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General stabilization of a thermoelastic systems with a boundary control of a memory type

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Abstract. In this paper we consider an n-dimensional thermoelastic system, in a bounded domain, where the memory-type damping is acting on a part of the boundary and where the resolvent kernel k of $-g'(t)/g(0)$ satisfies $k''(t) \geq \gamma(t)(-k'(t))^p$, $t \geq 0$, $1 < p < \frac{3}{2}$. We establish a general decay result, from which the usual exponential and polynomial decay rates are only special cases. This work generalizes and improves earlier results in the literature.

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1. Introduction

In [4], Messaoudi and Al-Khulaifi studied the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, ρ is a positive real number such that $0 < \rho \leq 2/(n-2)$ if $n \geq 3$ and $\rho > 0$ if $n = 1, 2$, and g is a positive nonincreasing function. They obtained a general decay rate where the relaxation functions satisfies

$$g'(t) \leq -\xi(t)\, g^p(t), \quad \forall t \geq 0, \quad 1 \leq p < \frac{3}{2}.$$

Stabilization of thermoelastic systems has been studied by many researchers. Different mechanisms have been utilized to stabilize such systems and several decay and stability results have been obtained. In this regard we mention, among many

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others, the work of Dafermos [2], Messaoudi and Al-Shehri [3], Muñoz Rivera [7], Rivera and Barreto [8], Rivera and Racke [9], Racke and Shibata [11].

In the present work, we are concerned with

$$\begin{cases} u_{tt} - \mu \triangle u - (\mu+\lambda)\nabla(divu) + \beta\nabla\theta = 0, & \text{in } \Omega\times(0,+\infty) \\ c\theta_t - \kappa\triangle\theta + \beta divu_t = 0, & \text{in } \Omega\times(0,+\infty) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_0\times[0,+\infty) \\ u(x,t) = -\int_0^t g(t-s)\left(\mu\frac{\partial u}{\partial\nu} + (\mu+\lambda)(divu)\nu\right)(s)\,ds, & \text{on } \Gamma_1\times[0,+\infty) \\ \theta = 0, & \text{on } \Gamma\times[0,+\infty), \end{cases} \tag{1.2}$$

which is a thermoelastic system subjected to the effect of a viscoelastic damping acting on a part of the boundary. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, ν is the unit outward normal vector to Γ, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature. The relaxation function g is positive and differentiable function and the boundary condition on Γ_1 is the nonlocal condition responsible for the memory effect. The coefficients $c,\ \kappa,\ \mu,\ \lambda$ are positive constants, where $\mu,\ \lambda$ are Lame moduli and $\beta \neq 0$ is a real number. By considering the resolvent kernel of $-g'/g(0)$, the boundary condition takes the form

$$\frac{\partial u}{\partial\nu} = -\frac{1}{g(0)}(u_t + k * u_t), \quad \text{on } \Gamma_1\times[0,+\infty),$$

where k is the resolvent kernel of $-g'/g(0)$.

Messaoudi and Al-Shehri [3] considered (1.2) for a wider class of kernels k satisfying

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq \gamma(t)(-k'(t)),$$

where $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma(t) \geq 0, \quad \gamma'(t) \leq 0, \quad \text{and} \int_0^\infty \gamma(t)dt = +\infty, \tag{1.3}$$

they proved a more general energy decay result.

Recently, Mustafa [10] treated system (1.2), for k satisfying

$$k(0) > 0, \quad \lim_{t\to\infty} k(t) = 0, \quad k'(t) \leq 0, \tag{1.4}$$

$$k''(t) \geq H(-k'(t)), \quad \forall t > 0, \tag{1.5}$$

where H is a positive function, which is linear or strictly increasing, strictly convex of class C^2 on $(0,r]$, $r < 1$, and $H(0) = 0$ and proved for $u_0 = 0$ on Γ_1, an explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases.

The aim of this work is to study problem (1.2) for k satisfies

$$k(0) > 0,\ k(t) \geq 0,\ k'(t) \leq 0,\ k''(t) \geq \gamma(t)(-k'(t))^p,\ t \geq 0,\ 1 < p < \frac{3}{2}, \tag{1.6}$$

where γ satisfies (1.3).

2. Notation and transformation

In this section we introduce our problem, as well as some notation and lemmas. The partition Γ_0 and Γ_1 of boundary are closed, disjoint, with $meas\,(\Gamma_0) > 0$ and satisfying

$$\Gamma_1 = \{x \in \Gamma : \ m(x).\nu \geq \delta > 0\}, \quad \Gamma_0 = \{x \in \Gamma : \ m(x).\nu \leq 0\}, \tag{2.1}$$

where $m(x) = x - x_0$, for some $x_0 \in \mathbb{R}^n$.

Similarly to [5, 3, 6], applying Volterra's inverse operator, the boundary condition

$$u(x,t) = -\int_0^t g(t-s)\left(\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)(divu)\,\nu\right)(s)\,ds, \quad \text{on } \ \Gamma_1 \times [0,+\infty),$$

can be transformed into

$$\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)(divu)\,\nu = -\frac{1}{g(0)}(u_t + k * u_t), \quad \text{on } \ \Gamma_1 \times [0,+\infty),$$

where $*$ denotes the convolution product

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-s)\psi(s)ds,$$

and k is the resolvent kernel of $-g'/g(0)$ which satisfies

$$k + \frac{1}{g(0)}(g' * k) = -\frac{1}{g(0)} g'.$$

Taking $\eta = 1/g(0)$ and assuming throughout the paper that $u_0 = 0$ on Γ_1, we arrive at

$$\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)(divu)\,\nu = -\eta\,(u_t + k(0)u + k' * u), \quad \text{on } \ \Gamma_1 \times [0,+\infty). \tag{2.2}$$

Therefore, we will use the boundary relation (2.2) instead of the third equation in (1.2).

Since we are interested in relaxation functions of more general decay, we would like to know if the resolvent kernel k, involved in (2.2), inherits some properties of the relaxation function involved in $(1.2)_3$. The following Lemma answers this question.

Let $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be continuous. Let k be its resolvent, i.e.

$$k(t) = h(t) + (k * h)(t), \tag{2.3}$$

It is well known that k is continuous and positive (see [1, 9]).

Lemma 2.1. *Let $p > 1$, $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a nonincreasing function satisfying $\gamma(0) > 0$, and*

$$\begin{aligned} C_p &= \sup_{t\geq 0} \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}} \left(1 + \int_0^{t-s} \gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}} \\ &\quad \times \left(1 + \int_0^{s} \gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}} ds. \end{aligned}$$

Assume that there exists C and $1-CC_p>0$ such that

$$h(t)\leq\frac{C}{\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}}.$$

Then there exists $\widetilde{C}$ such that

$$k(t)\leq\frac{\widetilde{C}}{\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}}.$$

Proof. We set

$$k_p(t)=k(t)\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}},$$

and

$$h_p(t)=h(t)\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}.$$

By multiplying (2.3) by $\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}$, we obtain

$$\begin{aligned}
k_p(t)&=h_p(t)+\int_0^t\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}k(t-s)\,h(s)\,ds\\
&=h_p(t)+\int_0^t\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}\left(1+\int_0^{t-s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}\\
&\times k_p(t-s)\,h(s)\,ds\\
&=h_p(t)+\int_0^t\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}\left(1+\int_0^{t-s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}\\
&\times\left(1+\int_0^{s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}k_p(t-s)\left(1+\int_0^{s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}h(s)\,ds\\
&=h_p(t)+\int_0^t\left(1+\int_0^t\gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}\left(1+\int_0^{t-s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}\\
&\times\left(1+\int_0^{s}\gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}k_p(t-s)\,h_p(s)\,ds,
\end{aligned}$$

Consequently,

$$\sup_{0\leq s\leq t}k_p(s)\leq\sup_{0\leq s\leq t}h_p(s)+CC_p\sup_{0\leq s\leq t}k_p(s)\leq C+CC_p\sup_{0\leq s\leq t}k_p(s),$$

which implies

$$\sup_{0\leq s\leq t}k_p(s)\leq\frac{C}{1-CC_p},\ \forall t>0.$$

Hence

$$k_p(t)\leq\frac{C}{1-CC_p}.$$

Therefore

$$k(t) \leq \frac{C}{1 - CC_p}\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}}.$$

Finally, we obtain the result of the lemma

$$k(t) \leq \frac{\widetilde{C}}{\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}}}.$$

□

Let us define

$$(\varphi \circ \psi)(t) = \int_0^t \varphi(t-s) \left|\psi(t) - \psi(s)\right|^2 ds,$$

$$(\varphi \Diamond \psi)(t) = \int_0^t \varphi(t-s)\left(\psi(t) - \psi(s)\right) ds.$$

By using Hölder's inequality, we have

$$\left|(\varphi \Diamond \psi)(t)\right|^2 \leq \left(\int_0^t |\varphi(s)|\, ds\right)\left(|\varphi| \circ \psi\right)(t). \tag{2.4}$$

Lemma 2.2 ([9])**.** *If* $\varphi,\ \psi \in C^1(\mathbb{R}^+)$, *then*

$$(\varphi * \psi)\psi_t = -\frac{1}{2}\varphi(t)\left|\psi(t)\right|^2 + \frac{1}{2}\varphi' \circ \psi - \frac{1}{2}\frac{d}{dt}\left(\varphi \circ \psi - \left(\int_0^t \varphi(s)ds\right)\left|\psi(t)\right|^2\right). \tag{2.5}$$

Let us define

$$V = \left\{u \in H^1(\Omega) :\ u = 0 \quad \text{on} \quad \Gamma_0\right\}.$$

The well-posedness of system (1.2) is presented in the following theorem, which can be proved, using the Galerkin method as in [9].

Theorem 2.3. *Let* $k \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $u_0 \in \left(H^2(\Omega) \cap V\right)^n$, $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *and* $u_1 \in V^n$, *with*

$$\frac{\partial u_0}{\partial \nu} + \eta u_0 = 0 \quad \text{on}\ \Gamma_1.$$

Then there exists a unique strong solution u *of system* (1.2), *such that*

$$\begin{aligned} u &\in C\left(\mathbb{R}^+; \left(H^2(\Omega) \cap V\right)^n\right) \cap C^1\left(\mathbb{R}^+; V^n\right) \cap C^2\left(\mathbb{R}^+; L^2(\Omega)^n\right), \\ \theta &\in C\left(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}^+; H_0^1(\Omega)\right). \end{aligned}$$

3. Decay of solutions

In this section we study the asymptotic behavior of the solutions of system (1.2) when the resolvent kernel k satisfies the assumption

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq \gamma(t)\left(-k'(t)\right)^p, \tag{3.1}$$

where $t \geq 0$, $1 < p < \frac{3}{2}$ and $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function satisfying

$$\gamma(t) > 0, \quad \gamma'(t) \leq 0. \tag{3.2}$$

By multiplying the first equation in (1.2) by u_t and the second equation in (1.2) by θ and integrating over Ω, using integration by parts and boundary conditions (2.2) and (2.5), one can easily find that the first order energy of system (1.2) is given by (see Lemma 3.1 below).

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega}\left[|u_t|^2+\mu|\nabla u|^2+(\mu+\lambda)(divu)^2+c\theta^2\right]dx \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k'\circ u)(t)d\Gamma_1+\frac{\eta}{2}\int_{\Gamma_1}k(t)|u|^2 d\Gamma_1. \end{aligned} \tag{3.3}$$

Lemma 3.1. *The energy of the solution of* (1.2) *satisfies*

$$\begin{aligned} E'(t) &= -\kappa\int_{\Omega}|\nabla\theta|^2 dx-\eta\int_{\Gamma_1}|u_t|^2 d\Gamma_1+\frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2 d\Gamma_1 \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k''\circ u)(t)\,d\Gamma_1\le 0. \end{aligned} \tag{3.4}$$

Proof. Direct differentiation, using Eqs. (1.2) and (2.2), gives

$$\begin{aligned} E'(t) &= -\kappa\int_{\Omega}|\nabla\theta|^2 dx-\eta\int_{\Gamma_1}|u_t|^2 d\Gamma_1+\frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2 d\Gamma_1 \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k''\circ u)(t)\,d\Gamma_1. \end{aligned}$$

and consequently, we obtain (3.4) for strong solutions. This result and all estimates below remain valid for weak solutions by a simple density argument. □

The following crucial lemmas will be used in the proof of our result.

Lemma 3.2. *The solution u of* (1.2) *satisfies*

$$\|u(t)-u(s)\|_{L^2(\Gamma_1)}^2\le CE(0),\quad \forall s\in[0,t].$$

Proof. Using the trace theorem and (3.3), we obtain, for all $s\in[0,t]$,

$$\begin{aligned} \|u(t)-u(s)\|_{L^2(\Gamma_1)}^2 &\le c\|\nabla u(t)-\nabla u(s)\|_2^2 \\ &\le c\left(\|\nabla u(t)\|_2^2+\|\nabla u(s)\|_2^2\right) \\ &\le c'(E(t)+E(s)) \\ &\le C(E(0)). \end{aligned}$$

□

Lemma 3.3. *Assume that k satisfies* (3.1)*. Then*

$$\int_0^{+\infty}\gamma(t)\left[-k'(t)\right]^{1-\sigma}dt<+\infty,\ \forall\sigma<2-p.$$

Proof. Recalling (3.1), we easily see that

$$\begin{aligned}\gamma(t)\left[-k'(t)\right]^{1-\sigma} &= \gamma(t)\left(-k'(t)\right)^{p}\left[-k'(t)\right]^{1-\sigma-p}\\ &\leq k''(t)\left[-k'(t)\right]^{1-\sigma-p}.\end{aligned}$$

Then, integration gives

$$\begin{aligned}\int_0^{+\infty}\gamma(t)\left[-k'(t)\right]^{1-\sigma}dt &\leq \int_0^{+\infty}k''(t)\left[-k'(t)\right]^{1-\sigma-p}dt\\ &= -\frac{\left[-k'(t)\right]^{2-p-\sigma}}{2-p-\sigma}\Bigg|_0^{+\infty} < +\infty, \qquad (3.5)\end{aligned}$$

since $\sigma < 2-p$ and $-k'$ is nonnegative and nonincreasing. □

Lemma 3.4. *Assume that k satisfies* (3.1)*. Then the solution u of* (1.2) *satisfies*

$$\left[\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}} \leq \left[\int_{\Gamma_1}\left(k''\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}.$$

Proof. Using the fact that γ is nonincreasing, we get

$$\left(-k'(t-s)\right)^p\gamma(t-s) \geq \left(-k'(t-s)\right)^p\gamma(t).$$

Multiplication by $|u(t)-u(s)|^2$ and integration over $(0,t)\times\Gamma_1$, we obtain

$$\int_{\Gamma_1}\int_0^t\left(-k'(t-s)\right)^p\gamma(t-s)\left|u(t)-u(s)\right|^2 dsd\Gamma_1$$

$$\geq \int_{\Gamma_1}\int_0^t\left(-k'(t-s)\right)^p\gamma(t)\left|u(t)-u(s)\right|^2 dsd\Gamma_1,$$

then, by using (3.1) , we find

$$\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1 \leq \int_{\Gamma_1}k''\circ ud\Gamma_1,$$

hence

$$\left[\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}} \leq \left[\int_{\Gamma_1}\left(k''\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}. \qquad □$$

Lemma 3.5. *Assume that k satisfies (*3.1*). Then there exists $C>0$ such that the solution u of* (1.2) *satisfies*

$$\int_{\Gamma_1}\gamma(t)\left(-k'\circ u\right)d\Gamma_1 \leq C\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

Proof. It easy to see that

$$\begin{aligned}
\int_{\Gamma_1} (-k' \circ u)\, d\Gamma_1 &= \int_{\Gamma_1} \int_0^t -k'(t-s)\, |u(t)-u(s)|^2\, ds d\Gamma_1 \\
&= \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{(1-\sigma)\frac{p-1}{p-1+\sigma}} \left(|u(t)-u(s)|^2\right)^{\frac{p-1}{p-1+\sigma}} \\
&\quad \times [-k'(t-s)]^{1-(1-\sigma)\frac{p-1}{p-1+\sigma}} \left(|u(t)-u(s)|^2\right)^{\frac{\sigma}{p-1+\sigma}} ds d\Gamma_1 \\
&= \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{(1-\sigma)\frac{p-1}{p-1+\sigma}} \left(|u(t)-u(s)|^2\right)^{\frac{p-1}{p-1+\sigma}} \\
&\quad \times [-k'(t-s)]^{\frac{\sigma p}{p-1+\sigma}} \left(|u(t)-u(s)|^2\right)^{\frac{\sigma}{p-1+\sigma}} ds d\Gamma_1.
\end{aligned}$$

Using Hölder's inequality, for

$$s = \frac{p-1+\sigma}{p-1} \text{ and } s' = \frac{p-1+\sigma}{\sigma},$$

and Lemma 3.2, we arrive at

$$\begin{aligned}
\int_{\Gamma_1} (-k' \circ u)\, d\Gamma_1 &\le \left[\int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1-\sigma} |u(t)-u(s)|^2\, ds d\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}} \\
&\quad \times \left[\int_{\Gamma_1} \int_0^t [-k'(t-s)]^{p} |u(t)-u(s)|^2\, ds d\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}} \\
&\le \left[\int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1-\sigma} |u(t)-u(s)|^2\, ds d\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}} \\
&\quad \times \left[\int_{\Gamma_1} \left((-k')^p \circ u\right) d\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}} \\
&\le C \left[\int_0^t [-k'(t-s)]^{1-\sigma}\, ds d\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}} \\
&\quad \times \left[\int_{\Gamma_1} \left((-k')^p \circ u\right) d\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}}.
\end{aligned}$$

By taking $\sigma = \frac{1}{2}$, we have

$$\int_{\Gamma_1} (-k' \circ u)\, d\Gamma_1 \le C \left[\int_0^t [-k'(s)]^{\frac{1}{2}}\, ds d\Gamma_1\right]^{\frac{2p-2}{2p-1}} \left[\int_{\Gamma_1} \left((-k')^p \circ u\right) d\Gamma_1\right]^{\frac{1}{2p-1}}. \quad (3.6)$$

Multiply both sides of (3.6) by $\gamma(t)$, recall Lemma 3.3 and Lemma 3.4 and use Lemma 3.1 to get

$$\begin{aligned}
&\gamma(t)\int_{\Gamma_1}(-k'\circ u)\,d\Gamma_1\\
\le\;& C\gamma(t)\left[\int_0^t[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\le\;& C\gamma(t)^{\frac{2p-2}{2p-1}}\left[\int_0^t[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\gamma(t)^{\frac{1}{2p-1}}\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\le\;& C\left[\int_0^t\gamma(s)[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}\left(\gamma(t)(-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\le\;& C\left[\int_0^{+\infty}\gamma(s)[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}(k''\circ u)\,d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\le\;& C[-E'(t)]^{\frac{1}{2p-1}}.
\end{aligned}$$

□

For completeness, we adopt without proof the following result from [3].

Lemma 3.6 ([3])**.** *There exist positive constants N, M, m, c, and t_0 such that the functional*

$$L(t)=NE(t)+\int_\Omega u_t.[M+(n-1)u]\,dx,$$

is equivalent to $E(t)$ and satisfies

$$L'(t)\le -mE(t)-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma_1,\ \ \forall t\ge t_0. \tag{3.7}$$

Theorem 3.7. *Given $(u_0,u_1,\theta_0)\in\left(V^n,\left(L^2(\Omega)\right)^n,H_0^1(\Omega)\right)$. Assume that (2.1) and (3.1) − (3.2) hold, with $\lim_{t\to\infty}k(t)=0$. Then for each $t_0>0$, there exists a strictly positive constant C' such that the solution u of (1.2) satisfies, for all $t\ge t_0$,*

$$E(t)\le C'\left[\frac{1}{\int_0^t\gamma^{2p-1}(s)\,ds+1}\right]^{\frac{1}{2p-2}} \tag{3.8}$$

Moreover,

$$\text{If}\ \int_0^{+\infty}E(t)<+\infty, \tag{3.9}$$

then

$$E(t)\le C'\left[\frac{1}{\int_0^t\gamma^{p}(s)\,ds+1}\right]^{\frac{1}{p-1}} \tag{3.10}$$

4. Proof of the main result

Multiplying (3.7) by $\gamma(t)$, and recall lemma 3.5, we obtain

$$\gamma(t)L'(t) \le -m\gamma(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

Multiplication of the last inequality by $\gamma^{\alpha}(t)E^{\alpha}(t)$, where $\alpha = 2p-2$, gives

$$\gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le -m\gamma^{\alpha+1}(t)E^{\alpha+1}(t) + C\gamma^{\alpha}(t)E^{\alpha}(t)\left[-E'(t)\right]^{\frac{1}{\alpha+1}}.$$

Use of Young's inequality, with $q = \alpha+1$ and $q^* = \frac{\alpha+1}{\alpha}$, yields, for any $\varepsilon > 0$,

$$\begin{aligned}\gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) &\le -m\gamma^{\alpha+1}(t)E^{\alpha+1}(t) + C\left[\varepsilon\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C_{\varepsilon}E'(t)\right] \\ &= -\left(m-\varepsilon C\right)\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C'E'(t)\end{aligned}$$

We then choose $\varepsilon < \dfrac{m}{C}$ and recall that $\gamma' \le 0$ and $E' \le 0$, to get

$$\left(\gamma^{\alpha+1}E^{\alpha}L\right)'(t) \le \gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) \le -c\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C'E'(t),$$

which implies that

$$\left(\gamma^{\alpha+1}(t)E^{\alpha}(t)L(t) + C'E(t)\right)' \le -c\gamma^{\alpha+1}(t)E^{\alpha+1}(t) \tag{4.1}$$

Let

$$F(t) = \gamma^{\alpha+1}(t)E^{\alpha}(t)L(t) + C'E(t), \tag{4.2}$$

where $F(t) \sim E(t)$. Then

$$F'(t) \le -c\gamma^{\alpha+1}(t)F^{\alpha+1}(t) = -c\gamma^{2p-1}(t)F^{2p-1}(t). \tag{4.3}$$

Integrating over $(0,t)$ and using the fact that $F \sim E$, we obtain

$$E(t) \le C'\left[\frac{1}{\int_0^t \gamma^{2p-1}(s)\,ds + 1}\right]^{\frac{1}{2p-2}}.$$

To establish (3.10), we consider (3.9). Let

$$\eta(t) = \int_0^t \|u(t) - u(t-s)\|_2^2 ds.$$

Assume that $\eta(t) > 0$. Then multiplying (3.7) by $\gamma(t)$, we obtain

$$\begin{aligned}\gamma(t)L'(t) &\le -m\gamma(t)E(t) - c\gamma(t)\int_{\Gamma_1}(k'\circ u)(t)d\Gamma_1 \\ &= -m\gamma(t)E(t) \\ &+ c\frac{\eta(t)}{\eta(t)}\int_{\Gamma_1}\int_0^t\left[\gamma^p(s)(-k')^p(s)\right]^{\frac{1}{p}}\|u(t) - u(t-s)\|_2^2\, dsd\Gamma_1,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\eta(t) = \int_0^t \|u(t) - u(t-s)\|_2^2 \, ds &\leq 2\int_0^t \|u(t)\|_2^2 + \|u(t-s)\|_2^2 \, ds \\
&\leq 2C_\Omega \int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 \, ds \\
&\leq 2C_\Omega \int_0^t [E(t) + E(t-s)] \, ds \\
&\leq 4C_\Omega \int_0^t E(t-s) \, ds = 4C_\Omega \int_0^t E(s) \, ds \\
&< 4C_\Omega \int_0^{+\infty} E(s) \, ds < +\infty.
\end{aligned}$$

Applying Jensen's inequality for the third term of (4.4), with $G(y) = y^{\frac{1}{p}}$, $y > 0$, $f(s) = \gamma^p(s)(-k')^p(s)$ and $h(s) = \|u(t) - u(t-s)\|_2^2$, for $y > 0$ and $s > 0$, we get

$$\begin{aligned}
&\gamma(t) L'(t) \leq -m\gamma(t) E(t) \\
&\qquad + c\eta(t) \left[\frac{1}{\eta(t)} \int_{\Gamma_1} \int_0^t \left[\gamma^p(s)(-k')^p(s)\right] \|u(t) - u(t-s)\|_2^2 \, ds d\Gamma_1 \right]^{\frac{1}{p}}.
\end{aligned}$$

If $\eta(t) = 0$, then previous inequality still has a sense because $p > 1$. By using the fact that γ is nonincreasing, to see that

$$\begin{aligned}
&\gamma(t) L'(t) \leq -m\gamma(t) E(t) \\
&\qquad + c\eta^{\frac{p-1}{p}}(t) \left[\gamma^{p-1}(0) \int_{\Gamma_1} \int_0^t \gamma(s)(-k')^p(s) \|u(t) - u(t-s)\|_2^2 \, ds d\Gamma_1 \right]^{\frac{1}{p}} \\
&\qquad \leq -m\gamma(t) E(t) + C' \left(\int_{\Gamma_1} (k'' \circ u) \, d\Gamma_1 \right)^{\frac{1}{p}} \\
&\qquad \leq -m\gamma(t) E(t) + C' \left(-E'(t)\right)^{\frac{1}{p}}.
\end{aligned}$$

Multiplying by $\gamma^\alpha(t) E^\alpha(t)$, for $\alpha = p - 1$, and repeating the same computations as in above, we arrive at

$$E(t) \leq C' \left[\frac{1}{\int_0^t \gamma^p(s) \, ds + 1} \right]^{\frac{1}{p-1}}.$$

This completes the proof of our main result.

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