that

$$f\in\mathbb{L}^2(0.T;V'),q\in\mathbb{L}^2(0.T;W).\tag{3.17}$$

We introduce the following bilinear

$$a_1:H^1(\Omega)\times H^1(\Omega)\to\mathbb{R},a_1(\zeta,\xi)=k_1\int_\Omega\nabla\zeta.\nabla\xi dx,\ \forall\zeta,\xi\in H^1(\Omega)\,.\tag{3.18}$$

We consider the wear functional $j:V\times V\to\mathbb{R}$,

$$j(u,v)=\int_{\Gamma_3}\alpha\,|u_\nu|\,(\mu\,|v_\tau-v^*|)\,da.\tag{3.19}$$

Finally, we consider $\phi:V\times V\to\mathbb{R}$,

$$\phi(u,v)=\int_{\Gamma_3}\alpha\,|u_\nu|\,v_\nu da,\forall v\in V.\tag{3.20}$$

We define for all $\varepsilon>0$

$$j_\varepsilon(g,v)=\textstyle\int_{\Gamma_3}\alpha\,|g_\nu|\left(\mu\sqrt{|v_\tau-v^*|^2+\varepsilon^2}\right)da,\forall v\in V.$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV**:** Find a displacement field $u:\Omega\times[0,T]\to V$, a stress field $\sigma:\Omega\times[0,T]\to\mathbb{S}^d$, an electric potential field $\varphi:\Omega\times[0.T]\to\mathbb{R}$, an electric displacement field $D:\Omega\times[0.T]\to\mathbb{R}^d$, the damage field $\beta:\Omega\times[0,T]\to\mathbb{R}^d$ and the wear $\omega:\Gamma_3\times[0,T]\to\mathbb{R}_+$

such that

$$\begin{aligned}&\sigma(t)=\mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon(u(t)),\beta(t))\\&+\int_0^t\mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(s))),\varepsilon(u(s))\right)ds-\xi^*E\left(\varphi\right)\ \text{, in }\Omega\text{ a.e. }t\in[0,T]\end{aligned}\tag{3.21}$$

$$\begin{aligned}&(\ddot{u}(t),w-\dot{u}(t))_{V'\times V}+(\sigma(t),\varepsilon(w-\dot{u}(t)))_{\mathcal{H}}+j(\dot{u},w)-j(\dot{u},\dot{u}(t))\\&+\phi(\dot{u},w)-\phi(\dot{u},\dot{u}(t))\geq(f(t),w-\dot{u}(t)),\ \forall u,w\in V\end{aligned}\tag{3.22}$$

$$(D(t),\nabla\psi)_{\mathbb{L}^2(\Omega)^d}+(q\left(t\right),\psi)_W=0\forall\psi\in W\tag{3.23}$$

$$\begin{aligned}&\left(\dot{\beta}\left(t\right),\zeta-\beta\left(t\right)\right)_{\mathbb{L}^2(\Omega)}+a_1\left(\beta\left(t\right),\zeta-\beta\left(t\right)\right)\geq\\&\left(S(\varepsilon(u\left(t\right)),\beta\right),\zeta-\beta\left(t\right))_{\mathbb{L}^2(\Omega)},\forall\zeta\in K,\text{ a.e. }t\in[0,T]\end{aligned}\tag{3.24}$$

$$\dot{\omega}=-kv^*\sigma_\nu,\ k>0\tag{3.25}$$

$$u\left(0\right)=u_0,v\left(0\right)=v_0,\beta\left(0\right)=\beta_0,\omega\left(0\right)=\omega_0,\text{ in }\Omega\tag{3.26}$$

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem are the following.

Theorem 4.1. *Let the assumptions* (3.7)−(3.15) *hold. Then, Problem PV has a unique solution* $(u,\sigma,\varphi,D,\beta,\omega)$ *which satisfies*

$$u\in C^1\left(0,T;H\right)\cap W^{1.2}\left(0,T;V\right)\cap W^{2.2}\left(0,T;V'\right)\tag{4.1}$$

$$\sigma\in\mathbb{L}^2(0,T;\mathcal{H}_1),Div\sigma\in\mathbb{L}^2(0,T;V')\tag{4.2}$$

$$\varphi\in W^{1.2}\left(0,T;W\right)\tag{4.3}$$

$$D\in W^{1.2}(0,T;\mathcal{W}_1)\tag{4.4}$$

$$\beta\in W^{1.2}\left(0,T;\mathbb{L}^2\left(\Omega\right)\right)\cap\mathbb{L}^2(0,T;H^1\left(\Omega\right))\tag{4.5}$$

$$\omega\in C^1(0,T;\mathbb{L}^2(\Gamma_3))\tag{4.6}$$

We conclude that under the assumptions (3.7) − (3.15), the mechanical problem (2.1) − (2.12) has a unique weak solution with the regularity (4.1) − (4.6).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g\in\mathbb{L}^2(0,T;V)$ and $\eta\in\mathbb{L}^2(0,T;V')$ are given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta}:[0,T]\to V$ such that

$$\begin{cases}u_{g\eta}\left(t\right)\in V\quad(\ddot{u}_{g\eta}(t),w-\dot{u}_{g\eta}(t))_{V'\times V}+(\mathcal{A}\varepsilon(\dot{u}_{g\eta}(t)),\varepsilon(w-\dot{u}_{g\eta}(t))_{\mathcal{H}}+\\\left(\eta,w-\dot{u}_{g\eta}(t)\right)_{V'\times V}+j(g,w)-j(g,\dot{u}_{g\eta}(t))\geq(f(t),w-\dot{u}_{g\eta}(t)),\quad\forall w\in V.\end{cases}\tag{4.7}$$

$$\dot{u}_{g\eta}\left(0\right)=v\left(0\right)=v_0.\tag{4.8}$$

We define $f_\eta(t)\in V$ for a.e. $t\in[0,T]$ by

$$(f_\eta(t),w)_{V'\times V}=(f(t)-\eta(t),w)_{V'\times V},\forall w\in V.\tag{4.9}$$

From (3.17), we deduce that

$$f_\eta\in\mathbb{L}^2(0,T;V')\tag{4.10}$$

Let now $u_{g\eta} : [0,T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s)\,ds + u_0, \quad \forall t \in [0,T]. \tag{4.11}$$

We define the operator $A : V' \to V$ by

$$(Av, w)_{V' \times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \forall v, w \in V. \tag{4.12}$$

Lemma 4.2. *For all $g \in \mathbb{L}^2(0,T;V)$ and $\eta \in \mathbb{L}^2(0,T;V')$, $PV_{g\eta}$ has a unique solution with the regularity*

$$v_{g\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \text{ and} \dot{v}_{g\eta} \in \mathbb{L}^2(0,T;V'). \tag{4.13}$$

Proof. The proof from nonlinear first order evolution inequalities(see [4, 6]). □

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.
Let us consider now the operator $\Lambda_\eta : \mathbb{L}^2(0,T;V) \to \mathbb{L}^2(0,T;V)$, defined by

$$\Lambda_\eta g = v_{g\eta} \tag{4.14}$$

We have the following lemma.

Lemma 4.3. *The operator Λ_η has a unique fixed point $g \in \mathbb{L}^2(0,T;V)$*

Proof. Let $g_1, g_2 \in \mathbb{L}^2(0,T;V)$ and let $\eta \in \mathbb{L}^2(0.T;V')$. Using similar arguments as those in (4.7), (4.11) we find

$$\begin{aligned}&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t)))\\ &\quad + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \le 0.\end{aligned} \tag{4.15}$$

From the definition of the functional j given by (3.17), we have

$$\begin{aligned}&j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t))\\ &= \int_{\Gamma_3} (\alpha|g_{1\nu}| - \alpha|g_{2\nu}|)(\mu|v_{1\tau} - v^*| - \mu|v_{2\tau} - v^*|)\,da.\end{aligned} \tag{4.16}$$

From (3.6) and (3.14), we find

$$j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) \le C|g_1 - g_2|_V |v_1 - v_2|_V. \tag{4.17}$$

Integrating the (4.15) inequality with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (3.7), (4.17)and the inequality $2ab \le \frac{C}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{C}b^2$ we find

$$|v_2(t) - v_1(t)|_V^2 \le C\int_0^t |g_2(s) - g_1(s)|_V^2\,ds. \tag{4.18}$$

Thus, for m sufficiently large, Λ_η^m is a contraction on $\mathbb{L}^2(0,T;V)$ and so Λ_η has a unique fixed point in this Banach space. □

Third step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.
Problem $PV_{g\eta}^{\varphi}$: Find an electric potential field $\varphi_{g\eta} : \Omega \times [0.T] \to W$ such that

$$(\beta\nabla\varphi_{g\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} = (q(t), \psi)_W, \ \forall\psi \in W, \ t \in [0,T]. \tag{4.19}$$

We have the following result for $PV_{g\eta}^{\varphi}$:

Lemma 4.4. *There exists a unique solution $\varphi_{g\eta} \in W^{1.2}(0.T;W)$ satisfies (4.19), moreover if φ_1 and φ_2 are two solutions to (4.19). Then, there exists a constants $c > 0$ sach that*

$$|\varphi_1(t) - \varphi_2(t)|_W \leq c\,|u_1(t) - u_2(t)|_V \quad \forall t \in [0,T]\,. \tag{4.20}$$

Proof. The proof given in Ref (see [1]). □

Fourth step: For $\phi \in C(0,T;\mathbb{L}^2(\Omega))$, we consider the following variational problem.
Problem PV_ϕ: Find the damage field $\beta_\phi : [0,T] \to K$ such that

$$\begin{aligned}&\left(\dot{\beta}_\phi(t), \zeta - \beta_\phi(t)\right)_{\mathbb{L}^2(\Omega)} + a_1(\beta_\phi(t), \zeta - \beta_\phi(t)) \geq \\ &(\phi, \zeta - \beta_\phi(t))_{\mathbb{L}^2(\Omega)}, \forall \zeta \in K, \text{ a.e. } t \in [0,T]\,,\end{aligned} \tag{4.21}$$

$$\beta_\phi(0) = \beta_0 \tag{4.22}$$

Lemma 4.5. *There exists a unique solution β_ϕ to the auxiliary problem PV_ϕ such that*

$$\beta_\phi \in W^{1.2}\left(0,T;\mathbb{L}^2(\Omega)\right) \cap \mathbb{L}^2(0,T;H^1(\Omega))$$

Proof. The proof given in Ref (see [3]). □

By taking into account the above results and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$ and of the functions ψ and S, we may consider the operator

$$\begin{aligned}&\Lambda : C(0,T;V' \times \mathbb{L}^2(\Omega)) \to C(0,T;V' \times \mathbb{L}^2(\Omega)),\\ &\Lambda(\eta,\phi)(t) = (\Lambda_1(\eta)(t), \Lambda_2(\phi)(t)),\end{aligned} \tag{4.23}$$

$$\begin{aligned}&(\Lambda_1(\eta), w)_{V'\times V} = (\mathcal{B}(\varepsilon(u_\eta(t)), \beta_\phi(t)), w)\\ &+\left(\int_0^t \mathcal{G}\left(\sigma_\eta(s) - \mathcal{A}(\varepsilon(\dot{u}_\eta(s))), \varepsilon(u_\eta(s))\right) ds + \xi^* \nabla(\varphi), w\right)\\ &+\phi(\dot{u}_\eta, w) \quad \forall w \in V,\end{aligned} \tag{4.24}$$

$$\Lambda_2(\phi)(t) = S(\varepsilon(u_\eta(t)), \beta_\phi). \tag{4.25}$$

We have the following result.

Lemma 4.6. *The mapping $\Lambda(\eta,\phi) : [0,T] \to V' \times \mathbb{L}^2(\Omega)$ has a unique element $(\eta^*,\phi^*) \in C(0,T;V' \times \mathbb{L}^2(\Omega))$ such that $\Lambda(\eta^*,\phi^*) = (\eta^*,\phi^*)$*

Proof. Let (η_1,ϕ_1), $(\eta_2,\phi_2) \in C(0,T;V' \times \mathbb{L}^2(\Omega))$ and $t \in [0,T]$. We use the notation $u_{\eta i} = u_i, \dot{u}_{\eta i} = v_{\eta i} = v_i, \beta_{\phi i} = \beta_i$, $\varphi_{\eta i} = \varphi_i$ and $\sigma_{\eta i} = \sigma_i$, for $i = 1,2$. Using (4.24) and the relations (3.7) – (3.9), we obtain

$$\begin{aligned}&|\eta_1(t) - \eta_2(t)|_{V'}^2 \leq C(|\beta_1(t) - \beta_2(t)|_{\mathbb{L}^2(\Omega)}^2\\ &+|u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2\\ &+\int_0^t (|\sigma_1(s) - \sigma_2(s)|_{\mathcal{H}_1}^2 + |v_1(s) - v_2(s)|_V^2\\ &+|u_1(s) - u_2(s)|_V^2)ds + |\varphi_1(t) - \varphi_2(t)|_W^2\\ &+\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)).\end{aligned} \tag{4.26}$$

From the definition of the functional ϕ given by (3.20), and using (3.6), (3.14) we have

$$\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)) \leq C\left|v_1(t) - v_2(t)\right|_V^2. \quad (4.27)$$

We have

$$\left|u_2(t) - u_1(t)\right|_V \leq \int_0^t \left|v_2(s) - v_1(s)\right|_V ds$$

Taking into account that

$$\sigma_i(t) = \mathcal{A}(\varepsilon(\dot{u}_i(t))) + \eta_i(t), \ \forall t \in [0, T]. \quad (4.28)$$

By (2.1), and using (3.7), we find

$$\left|\sigma_1(s) - \sigma_2(s)\right|_{\mathcal{H}_1}^2 \leq C\left(\left|v_1(t) - v_2(t)\right|_V^2 + \left|\eta_1 - \eta_2\right|_{V'}^2\right). \quad (4.29)$$

It follows that

$$\begin{aligned}&\left(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)\right)\\&+ \left(\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))\right) +\\&+ \left(\eta_1(s) - \eta_2(s), v_1(t) - v_2(t)\right) \leq j(v_1, v_2(t)) - j(v_1, v_1(t))\\&- j(v_2, v_2(t)) + j(v_2, v_1(t)).\end{aligned} \quad (4.30)$$

From the definition of the functional j given by (3.19), and using (3.6), (3.14) we have

$$j(v_1, v_2(t)) - j(v_1, v_1(t)) - j(v_2, v_2(t)) + j(v_2, v_1(t)) \leq C\left|v_1 - v_2\right|_V^2. \quad (4.31)$$

Integrating the (4.30) inequality with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (3.7), (4.31), using Cauchy-Schwartz's inequality and the inequality

$$2ab \leq m_{\mathcal{A}} a^2 + \frac{1}{m_{\mathcal{A}}} b^2,$$

by Gronwall's inequality we find

$$\left|v_1(t) - v_2(t)\right|_V^2 \leq C\int_0^t \left|\eta_1(s) - \eta_2(s)\right|_{V'}^2 ds. \quad (4.32)$$

Also

$$\begin{aligned}\int_0^t \left|u_1(s) - u_2(s)\right|_V^2 ds &\leq C\int_0^t \int_0^s \left|\eta_1(r) - \eta_2(r)\right|_{V'}^2 drds\\&\leq C\int_0^t \left|\eta_1(s) - \eta_2(s)\right|^2 ds.\end{aligned} \quad (4.33)$$

For the damage field, from (4.21) we deduce that

$$\left(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2\right)_{\mathbb{L}^2(\Omega)} + a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \leq (\phi_1 - \phi_2, \beta_1 - \beta_2)_{\mathbb{L}^2(\Omega)},$$

$$\text{a.e. } t \in [0, T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and the inequality $a_1(\beta_2 - \beta_1, \beta_2 - \beta_1) \geq 0$, by Gronwall's inequality we find

$$\left|\beta_1(t) - \beta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 \leq C\int_0^t \left|\phi_1(s) - \phi_2(s)\right|_{\mathbb{L}^2(\Omega)}^2 ds, \ \forall t \in [0, T]. \quad (4.34)$$

Applying the previous inequalities, the estimates (4.32) – (4.34), we obtain

$$|\Lambda(\eta_2,\phi_2)(t)-\Lambda(\eta_1,\phi_1)(t)|_{V'\times\mathbb{L}^2(\Omega))}\leq C\int_0^t|(\eta_2,\phi_2)(s)-(\eta_1,\phi_1)(s)|\,ds$$

Thus, for m sufficiently large, Λ^m is a contraction on $C(0,T;V'\times\mathbb{L}^2(\Omega)))$ and so Λ has a unique fixed point in this Banach space. □

We consider the operator $\mathcal{L}:C(0,T;\mathbb{L}^2(\Gamma_3))\to C(0,T;\mathbb{L}^2(\Gamma_3))$

$$\mathcal{L}\omega(t)=-kv^*\int_0^t\sigma_\nu(s)\,ds,\forall t\in[0,T]. \tag{4.35}$$

Lemma 4.7. *The operator $\mathcal{L}:C(0,T;\mathbb{L}^2(\Gamma_3))\to C(0,T;\mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^*\in C(0,T;\mathbb{L}^2(\Gamma_3))$, such that $\mathcal{L}\omega^*=\omega^*$.*

Proof. Using (4.35), we have

$$|\mathcal{L}\omega_1(t)-\mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}\leq kv^*\int_0^t|\sigma_1(s)-\sigma_2(s)|^2\,ds, \tag{4.36}$$

From (2.1) and using (3.7) – (3.9), we find

$$\begin{aligned}|\sigma_1(t)-\sigma_2(t)|^2_{\mathcal{H}_1}\leq C(&|\beta_1(t)-\beta_2(t)|^2_{\mathbb{L}^2(\Omega)}+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\\&+\int_0^t(|\sigma_1(s)-\sigma_2(s)|^2_{\mathcal{H}_1}+|v_1(s)-v_2(s)|^2_V+|u_1(s)-u_2(s)|^2_V)ds\\&+|\varphi_1(t)-\varphi_2(t)|^2_W\end{aligned} \tag{4.37}$$

By (4.26), (4.34), and by Gronwall's inequality we find

$$|\beta_1(t)-\beta_2(t)|^2_{\mathbb{L}^2(\Omega)}\leq C\int_0^t|u_1(s)-u_2(s)|^2_V\,ds,\ \forall t\in[0,T]. \tag{4.38}$$

And by Gronwall's inequality we find

$$|\sigma_1(t)-\sigma_2(t)|^2_{\mathcal{H}_1}\leq C\left(\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\right) \tag{4.39}$$

We have

$$\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V$$

$$\leq C\int_0^t|v_1(s)-v_2(s)|^2_V\,ds.$$

So

$$\begin{aligned}&\textstyle\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\\&\leq C\left(\int_0^t|v_1(s)-v_2(s)|^2_V\,ds++|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}\right)\end{aligned} \tag{4.40}$$

By Gronwall's inequality we find

$$\textstyle\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\leq C\,|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}$$

So, we have

$$|\sigma_1(t) - \sigma_2(t)|^2_{\mathcal{H}_1} \le C \int_0^t \; |\omega_1(s) - \omega_2(s)|^2_{\mathbb{L}^2(\Gamma_3)} \, ds \tag{4.41}$$

Using (4.41) , we find

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \le C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} \, ds$$

Thus, for m sufficiently large, $\mathcal{L}^m$ is a contraction on $C(0,T;\mathbb{L}^2(\Gamma_3))$ and so $\mathcal{L}$ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.

Existence. Let $g^* \in \mathbb{L}^2(0,T;V)$ be the fixed point of Λ_{η^*} defined by (4.14), let $(\eta^*, \phi^*) \in C(0,T;V' \times \mathbb{L}^2(\Omega)))$ be the fixed point of Λ defined by (4.23) – (4.25), let $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathcal{L}\omega^*$ defined by (4.36), and let

$$(u, \varphi, \beta) = (u_{g^*\eta^*}, \varphi_{g^*\eta^*}, \beta_{\phi^*})$$

be the solutions of Problems $PV_{g^*\eta^*}$, and respectively $PV^{\varphi}_{g^*\eta^*}$, PV_{ϕ^*}. It results from (4.7), (4.8), (4.19), (4.21), (4.22) that $(u_{g^*\eta^*}, \varphi_{g^*\eta^*}, \beta_{\phi^*})$ is the solutions of Problems PV. Properties (4.1) – (4.6) follow from Lemmas 1, 3 and 4.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators Λ_η, $\Lambda, \mathcal{L}$ defined by (4.14), (4.23) – (4.25), (4.36) and the unique solvability of the Problem $PV_{g\eta}$ and PV_ϕ which completes the proof.

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