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Fekete-Szegö inequality of bi-starlike and bi-convex functions of order b associated with symmetric q-derivative in conic domains

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Abstract. In this paper, two new subclasses of bi-univalent functions related to conic domains are defined by making use of symmetric q-differential operator. The initial bounds for Fekete-Szegö inequality for the functions f in these classes are estimated.

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1. Introduction

Let $\mathscr{A}$ denotes the set of all functions which are analytic in the unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

with Taylor's series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are normalized by $f(0) = 0, f'(0) = 1$. The subclass of $\mathscr{A}$ consisting of all univalent functions is denoted by $\mathscr{S}$. A function $f \in \mathscr{A}$ is said to be a starlike function if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \Delta).$$

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A function $f \in \mathscr{A}$ is said to be a convex function if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \Delta).$$

Goodman [10, 11, 12] introduced the classes uniformly starlike and uniformly convex functions as subclasses of starlike and convex functions. A starlike function (or convex function) is said to be uniformly starlike (or uniformly convex) if the image of every circular arc ζ contained in Δ, with center at ξ also in Δ is starlike (or convex) with respect to $f(\xi)$. The class of uniformly starlike functions is represented by $\mathscr{UST}$ and the class of uniformly convex functions is represented by $\mathscr{UCV}$. The class of parabolic starlike functions is represented by $\mathscr{S}_p$. Rønning [24] and Ma-Minda [18, 19] independently gave the characterization for the classes $\mathscr{S}_p$ and $\mathscr{UCV}$ as follows.
A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}_p$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \Delta).$$

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{UCV}$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \Delta).$$

Also, it is clear that

$$f \in \mathscr{UCV} \Leftrightarrow zf'(z) \in \mathscr{S}_p.$$

Kanas and Wisniowska [16, 15], introduced k-uniformly starlike functions and k-uniformly convex functions as follows.

$$k-\mathscr{ST} = \left\{f:\ f \in \mathscr{S}\ \text{ and }\ \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ \ z \in \Delta,\ \ k \geq 0\right\}$$
$$k-\mathscr{UCV} = \left\{f:\ f \in \mathscr{S}\ \text{ and }\ \Re\left(1+\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \Delta,\ k \geq 0\right\}.$$

Bharati, et al. [8], defined $k-\mathscr{ST}(\beta)$ and $k-\mathscr{UCV}(\beta)$ as follows. A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{ST}(\beta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) - \beta > k\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \Delta). \tag{1.2}$$

A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{UCV}(\beta)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) - \beta > k\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \Delta). \tag{1.3}$$

Sim et al.[26], generalized above classes and introduced $k-\mathscr{ST}(\alpha,\ \beta)$ and $k-\mathscr{UCV}(\alpha,\ \beta)$ as below:
A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{ST}(\alpha,\ \beta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) - \beta > k\left|\frac{zf'(z)}{f(z)} - \alpha\right| \qquad (z \in \Delta), \tag{1.4}$$

where $0 \leq \beta < \alpha \leq 1$ and $k(1-\alpha) < 1-\beta$.

A function $f \in \mathscr{A}$ is said to be in the class $k - \mathscr{UCV}(\alpha,\ \beta)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)-\beta > k\left|1+\frac{zf''(z)}{f'(z)}-\alpha\right| \qquad (z \in \Delta), \tag{1.5}$$

where $0 \le \beta < \alpha \le 1$ and $k(1-\alpha) < 1-\beta$.

In particular, for $\alpha = 1,\ \beta = 0$ the classes $k - \mathscr{ST}(\alpha,\ \beta)$ and $k - \mathscr{UCV}(\alpha,\ \beta)$ reduces to $k - \mathscr{ST}$ and $k - \mathscr{UCV}$ respectively. Further, for $\alpha = 1$ these classes coincides with the classes studied by Nishiwaki and Owa [20] and Shams et al. [25]. In 2017, Annamalai et al. [7], obtained second Hankel determinant of analytic functions involving conic domains.

Now we give the geometric interpretations of the classes $f \in k - \mathscr{ST}(\alpha,\ \beta)$ and $k - \mathscr{UCV}(\alpha,\ \beta)$ as follows:

A function $f \in k - \mathscr{ST}(\alpha,\ \beta)$ and $k - \mathscr{UCV}(\alpha,\ \beta)$ if and only if $\dfrac{zf'(z)}{f(z)}$ and $1+\dfrac{zf''(z)}{f'(z)}$, respectively takes all the values in the conic domain $\Omega_{k,\ \alpha,\ \beta}$

$$\Omega_{k,\ \alpha,\ \beta} = \{\omega :\ \omega \in \mathbb{C} \text{ and } k|\omega-\alpha| < \Re(\omega)-\beta\}$$

or

$$\Omega_{k,\ \alpha,\ \beta} = \left\{\omega :\ \omega \in \mathbb{C} \text{ and } k\sqrt{[\Re(\omega)-\alpha]^2+[\Im(\omega)]^2} < \Re(\omega)-\beta\right\},$$

where $0 \le \beta < \alpha \le 1$ and $k(1-\alpha) < 1-\beta$. Clearly $1 \in \Omega_{k,\ \alpha,\ \beta}$ and $\Omega_{k,\ \alpha,\ \beta}$ is bounded by the curve

$$\partial\Omega_{k,\ \alpha,\ \beta} = \left\{\omega :\ \omega = u+iv \text{ and } k^2(u-\alpha)^2+k^2v^2 = (u-\beta)^2\right\}.$$

The Caratheodory functions $p \in \mathscr{P}$ is said to be in the class $\mathcal{P}(p_{k,\ \alpha,\ \beta})$ if and only if p takes all the values in the conic domain $\Omega_{k,\ \alpha,\ \beta}$. Analytically it is defined as follows:

$$\mathcal{P}(p_{k,\ \alpha,\ \beta}) = \{p :\ p \in \mathscr{P} \text{ and } p(\Delta) \subset \Omega_{k,\ \alpha,\ \beta}\},$$
$$\mathcal{P}(p_{k,\ \alpha,\ \beta}) = \{p :\ p \in \mathscr{P} \text{ and } p(z) \prec p_{k,\ \alpha,\ \beta},\ z \in \Delta\}.$$

It is interesting to note that $\partial\Omega_{k,\ \alpha,\ \beta}$ represents conic section about real axis. In particular, $\Omega_{k,\ \alpha,\ \beta}$ represents an elliptic domain for $k > 1$, parabolic domain for $k = 1$, hyperbolic domain for $0 < k < 1$. Sim et al. [26] obtained the functions $p_{k,\alpha\ \beta}(z)$ which play the role of extremal functions of $\mathcal{P}(p_{k,\ \alpha,\ \beta})$ as

$$p_{k,\alpha\ \beta}(z) = \begin{cases} \dfrac{1+(1-2\beta)z}{1-z}, & \text{for } k = 0 \\ \alpha + \dfrac{2(\alpha-\beta)}{\pi^2}\log^2\left(\dfrac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right), & \text{for } k = 1 \\ \dfrac{\alpha-\beta}{1-k^2}\cosh\left\{\mathfrak{u}(k)\log\left(\dfrac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right)\right\} + \dfrac{\beta-\alpha k^2}{1-k^2}, & \text{for } 0 < k < 1 \\ \dfrac{\alpha-\beta}{k^2-1}\sin^2\left(\dfrac{\pi}{2K(k)}\displaystyle\int_0^{\omega}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-t^2k^2}}\right) + \dfrac{\alpha k^2-\beta}{k^2-1}, & \text{for } k > 1, \end{cases}$$

where $\mathfrak{u}(k) = \dfrac{2}{\pi}\cos^{-1} k$, $u_k(z) = \dfrac{z+\rho_k}{1+\rho_k z}$ and

$$\rho_k = \begin{cases} \left(\dfrac{e^A - 1}{e^A + 1}\right)^2, & \text{for } k = 1 \\ \left(\dfrac{\exp\left(\dfrac{1}{u_k(z)}\operatorname{arc\,cosh} B\right) - 1}{\exp\left(\dfrac{1}{u_k(z)}\operatorname{arc\,cosh} B\right) + 1}\right)^2, & \text{for } 0 < k < 1 \\ \sqrt{k}\sin\left[\dfrac{2K(\kappa)}{\pi}\arcsin C\right], & \text{for } k > 1 \end{cases}$$

with $A = \sqrt{\dfrac{1-\alpha}{2(\alpha-\beta)}}\pi$, $B = \dfrac{1}{\alpha-\beta}(1-k^2-\beta+\alpha k^2)$, $C = \dfrac{1}{\alpha-\beta}(k^2-1+\beta-\alpha k^2)$. Also

$$K(\kappa) = \int_0^{\omega} \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2\kappa^2}} \quad (0<\kappa<1),$$
$$K'(\kappa) = K(\sqrt{1-\kappa^2}) \quad (0<\kappa<1),$$
$$\kappa = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

According to Koebe's $\dfrac{1}{4}$ theorem, every analytic and univalent function f in Δ has an inverse f^{-1} and is defined as

$$f^{-1}(f(z)) = z \quad (z \in \Delta) \text{ and } f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

Also the function f^{-1} can be written as

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.6}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent if both f and analytic extension of f^{-1} in Δ are univalent in Δ. The class of all bi-univalent functions is denoted by Σ. That is a function f is said to be bi-univalent if and only if

1. f is an analytic and univalent function in Δ.
2. There exists an analytic and univalent function g in Δ such that $f(g(z)) = g(f(z)) = z$ in Δ.

The class of bi-univalent functions was introduced by Lewin [17] in 1967. Recently many researchers [1, 2, 4, 3, 14, 21, 22, 23, 28, 29, 30, 31, 33, 32, 34, 35] have introduced and investigated several interesting subclasses of the bi-univalent functions and they have found non-sharp estimates of two Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$, Fekete-Szegö inequalities and second Hankel determinants. In 2017, Altinkaya and Yalçin [5, 6] estimated the coefficients and Fekete-Szegö inequalities for some subclasses of bi-univalent functions involving symmetric q-derivative operator subordinate to the generating function of Chebyshev polynomials.

Jackson [13], defined $q-$derivative operator D_q of an analytic function f of the form (1.1)as follows:

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & for\ z \neq 0, \\ f'(0), & for\ z = 0 \end{cases}$$

$$D_q f(0) = f'(0) \text{ and } D_q^2 = D_q(D_q f(z)).$$

If $f(z) = z^n$ for any positive integer n, the q-derivative of $f(z)$ is defined by

$$D_q z^n = \frac{(qz)^n - z^n}{qz - z} = [n]_q z^{n-1},$$

where $[n]_q = \dfrac{q^n - 1}{q - 1}$. As $q \to 1^-$ and $k \in \mathbb{N}$, we have $[n]_q \to n$ and

$$\lim_{q \to 1} (D_q f(z)) = f'(z)$$

where f' is normal derivative of f. Therefore

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Brahim and Sidomou [9], defined the symmetric $q-$derivative operator $\widetilde{D}_q$ of an analytic function f of the form (1.1) as follows:

$$(\widetilde{D}_q f)(z) = \begin{cases} \dfrac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0 \end{cases}.$$

It is clear that $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$ and $\widetilde{D_q} f(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}$, where

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The relation between q-derivative operator and symmetric q-derivative operator is given by

$$(\widetilde{D_q} f)(z) = D_{q^2} f(q^{-1}z).$$

If g is the inverse of f then

$$\begin{aligned} (\widetilde{D_q} g)(w) &= \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w} \\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q (2a_2^2 - a_3) w^2 - \widetilde{[4]}_q (5a_2^3 - 5a_2 a_3 + a_4) w^3 + \cdots . \end{aligned}$$

One could refer [27], for more details of $q-$ calculus and fractional $q-$calculus and their applications in Geometric Function Theory.

Motivated by the above mentioned work, in this paper, bi-starlike functions of order b and bi-convex functions of order b involving q-derivative operator subordinate to the conic domains are defined and the Fekete-Szegö inequality for the function in these classes are obtained.

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$, where $0 \leq \beta < \alpha \leq 1$ and $k(1-\alpha) < 1-\beta$ and b is a non-zero complex number, if it satisfies the following conditions:

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) \prec p_{k,\, \alpha,\, \beta}(z) \qquad (z \in \Delta) \tag{1.7}$$

and for $g = f^{-1}$

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) \prec p_{k,\, \alpha,\, \beta}(w) \qquad (w \in \Delta). \tag{1.8}$$

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $k - \mathscr{U}\mathscr{C}\mathscr{V}_{\Sigma,\, b}(\alpha,\ \beta)$; where $0 \leq \beta < \alpha \leq 1$ and $k(1-\alpha) < 1-\beta$, and b is a non-zero complex number, if it satisfies the following conditions:

$$1 + \frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))} - 1\right) \prec p_{k,\, \alpha,\, \beta}(z) \qquad (z \in \Delta) \tag{1.9}$$

and for $g = f^{-1}$

$$1 + \frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))} - 1\right) \prec p_{k,\, \alpha,\, \beta}(w) \qquad (w \in \Delta). \tag{1.10}$$

2. Main results

In this section, initial estimates $|a_2|$, $|a_3|$ and Fekete-Szegö inequalities for the functions f in the classes $k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and $k - \mathscr{U}\mathscr{C}\mathscr{V}_{\Sigma,\, b}(\alpha,\ \beta)$ are obtained.

Theorem 2.1. *If $f \in k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and is of the form* (1.1) *then*

$$|a_2| \leq \frac{|P_1|\sqrt{|P_1|}b^2}{\sqrt{|P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + 2(P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2|}},$$

$$|a_3| \leq \frac{b^2 P_1^2}{\left(\widetilde{[2]}_q - 1\right)^2} + \frac{\left|bP_1\right|}{\widetilde{[3]}_q - 1}$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|P_1 b|}{\widetilde{[3]}_q - 1}, & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \dfrac{|P_1 b|\ |s(\mu)|}{\widetilde{[3]}_q - 1} & \text{if } |s(\mu)| \geq 1, \end{cases}$$

where

$$s(\mu) = \frac{P_1^2 b(1-\mu)}{[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2]}.$$

Proof. Let $f \in k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and g be an analytic extension of f^{-1} in Δ. Then there exist two Schwarz functions $u,\ v \in \Delta$ such that

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) = p_{k,\,\alpha,\,\beta}(u(z)), \tag{2.1}$$

and

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) = p_{k,\,\alpha,\,\beta}(v(w)). \tag{2.2}$$

Define two functions $h,\ q \in \mathscr{P}$ such that

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots .$$

Then

$$\begin{aligned} p_{k,\,\alpha,\,\beta}\left(\frac{h(z)-1}{h(z)+1}\right) =& 1 + \frac{P_1 h_1 z}{2} + \left(\frac{P_1}{2}(h_2 - \frac{h_1^2}{2}) + \frac{P_2 h_1^2}{4}\right) z^2 \\ &+ \left(\frac{P_1}{2}\left(\frac{h_1^3}{4} - h_1 h_2 + h_3\right) + \frac{P_2}{4}(2h_1h_2 - h_1^3) + \frac{P_3}{8}h_1^3\right) z^3 + \cdots \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} p_{k,\,\alpha,\,\beta}\left(\frac{q(w)-1}{q(w)+1}\right) =& 1 + \frac{P_1 q_1 w}{2} + \left(\frac{P_1}{2}(q_2 - \frac{q_1^2}{2}) + \frac{P_2 q_1^2}{4}\right) w^2 \\ &+ \left(\frac{P_1}{2}\left(\frac{q_1^3}{4} - q_1 q_2 + q_3\right) + \frac{P_2}{4}(2q_1q_2 - q_1^3) + \frac{P_3}{8}q_1^3\right) w^3 + \cdots . \end{aligned} \tag{2.4}$$

In view of (2.3) and (2.4), the equations (2.1) and (2.2) become

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) = p_{k,\,\alpha,\,\beta}\left(\frac{h(z)-1}{h(z)+1}\right) \tag{2.5}$$

and

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) = p_{k,\,\alpha,\,\beta}\left(\frac{v(w)-1}{v(w)+1}\right). \tag{2.6}$$

Comparing the coefficients of like powers of z in the equations (2.7) and (2.8), we get

$$\frac{1}{b}\left(\widetilde{[2]}_q - 1\right) a_2 = \frac{P_1 h_1}{2}, \tag{2.7}$$

$$\frac{1}{b}\left[\left(\widetilde{[3]}_q - 1\right) a_3 - \left(\widetilde{[2]}_q - 1\right) a_2^2\right] = \frac{P_1}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{P_2 h_1^2}{4}, \tag{2.8}$$

and

$$\frac{-1}{b}\left(\widetilde{[2]}_q - 1\right) a_2 = \frac{P_1 q_1}{2}, \tag{2.9}$$

$$\frac{1}{b}\left[\left(\widetilde{[3]}_q - 1\right)(2a_2^2 - a_3) - \left(\widetilde{[2]}_q - 1\right) a_2^2\right] = \frac{P_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{P_2 q_1^2}{4}. \tag{2.10}$$

From the equations (2.7) and (2.9)

$$h_1 = -q_1. \tag{2.11}$$

Now, squaring and adding the equations (2.7) from (2.9), we get

$$h_1^2 + q_1^2 = \frac{8\left(\widetilde{[2]}_q - 1\right)^2 a_2^2}{P_1^2 b^2}. \tag{2.12}$$

Next, adding (2.8) and (2.10), use the equation (2.12), one can get

$$a_2^2 = \frac{P_1^3 (h_2 + q_2) b^2}{4\left[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2\right]}. \tag{2.13}$$

Subtract the equation (2.10) from (2.8),

$$a_3 = a_2^2 + \frac{bP_1(h_2 - q_2)}{4\left(\widetilde{[3]}_q - 1\right)}. \tag{2.14}$$

Then using the equation (2.12), we get

$$a_3 = \frac{P_1^2 b^2 (h_1^2 + q_1^2)}{8\left(\widetilde{[2]}_q - 1\right)^2} + \frac{bP_1(h_2 - q_2)}{4\left(\widetilde{[3]}_q - 1\right)}. \tag{2.15}$$

Using the equations (2.13) and (2.14), we get

$$a_3 - \mu a_2^2 = \frac{bP_1}{4\left(\widetilde{[3]}_q - 1\right)}\left[h_2(1 + s(\mu)) + q_2(-1 + s(\mu))\right], \tag{2.16}$$

where

$$s(\mu) = \frac{P_1^2 b(1-\mu)}{\left[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2\right]}.$$

By applying the modulus for the equations (2.13), (2.15) and (2.16), we get the required results. □

Theorem 2.2. *If $f \in k - \mathscr{UCV}_{\Sigma,\, b}(\alpha,\ \beta)$ and is of the form (1.1), then*

$$|a_2| \leq \frac{|P_1|\,|b|\,\sqrt{|P_1|}}{\sqrt{\left|\left(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\right)bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)\right|}}$$

$$|a_3| \leq \frac{P_1^2 b^2}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2}+\frac{\left|bP_1\right|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}$$

and

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|P_1 b|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}, & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \dfrac{|P_1 b s(\mu)|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)} & \text{if } |s(\mu)| \geq 1, \end{cases}$$

where

$$s(\mu)=\frac{P_1^2 b(1-\mu)}{4\left[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right))bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)\right]}.$$

Proof. If $f \in k - \mathscr{UCV}_{\Sigma,\, b}(\alpha,\ \beta)$ and g is an analytic extension of f^{-1} in Δ, then there exist two Schwarz functions $u,\ v \in \Delta$ such that

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))}-1\right)=p_{k,\,\alpha,\,\beta}(u(z)), \tag{2.17}$$

and

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))}-1\right)=p_{k,\,\alpha,\,\beta}(v(w)). \tag{2.18}$$

Then in view of (2.3) and (2.4) the equations (2.17) and (2.18) reduces to

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))}-1\right)=p_{k,\,\alpha,\,\beta}\left(\frac{h(z)-1}{h(z)+1}\right), \tag{2.19}$$

and

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))}-1\right)=p_{k,\,\alpha,\,\beta}\left(\frac{v(w)-1}{v(w)+1}\right). \tag{2.20}$$

Comparing the coefficients of similar powers of z in equations (2.19) and (2.20)

$$\frac{1}{b}\widetilde{[2]}_q\left(\widetilde{[2]}_q-1\right)a_2=\frac{P_1h_1}{2}, \tag{2.21}$$

$$\frac{1}{b}\left[\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)a_3-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)a_2^2\right]=\frac{P_1}{2}(h_2-\frac{h_1^2}{2})+\frac{P_2h_1^2}{4}, \tag{2.22}$$

and

$$\frac{-1}{b}\widetilde{[2]}_q\left(\widetilde{[2]}_q-1\right)a_2=\frac{P_1q_1}{2}, \tag{2.23}$$

$$\frac{1}{b}(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)(2a_2^2-a_3)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)a_2^2)=\frac{P_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{P_2q_1^2}{4}. \tag{2.24}$$

From the equations (2.21) and (2.23), we get

$$h_1=-q_1. \tag{2.25}$$

Squaring and adding the equations (2.21) from (2.23), we get

$$h_1^2+q_1^2=\frac{8(\widetilde{[2]}_q)^2\left(\widetilde{[2]}_q-1\right)^2a_2^2}{P_1^2b^2}. \tag{2.26}$$

Adding (2.22) and (2.24), and using the equation (2.26), one can get

$$a_2^2=\frac{P_1^3(h_2+q_2)b^2}{4[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right))bP_1^2+(\widetilde{[2]}_q)^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)]}. \tag{2.27}$$

Subtracting the equation (2.24) from (2.22), we get

$$a_3=a_2^2+\frac{bP_1(h_2-q_2)}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{2.28}$$

Using the equation (2.26), we obtain

$$a_3=\frac{P_1^2b^2(h_1^2+q_1^2)}{8\widetilde{[2]}_q^2\left(\widetilde{[3]}_q-1\right)\left(\widetilde{[2]}_q-1\right)^2}+\frac{bP_1(h_2-q_2)}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{2.29}$$

Then using the equations (2.27) and (2.28), we get

$$a_3-\mu a_2^2=\frac{bP_1}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\Big[h_2(1+s(\mu))+q_2(-1+s(\mu))\Big], \tag{2.30}$$

where

$$s(\mu)=\frac{bP_1^2(1-\mu)}{4[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)]}.$$

By applying modulus for the equations (2.27), (2.29) and (2.30) on both sides we get the required results. □

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