Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 2, 183–195
DOI: 10.24193/subbmath.2017.2.05

Bounds on third Hankel determinant for certain classes of analytic functions

Jugal K. Prajapat, Deepak Bansal and Sudhananda Maharana

Abstract. In this paper, the estimate for the third Hankel determinant $H_{3,1}(f)$ of Taylor coefficients of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, belonging to certain classes of analytic functions in the open unit disk $\mathbb{D}$, are investigated.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic, starlike and convex functions, Fekete-Szegö functional, Hankel determinants.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathcal{A}$ be the class of functions $f \in \mathcal{H}(\mathbb{D})$, having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

with the standard normalization $f(0) = 0$, $f'(0) = 1$. We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$, and $\mathcal{P}$ denotes the class of functions $p \in \mathcal{H}(\mathbb{D})$ with $\Re(p(z)) > 0$, $z \in \mathbb{D}$.

A function $f \in \mathcal{A}$ is called starlike (with respect to origin 0), if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain. We denote this class of starlike functions by $\mathcal{S}^*$. A function $f \in \mathcal{S}$ maps the unit disk $\mathbb{D}$ onto a convex domain is called convex function, and this class of functions is denoted by $\mathcal{K}$. Let $\mathcal{M}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \lambda, \quad z \in \mathbb{D}, \tag{1.2}$$

for some $\lambda\,(\lambda > 1)$. And let $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ if and only if $zf'(z) \in \mathcal{M}(\lambda)$, i.e. $f(z)$ satisfy the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \lambda, \quad z \in \mathbb{D}, \tag{1.3}$$

for some $\lambda\,(\lambda > 1)$. These classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated recently by Nishiwaki and Owa [19] (see also [23]). For $1 < \lambda \leq 4/3$, the classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated by Uralegaddi *et al.* [32].

Throughout the present paper, by $\mathcal{M}$ we always mean the class of functions $\mathcal{M}(3/2)$, and by $\mathcal{N}$ we always mean the class of functions $\mathcal{N}(3/2)$. Ozaki [24] proved that functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Moreover, if $f \in \mathcal{N}$, then (see e.g. [11, Theorem 1] and [21, p. 196]) one have

$$\frac{zf'(z)}{f(z)} \prec g(z) = \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D},$$

where $\prec$ denotes the subordination [18]. We see that g above is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the disk $|w - (2/3)| < 2/3$. Thus, functions in $\mathcal{M}$ are starlike in $\mathbb{D}$.

For $f \in \mathcal{A}$ of the form (1.1), a classical problem settled by Fekete and Szegö [9] is to find the maximum value of the coefficient functional $\Phi_\lambda(f) := a_3 - \lambda a_2^2$ for each $\lambda \in [0, 1]$, over the function $f \in \mathcal{S}$. By applying the Löewner method they proved that

$$\max_{f\in\mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right), & \lambda \in [0,1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating the maximum of the coefficient functional $\Phi_\lambda(f)$ for various compact subfamilies of $\mathcal{A}$, as well as λ being an arbitrary real or complex number, has been studied by many authors (see e.g. [1, 12, 13, 17, 30, 31]).

We denote by $H_{q,n}(f)$ where n, $q \in \mathbb{N} = \{1, 2, \cdots\}$, the Hankel determinant of functions $f \in \mathcal{A}$ of the form (1.1), which is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (a_1 = 1). \tag{1.4}$$

The Hankel determinant $H_{q,n}(f)$ has been studied by several authors including Cantor [6], Noonan and Thomas [20], Pommerenke [26, 25], Hayman [10], Ehrenborg [8], which are useful, in showing that a function of bounded characteristic in $\mathbb{D}$.

Indeed, $H_{2,1}(f) = \Phi_1(f)$ is the Fekete-Szegö coefficient functional. Many authors have studied the problem of calculating $\max\limits_{f\in\mathcal{F}} |H_{2,2}(f)|$ for various subfamily $\mathcal{F}$ of the class $f \in \mathcal{A}$ (see e.g. [2, 4, 14]). Recently, several authors including Babalola [3], Bansal *et al.* [5], Prajapat *et al.* [28], Raza and Malik [29] have obtained the bounds on the third Hankel determinant $H_{3,1}(f)$ for certain families of analytic functions, which is defined by

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \end{aligned} \tag{1.5}$$

In the present paper, we investigate the bounds on $H_{3,1}(f)$ for the functions belonging to the classes $\mathcal{M}$ and $\mathcal{N}$ defined above. In order to get the main results, we need the following known results.

Lemma 1.1. ([16]) *If $p \in \mathcal{P}$ be of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

and

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

Lemma 1.2. ([22, Theorem 1]) *If $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_n| \le \frac{1}{n(n-1)}, \quad n \ge 2.$$

The result is sharp for the function f_n such that $f_n'(z) = (1 - z^{n-1})^{1/(n-1)}$, $n \ge 2$.

As it is known that, if $f(z) \in \mathcal{N}$ then $zf'(z) \in \mathcal{M}$, therefore from Lemma 1.2, we conclude that

Lemma 1.3. *If $f(z) \in \mathcal{M}$ be given by (1.1), then*

$$|a_n| \le \frac{1}{n-1}, \quad n \ge 2.$$

The result is sharp for the function $g_n(z) = z(1 - z^{n-1})^{1/(n-1)}$, $n \ge 2$.

Lemma 1.4. ([22, Corollary 2]) *If $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_3 - a_2^2| \le 1/4.$$

Equality is attained for the function f such that $f'(z) = (1 - z^2e^{i\theta})^{1/2}$, $\theta \in [0, 2\pi]$.

2. Main results

Our first main result is contained in the following theorem:

Theorem 2.1. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_3 - a_2^2| \le 1. \tag{2.1}$$

The result (2.1) is sharp and equality in (2.1) is attained for the function

$$e_1(z) = z - z^2.$$

Proof. If the function $f \in \mathcal{M}$ be given by (1.1), then we may write

$$\frac{zf'(z)}{f(z)} = \frac{3}{2} - \frac{1}{2}p(z), \tag{2.2}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in $\mathbb{D}$ and $\Re(p(z)) > 0$ in $\mathbb{D}$. Also, we have $|c_n| \leq 2$ for all $n \geq 1$ (see [7]). In terms of power series expansion, the last identity is equivalent to

$$\sum_{n=1}^{\infty} n a_n z^n = \left(1 - \frac{1}{2}\sum_{n=1}^{\infty} c_n z^n\right)\left(\sum_{n=1}^{\infty} a_n z^n\right),$$

where $a_1 = 1$. Equating the coefficients of z^n on both sides, we deduce that

$$a_2 = -\frac{1}{2}c_1, \quad a_3 = \frac{1}{8}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3). \tag{2.3}$$

Now using Lemma 1.1 for some x such that $|x| \leq 1$, we have

$$|a_3 - a_2^2| = \left|\frac{1}{8}(c_1^2 - 2c_2) - \frac{1}{4}c_1^2\right| = \frac{1}{8}|2c_1^2 + x(4 - c_1^2)|.$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_3 - a_2^2| &\leq \frac{1}{8}[2c^2 + \mu(4 - c^2)] \\ &= F_1(c, \mu). \end{aligned}$$

Let $\Omega = \{(c, \mu) : 0 \leq c \leq 2 \text{ and } 0 \leq \mu \leq 1\}$. Differentiating F_1 with respect to μ, we get

$$\frac{\partial F_1}{\partial \mu} = \frac{1}{8}(4 - c^2) \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_1(c, \mu)$ is a non-decreasing function of μ on the closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_1(c, \mu) = F_1(c, 1) = \frac{c^2 + 4}{8} = G_1(c).$$

We observe that $G_1(c)$ is an increasing function in $[0, 2]$, so it will attains maximum value at $c = 2$. Next, to find the critical point on the boundary of Ω, we examine all the four line segments of Ω. Along the line segment $c = 2$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(2, \mu) = 1$, which is a constant, thus every point on the line segment is the critical point. For the line segment $c = 0$ with $0 \leq \mu \leq 1$, we have $F_1(c, \mu) = F_1(0, \mu) = \mu/2$. For the line segment $\mu = 0$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 0) = c^2/4$, which gives the critical point $(0, 0)$ and $F_1(0, 0) = 0$. Also, for the line segment $\mu = 1$ with $0 \leq c \leq 2$, we have $F_1(c, \mu) = F_1(c, 1) = (c^2 + 4)/8$, which gives another critical point $(0, 1)$ and $F_1(0, 1) = 1/2$.

Putting this all together we can conclude that the maximum of $F_1(c, \mu)$ lie at each point along the line segment $c = 2$ with $0 \leq \mu \leq 1$, which can also be verified through the mathematica plot of $F_1(c,\mu)$ over the region Ω given below in the Figure 1. Hence

$$\max_{\Omega} F_1(c,\mu) = F_1(2,\mu) = 1.$$

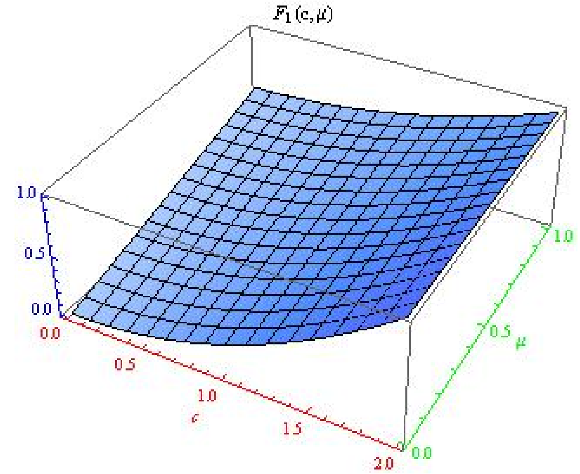

FIGURE 1. Mapping of $F_1(c,\mu)$ over Ω

To find the extremal function, setting $c_1 = 2$ and $x = 1$ in Lemma 1.1, we find that $c_2 = c_3 = 2$, using these values in (2.3), we get that $a_2 = -1$ and $a_3 = a_4 = 0$, therefore the extremal function would be $e_1(z) = z - z^2$. A simple calculation shows that $e_1(z) \in \mathcal{M}$. This complete the proof of Theorem 2.1. □

Theorem 2.2. *Let the function $f \in \mathcal{M}$ be given by* (1.1), *then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \tag{2.4}$$

The result (2.4) *is sharp and equality is attained for the function*

$$e_2(z) = z - \frac{1}{2}z^3 \text{ and } e_3(z) = z(1-z^2)^{1/2}.$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| -\frac{1}{96}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{64}\left(c_1^2 - 2c_2\right)^2 \right| \\ &= \frac{1}{192}\left| -3x^2(4-c_1^2)^2 + 2c_1^2x(4-c_1^2) - 4c_1^2x^2(4-c_1^2) \right. \\ &\quad \left. +8c_1(4-c_1^2)(1-|x|^2)z \right|. \end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{1}{192}\left[(4-c^2)\{3\mu^2(4-c^2) + 2c^2\mu + 4\mu^2c^2 + 8c(1-\mu^2)\}\right] \\ &= \frac{1}{192}\left[(4-c^2)\{(12-8c+c^2)\mu^2 + 2c^2\mu + 8c\}\right] \\ &= F_2(c,\mu). \end{aligned}$$

Differentiating $F_2(c,\mu)$ in the above equation with respect to μ, we get

$$\frac{\partial F_2}{\partial \mu} = \frac{(4-c^2)}{96}\left\{(12-8c+c^2)\mu + c^2\right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_2(c,\mu)$ is a non-decreasing function of μ on closed interval $[0,1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0\leq\mu\leq1} F_2(c,\mu) = F_2(c,1) = \frac{16-c^4}{64} = G_2(c).$$

We observe that $G_2(c)$ is a decreasing function in $[0,2]$, so it will attains maximum value at $c = 0$. Next, to find the critical point on the boundary of Ω, we examine all the four line segments of Ω by the earlier method used in Theorem 2.1, and we are getting $(0,0)$, $(2/\sqrt{3},0)$ and $(0,1)$ are the critical points and $F_2(0,0) = 0$, $F_2(2/\sqrt{3},0) = 2/9\sqrt{3}$ *and* $F_2(0,1) = 1/4$. Therefore maximum value of $F_2(c,\mu)$ is obtained by putting $c = 0$ and $\mu = 1$, which can also verified through the mathematica plot of $F_2(c,\mu)$ over Ω given below in Figure 2. Hence

$$\max_{\Omega} F_2(c,\mu) = F_2(0,1) = \frac{1}{4}.$$

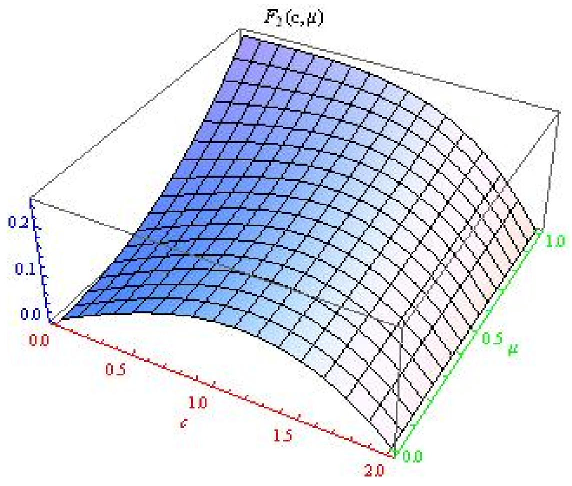

FIGURE 2. Mapping of $F_2(c,\mu)$ over Ω

Now, to find extremal function, set $c_1 = 0$ and selecting $x = 1$ in Lemma 1.1, we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (2.3), we get $a_2 = a_4 = 0$ and $a_3 = 1/2$, therefore one of the extremal function of (2.4) would be $e_2(z) = z - \frac{1}{2}z^3$. We can also see that equality in (2.4) is attended for the function $e_3(z) = z(1-z^2)^{1/2} \in \mathcal{M}$. A simple calculation shows that $e_2 \in \mathcal{M}$ and $e_3 \in \mathcal{M}$. This complete the proof of Theorem 2.2. □

Theorem 2.3. *Let the function $f \in \mathcal{M}$ be given by* (1.1), *then*

$$|a_2a_3 - a_4| \le \frac{2\sqrt{3}}{9}. \tag{2.5}$$

Proof. Using (2.3) and applying Lemma 1.1 for some x and z such that $|x| \le 1$ and $|z| \le 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left|\frac{1}{16}c_1(c_1^2 - 2c_2) + \frac{1}{48}(6c_1c_2 - 8c_3 - c_1^3)\right| \\ &= \frac{1}{24}\left|2c_1x(4-c_1^2) - c_1x^2(4-c_1^2) + 2(4-c_1^2)(1-|x|^2)z\right|. \end{aligned}$$

As $|c_1| \le 2$, letting $c_1 = c$, assume without restriction that $c \in [0,2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\le \frac{(4-c^2)}{24}[2 + 2c\mu + (c-2)\mu^2] \\ &= F_3(c,\mu). \end{aligned}$$

To find the maximum of F_3 over the region Ω, differentiating F_3 with respect to μ and c, we get

$$\frac{\partial F_3}{\partial \mu} = \frac{(4-c^2)}{12}\left[c + (c-2)\mu\right] \tag{2.6}$$

$$\frac{\partial F_3}{\partial c} = \frac{1}{24}\left[-4c + (8-6c^2)\mu + \left(4 + 4c - 3c^2\right)\mu^2\right]. \tag{2.7}$$

A critical point of $F_3(c,\mu)$ must satisfy $\frac{\partial F_3}{\partial \mu} = 0$ and $\frac{\partial F_3}{\partial c} = 0$. The condition $\frac{\partial F_3}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -c/(c-2)$. The interior point (c,μ) of Ω satisfying such condition in only $(0,0)$, and at that point $(0,0)$, we have

$$\left(\frac{\partial^2 F_3}{\partial \mu^2}\right)\left(\frac{\partial^2 F_3}{\partial c^2}\right) - \left(\frac{\partial^2 F_3}{\partial c\,\partial \mu}\right)^2 = 0.$$

Hence, it is not certain that at $(0,0)$ function have maximum value in Ω. Since Ω is closed and bounded and F_3 is continuous, the maximum of F_3 shall be attained on the boundary of Ω. Along the line segment $c = 2$ with $0 \le \mu \le 1$, we have $F_3(c,\mu) = F_3(2,\mu) = 0$, which is a constant. For the line segment $c = 0$ with $0 \le \mu \le 1$, we have $F_3(c,\mu) = F_3(0,\mu) = (1-\mu^2)/3$, which gives the same critical point $(0,0)$ and $F_3(0,0) = 1/3$. For the line segment $\mu = 0$ with $0 \le c \le 2$, we have $F_3(c,\mu) = F_3(c,0) = (4-c^2)/12$, which gives the same critical point $(0,0)$. Also, for the line segment $\mu = 1$ with $0 \le c \le 2$, we have $F_3(c,\mu) = F_3(c,1) = (4c-c^3)/8$, which gives another critical point $(2/\sqrt{3}, 1)$ on this line and $F_3(2/\sqrt{3}, 1) = 2\sqrt{3}/9$. Therefore, the point $(0, 0)$ and $(2/\sqrt{3}, 1)$ are the only critical points of F_3 over Ω. Hence, the largest value of $F_3(c, \mu)$ over the region Ω lies at $(2/\sqrt{3}, 1)$ and

$$\max_{\Omega} F_3(c, \mu) = F_3(2/\sqrt{3}, 1) = \frac{2\sqrt{3}}{9}. \qquad \square$$

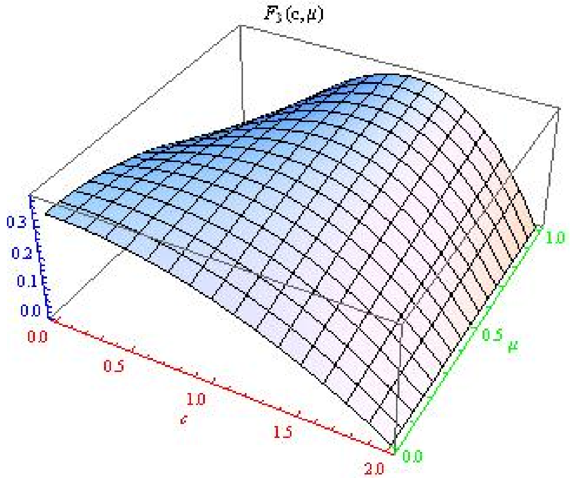

FIGURE 3. Mapping of $F_3(c, \mu)$ over Ω

Theorem 2.4. *Let the function $f \in \mathcal{M}$ be given by* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{81 + 16\sqrt{3}}{216}.$$

Proof. Using Lemma 1.3, Theorem 2.1, Theorem 2.2, Theorem 2.3 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2\sqrt{3}}{9} + \frac{1}{4} \cdot 1 = \frac{81 + 16\sqrt{3}}{216}. \end{aligned}$$

This completes the proof of Theorem 2.4. $\square$

Theorem 2.5. *Let the function $f \in \mathcal{N}$ be given by* (1.1)*, then*

$$|a_2a_3 - a_4| \leq \frac{1}{12}. \tag{2.8}$$

The result (2.8) *is sharp and equality in* (2.8) *is attained for the function e_4 where $e_4'(z) = (1 - z^3)^{1/3}$.*

Proof. Let the function $f \in \mathcal{N}$ be given by (1.1), then by definitions it is clear that $f(z) \in \mathcal{N}$ *if and only if* $zf'(z) \in \mathcal{M}$, thus replacing a_n by na_n in (2.3), we get

$$a_2 = -\frac{1}{4}c_1, \quad a_3 = \frac{1}{24}(c_1^2 - 2c_2), \quad a_4 = \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3). \tag{2.9}$$

Now using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \le 1$ and $|z| \le 1$, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left| -\frac{1}{96}c_1(c_1^2 - 2c_2) - \frac{1}{192}(6c_1c_2 - 8c_3 - c_1^3) \right| \\ &= \frac{1}{192}\left|3c_1x(4-c_1^2) - 2c_1x^2(4-c_1^2) + 4(4-c_1^2)(1-|x|^2)z\right|. \end{aligned}$$

As $|c_1| \le 2$, taking $c_1 = c$, assume without restriction that $c \in [0,2]$. Hence applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\le \frac{(4-c^2)}{192}[4 + 3c\mu + 2(c-2)\mu^2] \\ &= F_4(c,\mu). \end{aligned}$$

Following the earlier method used in Theorem 2.3, we can show that the global maximum of $F_4(c,\mu)$ over the region Ω is achieved at $(0,0)$ and $F_4(0,0) = 1/12$. This can also be verified through the mathematica plot of $F_4(c,\mu)$ over Ω given below in Figure 4.

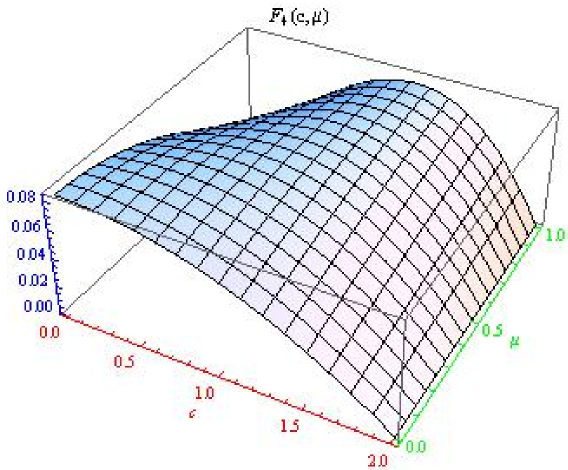

FIGURE 4. Mapping of $F_4(c,\mu)$ over Ω

Also observe that equality in (2.8) is attained for the function e_4 where

$$e_4'(z) = (1 - z^3)^{1/3}.$$

A computation shows that $e_4 \in \mathcal{N}$. Hence the result is obtained. □

Theorem 2.6. *Let the function $f \in \mathcal{N}$ be given by* (1.1)*, then*

$$|a_2a_4 - a_3^2| \leq \frac{9}{320}. \tag{2.10}$$

Proof. Using (2.9) and applying Lemma 1.1 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \frac{1}{192}\left|-\frac{1}{4}c_1(6c_1c_2 - 8c_3 - c_1^3) - \frac{1}{3}\left(c_1^2 - 2c_2\right)^2\right| \\
&= \frac{1}{192}\left|\frac{1}{12}c_1^4 + \frac{1}{6}c_1^2c_2 + \frac{4}{3}c_2^2 - 2c_1c_3\right| \\
&= \frac{1}{2304}\left|3xc_1^2(4 - c_1^2) - 6x^2c_1^2(4 - c_1^2) + 12zc_1(4 - c_1^2)(1 - |x|^2)\right. \\
&\quad \left. -4(4 - c_1^2)^2x^2\right|.
\end{aligned}$$

As $|c_1| \leq 2$, taking $c_1 = c$, assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &\leq \frac{(4 - c^2)}{2304}\left\{12c + 3c^2\mu + 2(8 - 6c + c^2)\mu^2\right\} \\
&= F_5(c, \mu).
\end{aligned}$$

Differentiating $F_5(c, \mu)$ with respect to μ, we get

$$\frac{\partial F_5}{\partial \mu} = \frac{(4 - c^2)}{2304}\left\{4\mu(c^2 - 6c + 8) + 3c^2\right\} \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Therefore $F_5(c, \mu)$ is a non-decreasing function of μ on closed interval $[0, 1]$. Thus, it attains maximum value at $\mu = 1$. Let

$$\max_{0 \leq \mu \leq 1} F_5(c, \mu) = F_5(c, 1) = \frac{1}{2304}(64 + 4c^2 - 5c^4) = G_5(c).$$

We can see that $G_5(c)$ is an increasing function in $[0, \sqrt{2/5}]$, so $G_5(c)$ attains maximum value at $c = \sqrt{2/5}$. Next, to find the critical points on the boundary of Ω, we examine all the four line segments of Ω by the earlier method used in Theorem 2.1 and 2.3, and we get $(0, 0)$, $(2/\sqrt{3}, 0)$ and $(0, 1)$ are the critical points and $F_5(0, 0) = 0$, $F_5(2/\sqrt{3}, 0) = 1/36\sqrt{3}$ and $F_5(0, 1) = 1/36$. Therefore $F_5(c, \mu)$ have maximum value at $\mu = 1$ and $c = \sqrt{2/5}$ in the region Ω. Thus

$$\max_{\Omega} F_5(c, \mu) = F_5(\sqrt{2/5}, 1) = \frac{9}{320}.$$

This completes the proof of Theorem 2.6. □

Remark 2.7. For $f \in \mathcal{S}$, Thomas [27, p. 166] conjectured that

$$|H_{2,n}(f)| = |a_na_{n+2} - a_{n+1}^2| \leq 1, \quad n = 2, 3, 4 \cdots.$$

Subsequently, Li and Srivastava [15, p. 1040] shown that this conjecture is not valid for $n \geq 4$, *i.e.* conjecture is valid only for $n = 2, 3$. From Theorem 2.6, we found that, if function f is member of class $\mathcal{N}$ and having form (1.1), then $|H_{2,2}(f)| \leq 9/320$.

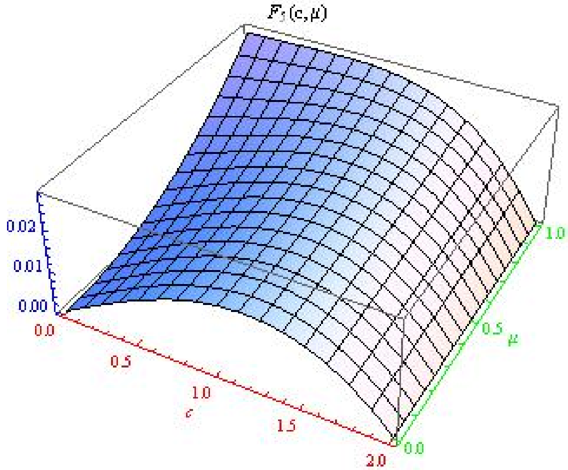

FIGURE 5. Mapping of $F_5(c,\mu)$ over Ω

Since all functions in $\mathcal{N}$ are univalent in $\mathbb{D}$. Therefore, Theorem 2.6 validates the Thomas conjecture when $n = 2$ for the function belonging to the classes $\mathcal{N}$.

Theorem 2.8. *Let the function $f \in \mathcal{N}$ be given by* (1.1)*, then*

$$|H_{3,1}(f)| \leq \frac{139}{5760}.$$

Proof. Using Lemma 1.2, Lemma 1.4, Theorem 2.5, Theorem 2.6 and the triangle inequality on $H_{3,1}(f)$, we get

$$\begin{aligned} |H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{1}{6}\frac{9}{320} + \frac{1}{12}\frac{1}{12} + \frac{1}{20}\frac{1}{4} = \frac{139}{5760}. \end{aligned}$$

This completes the proof of Theorem 2.8. □

References

[1] Abdet Gawad, H.R., Thomas, D.K., *The Fekete-Szegö problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc., **114**(1992), 345-349.

[2] Al-Abbadi, M.H., Darus, M., *Hankel determinant for certain class of analytic function defined by generalized derivative operators*, Tamkang J. Math., **43**(2012), 445-453.

[3] Babalola, K.O., *On third order Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6**(2010), 1-7.

[4] Bansal, D., *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett., **26**(2013), 103-107.

[5] Bansal, D., Maharana, S., Prajapat, J.K., *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc., **52**(2015), no. 6, 1139-1148.

[6] Cantor, D.G., *Power series with the integral coefficients*, Bull. Amer. Math. Soc., **69**(1963), 362-366.

[7] Duren, P.L., *Univalent Functions*, Springer Verlag, New Yark Inc. 1983.

[8] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, **107**(2000), 557-560.

[9] Fekete, M., Szegö, G., *Eine Benberkung uber ungerada Schlichte funktionen*, J. Lond. Math. Soc., **8**(1933), 85-89.

[10] Hayman, W.K., *On second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc., **18**(1968), 77-94.

[11] Jovanovic, I., Obradoviȅ, M., *A note on certain classes of univalent functions*, Filomat, **9**(1995), 69-72.

[12] Keogh, F.R., Merkes, E.P., *A Coefficient Inequality for Certain Classes of Analytic Functions*, Proc. Amer. Math. Soc., **20**(1969), 8-12.

[13] Koepf, W., *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc., **101**(1987), 85-95.

[14] Lee, S.K., Ravichandran, V., Supramaniam, S., *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., **2013**(2013), art. 281.

[15] Li, J.L., Srivastava, H.M., *Some questions and conjectures in the theory of univalent functions*, Rocky Mountain J. Math., **28**(1998), 1035-1041.

[16] Libera, R.J., Zlotkiewicz, E.J., *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85**(1982), 225-230.

[17] London, R.R., *Fekete-Szegö inequalities for close-to-convex functions*, Proc. Amer. Math. Soc., **117**(1993), 947-950.

[18] Miller, S.S., Mocanu, P.T., *Differential Subordinations. Theory and Applications*, New York and Basel, Marcel Dekker, 2000.

[19] Nishiwaki, J., Owa, S., *Coefficient inequalities for certain analytic functions*, Int. J. Math. Math. Sci., **29**(2002), 285-290.

[20] Noonan, J.W., Thomas, D.K., *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223**(1976), 337-346.

[21] Obradović, M., Ponnusamy, S., *Injectivity and starlikeness of section of a class of univalent functions*, Contemporary Math., **591**(2013), 195-203.

[22] Obradović, M., Ponnusamy, S., Wirths, K.-J., *Coefficient characterizations and sections for some univalent functions*, Sib. Math. J., **54**(2013), 679-696.

[23] Owa, S., Srivastava, H.M., *Some generalized convolution properties associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), no. 3, art. 42, 13 pages.

[24] Ozaki, S., *On the theory of multivalent functions II*, Sci. Rep. Tokyo Bunrika Daigaku. Sect A, **4**(1941), 45-87.

[25] Pommerenke, C., *On the coefficients and Hankel determinant of univalent functions*, J. Lond. Math. Soc., **41**(1966), 111-122.

[26] Pommerenke, C., *On the Hankel determinant of univalent functions*, Mathematika, **14**(1967), 108-112.

[27] Parvatham, R., Ponnusamy, S., (eds.) *New trends in geometric function theory and application*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1981.

[28] Prajapat, J.K., Bansal, D., Singh, A., Mishra, A.K., *Bounds on third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math., **7**(2015), no. 2, 210-219.

[29] Raza, M., Malik, S.N., *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl., **2013**(2013), art. 412.

[30] Shanmugam, T.N., Ramachandran, C., Ravichandran, V., *Fekete-Szegö problem for subclass of starlike functions with respect to symmetric points*, Bull. Korean Math. Soc., **43**(2006), no. 3, 589-598.

[31] Srivastava, H.M., Mishra, A.K., Das, M.K., *Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Var. Theory Appl., **44**(2001), 145-163.

[32] Uralegaddi, B.A., Ganigi, M.D., Sarangi, S.M., *Univalent functions with positive coefficients*, Tamkang J. Math., **25**(1994), 225-230.

Jugal K. Prajapat
Department of Mathematics
Central University of Rajasthan
NH-8, Bandarsindri, Kishangarh-305817
Dist.-Ajmer, Rajasthan, India
e-mail: jkprajapat@gmail.com

Deepak Bansal
Department of Mathematics
Govt. College of Engineering and Technology
Bikaner-334004, Rajasthan, India
e-mail: deepakbansal_79@yahoo.com

Sudhananda Maharana
Department of Mathematics
Central University of Rajasthan
NH-8, Bandarsindri, Kishangarh-305817
Dist.-Ajmer, Rajasthan, India
e-mail: snmmath@gmail.com