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Some applications of Maia's fixed point theorem for Fredholm integral equation systems

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Abstract. The aim of this paper is to study the existence and uniqueness of solutions for some Fredholm integral equation systems by applying the vectorial form of Maia's fixed point theorem. Some abstract Gronwall lemmas and an abstract comparison lemma are also obtained.

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1. Introduction

Let $a, b \in \mathbb{R}_+$, with $a < b$. Let $C[a,b]$ be the set of all real valued functions which are continuous on the interval $[a,b]$. Using a vectorial form of Maia's fixed point theorem, we study the existence and uniqueness of solutions $(x_1, x_2) \in (C[a,b])^2$ for the following Fredholm integral equation systems:

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases} \tag{1.1}$$

and

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases} \tag{1.2}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ are given functions.

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2. Preliminaries

We recall here some notions, notations and results which will be used in the sequel of this paper.

2.1. L-space

The notion of L-space was introduced in 1906 by M. Fréchet ([4]). It is an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method.

Let X be a nonempty set. Let $s(X) := \big\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

- (i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$.
- (ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, we have that $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

A simple example of an L-space is the pair $(X, \stackrel{d}{\to})$, where X is a nonempty set and $\stackrel{d}{\to}$ is the convergence structure induced by a metric d on X.

In general, an L-space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L-spaces are: Hausdorff topological spaces, generalized metric spaces in Perov' sense (i.e. $d(x,y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e. $d(x,y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces.

2.2. Picard operators and weakly Picard operators on L-spaces

Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f.

If an operator f is WPO and the fixed point set of f is a singleton, $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

For a WPO, $f : X \to X$, we define the operator $f^\infty : X \to X$, by

$$f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f.

If X is a nonempty set, then the triple $(X, \to, \leq)$ is an ordered L-space if $(X, \to)$ is an L-space and $\leq$ is a partial order relation on X which is closed with respect to the convergence structure of the L-space.

In the setting of ordered L-spaces, we have some properties concerning WPOs and POs.

Theorem 2.2.1 (Abstract Gronwall Lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an increasing WPO. Then:*

- (i) $x \in X,\ x \leq f(x) \Rightarrow x \leq f^\infty(x)$;
- (ii) $x \in X,\ x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

In particular, if f is a PO and we denote $F_f = \{x^*\}$, then:

(i') $\forall\ x \in X,\ x \le f(x) \Rightarrow x \le x^*$;
(ii') $\forall\ x \in X,\ x \ge f(x) \Rightarrow x \ge x^*$.

Theorem 2.2.2 (Abstract Comparison Lemma). *Let $(X, \to, \le)$ be an ordered L-space and the operators $f, g, h : X \to X$ be such that:*

(1) $f \le g \le h$;
(2) f, g, h *are* $WPOs$;
(3) g *is increasing.*

Then:

$$x, y, z \in X,\ x \le y \le z \Rightarrow f^\infty(x) \le g^\infty(y) \le h^\infty(z).$$

In particular, if f, g, h are POs and we denote $F_f = \{x^*\}$, $F_g = \{y^*\}$, $F_h = \{z^*\}$, then

$$\forall\ x, y, z \in X,\ x \le y \le z \Rightarrow x^* \le y^* \le z^*.$$

Regarding the theory of $WPOs$ and POs see [12], [13], [15], [16], [18], [11], [17], [3].

2.3. Maia's fixed point theorem

The following result was proved by M.G. Maia in [5].

Theorem 2.3.1. *Let X be a nonempty set, d and ρ be two metrics on X and $V : X \to X$ be an operator. We suppose that:*

(1) *there exists $c > 0$ such that, $d(x, y) \le c\rho(x, y)$, $\forall\ x, y \in X$;*
(2) *(X, d) is a complete metric space;*
(3) *$V : (X, d) \to (X, d)$ is continuous;*
(4) *$V : (X, \rho) \to (X, \rho)$ is an l-contraction, i.e.,*

$$\exists\ l \in [0, 1) \text{ such that } \rho(V(x), V(y)) \le l\rho(x, y),\ \forall\ x, y \in X.$$

Then:

(i) $F_V = \{x^*\}$;
(ii) $V : (X, d) \to (X, d)$ *is* PO.

Maia's Theorem 2.3.1 remains true if we replace the condition (1) with the following one:

$(1')$ there exists $c > 0$ such that, $d(V(x), V(y)) \le c\rho(x, y)$, $\forall\ x, y \in X$.

Hence, we obtain the so called Rus' variant of Maia's fixed point theorem. More considerations can be found in [11], [9], [10], [14].

2.4. Matrices which converge to zero

We denote by $M_m(\mathbb{R}_+)$ the set of all $m \times m$ square matrices with positive real elements, by I_m the identity $m \times m$ matrix and by O_m the zero $m \times m$ matrix.

$A \in M_m(\mathbb{R}_+)$ is said to be convergent to zero if $A^n \to O_m$ as $n \to \infty$.

Some examples of matrices that converge to zero are the following:

a) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2(\mathbb{R}_+)$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

A classical result in matrix analysis is the following theorem (see [19], [1]), which characterizes the matrices that converge to zero.

Theorem 2.4.1. *Let $A \in M_m(\mathbb{R}_+)$. The following assertions are equivalent:*

(1) *A is convergent to zero;*
(2) *its spectral radius $\rho(A)$ is strictly less than 1; that is, $|\lambda| < 1$, for any $\lambda \in \mathbb{C}$ with $\det(A - \lambda I_m) = 0$;*
(3) *the matrix $(I_m - A)$ is nonsingular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

(4) *the matrix $(I_m - A)$ is nonsingular and $(I_m - A)^{-1}$ has nonnegative elements.*

Throughout this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

2.5. Vector-valued metric spaces

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m_+$ is called a vector-valued metric on X if the following conditions are satisfied:

(1) $d(x, y) = 0 \in \mathbb{R}^m \Leftrightarrow x = y$, for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

On $\mathbb{R}^m_+$, the relation $\leq$ is defined in the component-wise sense.

Some examples of vector-valued metrics are the following:

Example 2.5.1. Let $X := (C[a, b])^2$ and $d : (C[a, b])^2 \times (C[a, b])^2 \to \mathbb{R}^2_+$, defined by

$$d(x, y) := \left(\max_{t \in [a,b]} |x_1(t) - y_1(t)|, \max_{t \in [a,b]} |x_2(t) - y_2(t)| \right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a, b])^2$.

Example 2.5.2. Let $X := (C[a, b])^2$ and $\rho : (C[a, b])^2 \times (C[a, b])^2 \to \mathbb{R}^2_+$, defined by

$$\rho(x, y) := \left(\left(\int_a^b |x_1(t) - y_1(t)|^2 dt \right)^{\frac{1}{2}}, \left(\int_a^b |x_2(t) - y_2(t)|^2 dt \right)^{\frac{1}{2}} \right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a, b])^2$.

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space in Perov' sense (or a $\mathbb{R}^m_+$-metric space) and it is denoted by the pair (X, d). The notions of convergent sequence, Cauchy sequence, completeness, open and closed subset and so forth are similar to those defined for usual metric spaces. The basic fixed point result which holds in generalized metric spaces in Perov' sense is the following (see [6], [7]).

Theorem 2.5.3 (Perov's fixed point theorem). *Let (X, d) be a complete generalized metric space, where $d : X \times X \to \mathbb{R}^m_+$. Let $f : X \to X$ be an A-contraction, i.e. there exists a matrix $A \in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$d(f(x), f(y)) \leq Ad(x, y), \forall\ x, y \in X.$$

Then f is PO in the L-space $(X, \xrightarrow{d})$.

Remark 2.5.4. It would be of interest to extend the study from [8] and [2] to the case of vector-valued metric spaces.

3. Vectorial Maia's fixed point theorems

In this section we present the Rus' variant of Maia's fixed point theorem in the setting of generalized metric spaces in Perov's sense.

Theorem 3.1. *Let X be a nonempty set, endowed with two vector-valued metrics, $d, \rho : X \times X \to \mathbb{R}^m_+$. Let $T : X \to X$ be an operator. We assume that:*

(1) *there exists a matrix $C \in M_m(\mathbb{R}_+)$ such that*

$$d(T(x), T(y)) \leq C\rho(x, y),\ \forall\ x, y \in X;$$

(2) *(X, d) is a complete generalized metric space;*
(3) *$T : (X, d) \to (X, d)$ is continuous;*
(4) *$T : (X, \rho) \to (X, \rho)$ is an A-contraction, i.e. there exists a matrix $A \in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$\rho(T(x), T(y)) \leq A\rho(x, y),\ \forall\ x, y \in X.$$

Then T is PO in the L-spaces $(X, \xrightarrow{d})$ and $(X, \xrightarrow{\rho})$.

Proof. Let $x_0 \in X$. By (4), the sequence of successive approximations $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ). Indeed, for $n, p \in \mathbb{N}$ we have

$$\begin{aligned}\rho(T^n(x_0), T^{n+p}(x_0)) &\leq \sum_{k=n}^{n+p-1} \rho(T^k(x_0), T^{k+1}(x_0)) \leq \sum_{k=n}^{n+p-1} A^k \rho(x_0, T(x_0)) \\ &\leq A^n(I_m - A)^{-1}\rho(x_0, T(x_0)) \to 0 \text{ as } n, p \to \infty.\end{aligned}$$

By (1), we get that $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). By (2), there exists $x^* \in X$, such that $T^n(x_0) \overset{d}{\to} x^*$ as $n\to\infty$. By (3), it follows that $x^*\in F_T$, since

$$\begin{aligned} d(x^*,T(x^*)) &\le d(x^*,T^n(x_0)) + d(T^n(x_0),T(x^*)) \\ &= d(x^*,T^n(x_0)) + d(T(T^{n-1}(x_0)),T(x^*)) \\ &\to d(x^*,x^*) + d(T(x^*),T(x^*)) = 0, \text{ as } n\to\infty. \end{aligned}$$

By (4), we obtain the uniqueness of the fixed point x^*. Hence T is PO in $(X,\overset{d}{\to})$.
We show next that T is PO in $(X,\overset{\rho}{\to})$.
For any $x_0\in X$, since $x^*\in F_T$, by (4) we have

$$\rho(x^*,T^n(x_0)) = \rho(T^n(x^*),T^n(x_0)) \le A^n\rho(x^*,x_0) \to 0 \text{ as } n\to\infty$$

which implies that $T^n(x_0)\overset{\rho}{\to} x^*$ as $n\to\infty$. Since x^* is the unique fixed point, we get that T is PO in $(X,\overset{\rho}{\to})$. □

Remark 3.2. Notice that, in the proof of the above result, Perov's Theorem cannot be applied for $T:(X,\rho)\to(X,\rho)$, because the lack of completeness of the generalized metric space (X,ρ).

Remark 3.3. From the proof of the above result, we can deduce the following weak Perov's contraction principle:

Theorem 3.4. *Let (X,ρ) be a generalized metric space, where $\rho: X\times X\to\mathbb{R}^m_+$. Let $T:X\to X$ be an operator. We assume that:*

(i) $F_T\neq\emptyset$;

(ii) there exists a matrix $A\in M_m(\mathbb{R}_+)$ which converges to zero, such that $\rho(T(x),T(y))\le A\rho(x,y)$, for all $x,y\in X$.

Then T is PO in the L-space $(X,\overset{\rho}{\to})$.

Another fixed point result of Maia type in vectorial form is the following.

Theorem 3.5. *Let X be a nonempty set, endowed with two vector-valued metrics, $d,\rho: X\times X\to\mathbb{R}^m_+$. Let $T:X\to X$ be an operator. We assume that:*

(1) *$F_T\neq\emptyset$;*

(2) *there exists a matrix $C\in M_m(\mathbb{R}_+)$ such that*

$$d(T(x),T(y)) \le C\rho(x,y),\ \forall\ x,y\in X;$$

(3) *$T:(X,\rho)\to(X,\rho)$ is an A-contraction, i.e. there exists a matrix $A\in M_m(\mathbb{R}_+)$ convergent to zero, such that*

$$\rho(T(x),T(y)) \le A\rho(x,y),\ \forall\ x,y\in X.$$

Then T is PO in the L-spaces $(X,\overset{d}{\to})$ and $(X,\overset{\rho}{\to})$.

Proof. By applying Theorem 3.4, T is PO in $(X,\overset{\rho}{\to})$. So $F_T=\{x^*\}$. For any $x_0\in X$,

$$\begin{aligned} d(x^*,T^{n+1}(x_0)) &= d(T^{n+1}(x^*),T^{n+1}(x_0)) \\ &\le C\rho(T^n(x^*),T^n(x_0)) \\ &\le CA^n\rho(x^*,x_0)\to 0 \end{aligned}$$

as $n \to \infty$. So T is PO in $(X, \xrightarrow{d})$. $\square$

4. Applications of vectorial Maia's fixed point theorem

In this section we study the existence and uniqueness of solutions for Fredholm integral equations systems (1.1) and (1.2), by applying the vectorial Maia's fixed point theorem.

First, let us consider the system (1.1)

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$, are given functions. We are searching the conditions in which the system (1.1) has a unique solution $(x_1, x_2) \in (C[a,b])^2$.

We assume that there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u - v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u - v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$.

On $X := (C[a,b])^2$ we consider the metrics $d, \rho : X \times X \to \mathbb{R}^2_+$, where

$$d(x,y) := \begin{pmatrix} \max\limits_{t \in [a,b]} |x_1(t) - y_1(t)| \\ \max\limits_{t \in [a,b]} |x_2(t) - y_2(t)| \end{pmatrix} \tag{4.1}$$

and

$$\rho(x,y) := \begin{pmatrix} \left(\int_a^b |x_1(t) - y_1(t)|^2 dt\right)^{\frac{1}{2}} \\ \left(\int_a^b |x_2(t) - y_2(t)|^2 dt\right)^{\frac{1}{2}} \end{pmatrix}, \tag{4.2}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in (C[a,b])^2$.

We consider the operator $T : (C[a,b])^2 \to (C[a,b])^2$, defined by

$$\begin{aligned} T(x)(t) &= \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \end{pmatrix} \\ &:= \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix} \end{aligned} \tag{4.3}$$

for all $x = (x_1, x_2) \in (C[a,b])^2$.

We have,

$$\rho(T(x), T(y)) = \begin{pmatrix} \left(\int_a^b |T_1(x)(t) - T_1(y)(t)|^2 dt\right)^{\frac{1}{2}} \\ \left(\int_a^b |T_2(x)(t) - T_2(y)(t)|^2 dt\right)^{\frac{1}{2}} \end{pmatrix}$$

and

$$\begin{aligned}
\begin{pmatrix}|T_1(x)(t)-T_1(y)(t)|\\|T_2(x)(t)-T_2(y)(t)|\end{pmatrix} &\le \begin{pmatrix}\int_a^b \big|K_1(t,s,x_1(s))-K_1(t,s,y_1(s))\big|ds\\ \int_a^b \big|K_2(t,s,x_2(s))-K_2(t,s,y_2(s))\big|ds\end{pmatrix}\\
&+\begin{pmatrix}\int_a^b \big|H_1(t,s,x_1(s))-H_1(t,s,y_1(s))\big|ds\\ \int_a^b \big|H_2(t,s,x_2(s))-H_2(t,s,y_2(s))\big|ds\end{pmatrix}\\
&\le \begin{pmatrix}\int_a^b L_{K_1}|x_1(s)-y_1(s)|ds\\ \int_a^b L_{K_2}|x_2(s)-y_2(s)|ds\end{pmatrix}+\begin{pmatrix}\int_a^b L_{H_1}|x_1(s)-y_1(s)|ds\\ \int_a^b L_{H_2}|x_2(s)-y_2(s)|ds\end{pmatrix}\\
&\overset{\substack{\text{Hölder's}\\\text{inequality}}}{\le} \begin{pmatrix}\big[\big(\int_a^b|L_{K_1}|^2ds\big)^{\frac12}+\big(\int_a^b|L_{H_1}|^2ds\big)^{\frac12}\big]\big(\int_a^b|x_1(s)-y_1(s)|^2ds\big)^{\frac12}\\ \big[\big(\int_a^b|L_{K_2}|^2ds\big)^{\frac12}+\big(\int_a^b|L_{H_2}|^2ds\big)^{\frac12}\big]\big(\int_a^b|x_2(s)-y_2(s)|^2ds\big)^{\frac12}\end{pmatrix}\\
&=\begin{pmatrix}\big(L_{K_1}+L_{H_1}\big)\sqrt{b-a}\,\tilde\rho(x_1,y_1)\\ \big(L_{K_2}+L_{H_2}\big)\sqrt{b-a}\,\tilde\rho(x_2,y_2)\end{pmatrix},
\end{aligned}$$

where

$$\tilde\rho(x_1,y_1):=\left(\int_a^b|x_1(s)-y_1(s)|^2ds\right)^{\frac12},\ \tilde\rho(x_2,y_2):=\left(\int_a^b|x_2(s)-y_2(s)|^2ds\right)^{\frac12}.$$

Hence,

$$\begin{aligned}
\rho(T(x),T(y)) &\le \begin{pmatrix}\big(\int_a^b[(L_{K_1}+L_{H_1})\sqrt{b-a}\,\tilde\rho(x_1,y_1)]^2dt\big)^{\frac12}\\ \big(\int_a^b[(L_{K_2}+L_{H_2})\sqrt{b-a}\,\tilde\rho(x_2,y_2)]^2dt\big)^{\frac12}\end{pmatrix}\\
&=\begin{pmatrix}(L_{K_1}+L_{H_1})(b-a)\tilde\rho(x_1,y_1)\\ (L_{K_2}+L_{H_2})(b-a)\tilde\rho(x_2,y_2)\end{pmatrix}=A\rho(x,y),
\end{aligned}$$

where

$$A:=\begin{pmatrix}(L_{K_1}+L_{H_1})(b-a) & 0\\ 0 & (L_{K_2}+L_{H_2})(b-a)\end{pmatrix}\in M_2(\mathbb{R}_+)$$

is a matrix that converges to zero if $(L_{K_1}+L_{H_1})(b-a)<1$ and $(L_{K_2}+L_{H_2})(b-a)<1$. So, if we add these two conditions, T becomes an A-contraction with respect to ρ. In addition, for all $x,y\in C[a,b]$, we have $d(T(x),T(y))\le C\rho(x,y)$, where

$$C:=\begin{pmatrix}(L_{K_1}+L_{H_1})\sqrt{b-a} & 0\\ 0 & (L_{K_2}+L_{H_2})\sqrt{b-a}\end{pmatrix}\in M_2(\mathbb{R}_+).$$

By applying Theorem 3.1, the system (1.1) has a unique solution in $(C[a,b])^2$. Hence, we have obtained the following result:

Theorem 4.1. *Let $a,b\in\mathbb{R}_+$ with $a<b$. We consider the system of Fredholm integral equations*

$$\begin{cases}x_1(t)=g_1(t)+\int_a^b K_1(t,s,x_1(s))ds+\int_a^b H_1(t,s,x_1(s))ds\\ x_2(t)=g_2(t)+\int_a^b K_2(t,s,x_2(s))ds+\int_a^b H_2(t,s,x_2(s))ds\end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(*i*) *there exist* $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(L_{K_1} + L_{H_1})(b-a) < 1$ *and* $(L_{K_2} + L_{H_2})(b-a) < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Remark 4.2. By Theorem 3.1, the operator T defined in (4.3) is PO. Hence, for all $t \in [a,b]$ we have $x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 = (x_0^1, x_0^2) \in (C[a,b])^2$, where $\{x_n\}_{n\in\mathbb{N}} \subset (C[a,b])^2$ is defined by

$$x_{n+1}(t) = \begin{pmatrix} x_{n+1}^1(t) \\ x_{n+1}^2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_n^1(s))ds + \int_a^b H_1(t,s,x_n^1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_n^2(s))ds + \int_a^b H_2(t,s,x_n^2(s))ds \end{pmatrix}.$$

Corollary 4.3. *Let* $a, b \in \mathbb{R}_+$ *with* $a < b$. *We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(*i*) *there exist* $L_{K_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $L_{K_1}(b-a) < 1$ *and* $L_{K_2}(b-a) < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Proof. We apply Theorem 4.1, by considering H_1 and H_2 as zero functions and by taking $L_{H_1} = 0$ and $L_{H_2} = 0$. ☐

Now, let us consider the system (1.2)

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ are given functions.

We assume that there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$.

On $X := (C[a,b])^2$ we consider the metrics $d, \rho : X \times X \to \mathbb{R}^2_+$ defined as in (4.1) and (4.2). Also, we consider the operator $T : (C[a,b])^2 \to (C[a,b])^2$, defined by

$$\begin{aligned} T(x)(t) &= \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \end{pmatrix} \\ &:= \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix} \end{aligned} \tag{4.4}$$

for all $x = (x_1, x_2) \in (C[a,b])^2$.
In a similar manner as shown for the system (1.1), we get $\rho(T(x), T(y)) \le A\rho(x,y)$, for all $x, y \in (C[a,b])^2$, where

$$A := \begin{pmatrix} L_{K_1}(b-a) & L_{H_1}(b-a) \\ L_{H_2}(b-a) & L_{K_2}(b-a) \end{pmatrix} \in M_2(\mathbb{R}_+).$$

The matrix A converges to zero if

$$\frac{\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right|}{2}(b-a) < 1.$$

So, if we add this condition, T becomes an A-contraction with respect to ρ.
In addition, for all $x, y \in (C[a,b])^2$, we obtain $d(T(x), T(y)) \le C\rho(x,y)$, where

$$C := \begin{pmatrix} L_{K_1}\sqrt{b-a} & L_{H_1}\sqrt{b-a} \\ L_{H_2}\sqrt{b-a} & L_{K_2}\sqrt{b-a} \end{pmatrix} \in M_2(\mathbb{R}_+).$$

By applying Theorem 3.1, the system (1.2) has a unique solution in $(C[a,b])^2$. Hence, we have obtained the following result:

Theorem 4.4. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ are given functions.

We assume that:

(i) there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:

$$\begin{aligned} |K_j(t,s,u) - K_j(t,s,v)| &\le L_{K_j}|u-v|, \\ |H_j(t,s,u) - H_j(t,s,v)| &\le L_{H_j}|u-v|, \end{aligned}$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j \in \{1,2\}$;

(ii) $\frac{b-a}{2}\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right| < 1.$

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Remark 4.5. By Theorem 3.1, the operator T defined in (4.4) is PO. Hence, for all $t \in [a,b]$ we have $x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 = (x_0^1, x_0^2) \in (C[a,b])^2$, where $\{x_n\}_{n\in\mathbb{N}} \subset (C[a,b])^2$ is defined by

$$x_{n+1}(t) = \begin{pmatrix} x_{n+1}^1(t) \\ x_{n+1}^2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_n^1(s))ds + \int_a^b H_1(t,s,x_n^2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_n^2(s))ds + \int_a^b H_2(t,s,x_n^1(s))ds \end{pmatrix}.$$

Corollary 4.6. *Let $a,b \in \mathbb{R}_+$ with $a<b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $H_1, H_2 \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ are given functions.

We assume that:

(*i*) *there exist $L_{H_j} > 0$, $j \in \{1,2\}$ such that:*

$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(b-a)\sqrt{L_{H_1}L_{H_2}} < 1$.

Then the system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$.

Proof. We apply Theorem 4.4, by considering K_1 and K_2 as zero functions and by taking $L_{K_1} = 0$ and $L_{K_2} = 0$. □

5. Abstract Gronwall lemmas

Since the operators T, defined in (4.3) and (4.4), are POs, by using Theorem 2.2.1 we can establish the following abstract Gronwall lemmas for our systems (1.1) and (1.2).

Theorem 5.1. *Let $a,b \in \mathbb{R}_+$ with $a<b$. We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ are given functions.

We assume that:

(*i*) *there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $(L_{K_1} + L_{H_1})(b-a) < 1$ *and* $(L_{K_2} + L_{H_2})(b-a) < 1$;

(*iii*) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *are increasing functions, for all* $t,s \in [a,b]$ *and* $j \in \{1,2\}$.

Let $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$ *be the unique solution of the system.*
Then the following implications hold:

(1) *for all* $x = (x_1,x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \leq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \leq x^*$;

(2) *for all* $x = (x_1,x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \geq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \geq x^*$.

Theorem 5.2. *Let* $a,b \in \mathbb{R}_+$ *with* $a < b$. *We consider the system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{cases}$$

where $g_1, g_2 \in C[a,b]$, $K_1, K_2, H_1, H_2 \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ *are given functions.*

We assume that:

(*i*) *there exist* $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ *such that:*

$$|K_j(t,s,u) - K_j(t,s,v)| \leq L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \leq L_{H_j}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $j \in \{1,2\}$;

(*ii*) $\frac{b-a}{2}\left|(L_{K_1} + L_{K_2}) \pm \sqrt{(L_{K_1} + L_{K_2})^2 - 4(L_{K_1}L_{K_2} - L_{H_1}L_{H_2})}\right| < 1$;

(*iii*) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *are increasing functions, for all* $t,s \in [a,b]$ *and* $j \in \{1,2\}$.

Let $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$ *be the unique solution of the system.*
Then the following implications hold:

(1) *for all* $x = (x_1,x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \leq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \leq x^*$;

(2) *for all* $x = (x_1,x_2) \in (C[a,b])^2$ *with*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \geq \begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_2(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_1(s))ds \end{pmatrix},$$

for all $t \in [a,b]$, *we have* $x \geq x^*$.

6. Abstract comparison lemmas

We can establish also some abstract comparison results, taking into account Theorem 2.2.2. One of them is the following.

Theorem 6.1. *Let $a, b \in \mathbb{R}_+$ with $a < b$. We consider the systems of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ x_2(t) = g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{cases} \tag{6.1}$$

$$\begin{cases} y_1(t) = g_3(t) + \int_a^b K_3(t,s,y_1(s))ds + \int_a^b H_3(t,s,y_1(s))ds \\ y_2(t) = g_4(t) + \int_a^b K_4(t,s,y_2(s))ds + \int_a^b H_4(t,s,y_2(s))ds \end{cases} \tag{6.2}$$

$$\begin{cases} z_1(t) = g_5(t) + \int_a^b K_5(t,s,z_1(s))ds + \int_a^b H_5(t,s,z_1(s))ds \\ z_2(t) = g_6(t) + \int_a^b K_6(t,s,z_2(s))ds + \int_a^b H_6(t,s,z_2(s))ds \end{cases} \tag{6.3}$$

where $g_i \in C[a,b]$, for all $i = \overline{1,6}$ and $K_j, H_j \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$, for all $j = \overline{1,6}$, are given functions.

We assume that:

(i) there exist $L_{K_j}, L_{H_j} > 0$, $j = \overline{1,6}$ such that:

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|,$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $j = \overline{1,6}$;

(ii) $(L_{K_j} + L_{H_j})(b-a) < 1$, for all $j = \overline{1,6}$;

(iii) $K_j(t,s,\cdot), H_j(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing functions, for all $t, s \in [a,b]$ and $j = \overline{3,4}$;

(iv) for all $t \in [a,b]$,

$$\begin{aligned} &\begin{pmatrix} g_1(t) + \int_a^b K_1(t,s,x_1(s))ds + \int_a^b H_1(t,s,x_1(s))ds \\ g_2(t) + \int_a^b K_2(t,s,x_2(s))ds + \int_a^b H_2(t,s,x_2(s))ds \end{pmatrix} \\ \le &\begin{pmatrix} g_3(t) + \int_a^b K_3(t,s,x_1(s))ds + \int_a^b H_3(t,s,x_1(s))ds \\ g_4(t) + \int_a^b K_4(t,s,x_2(s))ds + \int_a^b H_4(t,s,x_2(s))ds \end{pmatrix} \\ \le &\begin{pmatrix} g_5(t) + \int_a^b K_5(t,s,x_1(s))ds + \int_a^b H_5(t,s,x_1(s))ds \\ g_6(t) + \int_a^b K_6(t,s,x_2(s))ds + \int_a^b H_6(t,s,x_2(s))ds \end{pmatrix}. \end{aligned}$$

Let $x^* = (x_1^*, x_2^*), y^* = (y_1^*, y_2^*), z^* = (z_1^*, z_2^*) \in (C[a,b])^2$ be the unique solutions of the systems (6.1), (6.2) *and respectively* (6.3) *.*

Then for any $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in (C[a,b])^2$ we have

$$x \le y \le z \Rightarrow x^* \le y^* \le z^*.$$

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