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Classes of an univalent integral operator

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Abstract. In this paper we introduce a new general integral operator for analytic functions in the open unit disk **U** and we obtain sufficient conditions for univalence of this integral operator.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions f which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} :\mid z \mid< 1\}$ and $f(0) = f^{'}(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}} \tag{1.1}$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i$, with $\delta \neq 0$, $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$.

Remark 1.1. The integral operator $\mathcal{T}_n$ defined by (1.1), is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7].

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} \mathrm{dt},$$

ii) For $n=1$, $\delta=1$ and $\alpha_1-1=\gamma_1=\delta_1=0$ we obtain the integral operator which was studied by Pfaltzgraff [18].

$$\mathcal{G}_\alpha(z)=\int_0^z (f'(t))^\alpha \, \mathrm{dt},$$

iii) For $\alpha_i-1=\alpha_i$ and $\beta_i=\gamma_i=\delta_i=0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2].

$$\mathcal{D}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i-1=\gamma_i=\delta_i=0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n[f_i'(t)]^{\alpha_i}\,\mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17] .

v) For $\alpha_i-1=\alpha_i$ and $\gamma_i=\delta_i=0$ we obtain the integral operator which was defined and studied by Frasin [5]

$$\mathcal{F}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\left(f_i'(t)\right)^{\beta_i}\mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [9].

vi) For $\alpha_i-1=\beta_i=0$ we obtain the integral operator which was defined and studied by Pescar [13].

$$\mathcal{I}_n(z)=\left[\delta\int_0^z t^{\delta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i}\left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i}\mathrm{dt}\right]^{\frac{1}{\delta}},$$

Thus, the integral operator $\mathcal{T}_n$, introduced here by the formula (1.1), can be considered as an extension and a generalization of these operators above mentioned.

We need the following lemmas.

Lemma 1.2. [11] *Let γ,δ be complex numbers, $Re\gamma>0$ and $f\in\mathcal{A}$. If*

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zf''(z))}{f'(z)}\right|\leq 1,$$

for all $z\in\mathbb{U}$, then for any complex number δ, $Re\delta\geq Re\gamma$, the function F_δ defined by

$$F_\delta(z)=\left(\delta\int_0^z t^{\delta-1}f'(t)dt\right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathbb{U}$.

Lemma 1.3. [14] *Let* δ *be complex number,* $Re\delta > 0$ *and* c *a complex number,* $|c| \le 1$, $c \neq -1$, *and* $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. *If*

$$\left| c\left|z\right|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zf''(z))}{\delta f'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, *then the function* F_δ *defined by*

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)\,dt\right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathbb{U}$.

Lemma 1.4. [8] *Let* f *be the function regular in the disk* $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$, M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\ge m$, *then*

$$|f(z)| \le \frac{M}{R^m} z^m,$$

the equality for $z \neq 0$ *can hold only if*

$$f(z) = e^{i\theta}\frac{M}{R^m} z^m,$$

where θ *is constant.*

2. Main results

Theorem 2.1. *Let* $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $c = Re\gamma > 0$, M_i, N_i, P_i, Q_i, R_i, S_i *real positive numbers and* $f_i, g_i, h_i, k_i \in \mathcal{A}$,

$$\begin{aligned} f_i(z) &= z + a_{2i}z^2 + a_{3i}z^3 + \ldots, \\ g_i(z) &= z + b_{2i}z^2 + b_{3i}z^3 + \ldots, \\ h_i(z) &= z + c_{2i}z^2 + c_{3i}z^3 + \ldots, \\ k_i(z) &= z + d_{2i}z^2 + d_{3i}z^3 + \ldots,\ i = \overline{1,n}. \end{aligned}$$

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \le M_i, \quad \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \le N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \le P_i,$$
$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \le Q_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n}\left[\left|\alpha_i - 1\right| M_i + \left|\beta_i\right| N_i + \left|\gamma_i\right|(P_i + Q_i) + \left|\delta_i\right|(R_i + S_i)\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{2.1}$$

then, for all δ *complex numbers,* $Re\delta \ge Re\gamma$, *the integral operator* $\mathcal{T}_n$, *given by (1.1) is in the class* $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t)}{k_i{}'(t)} \right)^{\delta_i} \right] \mathrm{dt},$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function H_n is regular in $\mathbb{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n^{'}(0) - 1 = 0$.

Now

$$H_n{}'(z) = \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(z)}{k_i(z)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(z)}{k_i'(z)} \right)^{\delta_i} \right].$$

We have

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right]$$
$$+ \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathbb{U}$.

Thus, we have

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| = \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right]$$
$$+ \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathbb{U}$.

Therefore

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \le \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right]$$
$$+ \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| + \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \right) \right]$$
$$+ \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right],$$

for all $z \in \mathbb{U}$.

By applying the General Schwarz Lemma (1.4) we obtain

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \le M_i |z|, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \le N_i |z|, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \le P_i |z|,$$
$$\left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \le Q_i |z|, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \le R_i |z|, \quad \left| \frac{zK_i''(z)}{K_i'(z)} \right| \le S_i |z|,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Using these inequalities we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|$$

$$\leq \frac{1-|z|^{2c}}{c}|z|\sum_{i=1}^{n}\left[|\alpha_i-1|M_i+|\beta_i|N_i+|\gamma_i|(P_i+Q_i)+|\delta_i|(R_i+S_i)\right], \qquad (2.2)$$

for all $z\in\mathbb{U}$.

Since

$$\max_{|z|\leq 1}\frac{\left(1-|z|^{2c}\right)|z|}{c}=\frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (2.2), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|$$

$$\leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\alpha_i-1|M_i+|\beta_i|N_i+|\gamma_i|(P_i+Q_i)+|\delta_i|(R_i+S_i)\right],$$

and hence, by (2.1) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\cdot\frac{(2c+1)^{\frac{2c+1}{2c}}}{2}=1,$$

for all $z\in\mathbb{U}$.

So,

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \qquad (2.3)$$

and using (2.3), by Lemma 1.2, it results that the integral operator $\mathcal{T}_n$, given by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta=1$ in Theorem 2.1, obtain the next corollary:

Corollary 2.2. *Let $\gamma,\alpha_i,\beta_i,\gamma_i,\delta_i$ be complex numbers, $0<Re\gamma\leq 1$, $c=Re\gamma$, M_i,N_i,P_i,Q_i,R_i,S_i real positive numbers and $f_i,g_i,h_i,k_i\in\mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\leq M_i,\quad\left|\frac{zg_i''(z)}{g_i'(z)}\right|\leq N_i,\quad\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\leq P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\leq Q_i,\quad\left|\frac{zh_i''(z)}{h_i'(z)}\right|\leq R_i,\quad\left|\frac{zk_i''(z)}{k_i'(z)}\right|\leq S_i,$$

$$\sum_{i=1}^{n}\left[|\alpha_i-1|M_i+|\beta_i|N_i+|\gamma_i|(P_i+Q_i)+|\delta_i|(R_i+S_i)\right]\leq\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{F}_n$ defined by

$$\mathcal{F}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(g_i(t)'\right)^{\beta_i}\cdot\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\cdot\left(\frac{{h_i}'(t))}{{k_i}'(t)}\right)^{\delta_i}\right]dt, \qquad (2.4)$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.3. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, P_i, Q_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq N_i,$$

$$\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{S}_n$ defined by

$$\mathcal{S}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \right] dt, \tag{2.5}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.4. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, P_i, Q_i, R_i, S_i real positive numbers and $f_i, h_i, k_i \in \mathcal{A}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \leq P_i, \quad \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \leq Q_i,$$

$$\left| \frac{zh_i''(z)}{h_i'(z)} \right| \leq R_i, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{X}_n$ defined by

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] dt, \tag{2.6}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.5. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, N_i, P_i, Q_i, R_i, S_i real positive numbers and $g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{D}_n$ defined by

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.7}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.6. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, R_i, S_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{Y}_n$ defined by

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.8}$$

is in the class $\mathcal{S}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.1, obtain the next corollary:

Corollary 2.7. *Let α be complex number, $Re\alpha > 0$, M, N, P, Q, R, S real positive numbers and $f, g, h, k \in \mathcal{A}$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \left|\frac{zg''(z)}{g(z)'}\right| \leq N, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq P,$$

$$\left|\frac{zk'(z)}{k(z)} - 1\right| \leq Q, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq R, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq S,$$

for all $z \in \mathbb{U}$, and

$$|\alpha - 1| (M + N + P + Q + R + S) \leq \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{T}$ defined by

$$\mathcal{T}(z) = \left[\alpha \int_0^z \left(f(t) \cdot g'(t) \cdot \frac{h(t)}{k(t)} \cdot \frac{h'(t))}{k'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}, \tag{2.9}$$

is in the class $\mathcal{S}$.

Theorem 2.8. *Let $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = Re\gamma > 0$ and $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{c}{2}, \quad for \quad 0 < c < 1 \tag{2.10}$$

or

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{1}{2}, \quad for \quad c \geq 1 \tag{2.11}$$

then, for any complex numbers δ, $Re\delta \geq c$, the integral operator $\mathcal{T}_n$ defined in (1.1) is in the class $\mathcal{S}$.

Proof. After the same steps as in the proof of Theorem 2.1., we get

$$\begin{aligned}
\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq{}& \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + |\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right] \\
&+ \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}\right| + 1 + \left|\frac{zk_i'(z)}{k_i(z)}\right| + 1\right)\right] \\
&+ \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right| + \left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right],
\end{aligned}$$

for all $z \in \mathbb{U}$.

Since $f_i, h_i, k_i \in \mathcal{S}$ we have

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq \frac{1+|z|}{1-|z|},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

For ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$ we have

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Using these relations we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|} + 1\right)\sum_{i=1}^{n}|\alpha_i - 1|$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|+\frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|}+1+\frac{1+|z|}{1-|z|}+1\right)\sum_{i=1}^{n}|\gamma_i|$$

$$+\frac{1-|z|^{2c}}{c}\left(\frac{2|z|}{1-|z|^2}+\frac{2|z|}{1-|z|^2}\right)\sum_{i=1}^{n}|\delta_i|$$

$$\leq\frac{1-|z|^{2c}}{c}\cdot\frac{2}{1-|z|}\sum_{i=1}^{n}|\alpha_i-1|++\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{4}{1-|z|}\sum_{i=1}^{n}|\gamma_i|+\frac{1-|z|^{2c}}{c}\cdot\frac{4|z|}{1-|z|^2}\sum_{i=1}^{n}|\delta_i|, \tag{2.12}$$

for all $z\in\mathbb{U}$.

For $0<c<1$, we have $1-|z|^{2c}\leq 1-|z|^2$, $z\in\mathbb{U}$ and by (2.12), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq\frac{4}{c}\sum_{i=1}^{n}|\alpha_i-1|+\frac{2}{c}\sum_{i=1}^{n}|\beta_i|+\frac{8}{c}\sum_{i=1}^{n}|\gamma_i|+\frac{4}{c}\sum_{i=1}^{n}|\delta_i|, \tag{2.13}$$

for all $z\in\mathbb{U}$.

From (2.10) and (2.13) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \tag{2.14}$$

for all $z\in\mathbb{U}$ and $0<c<1$.

For $c\geq 1$ we have $\frac{1-|z|^{2c}}{c}\leq 1-|z|^2$, for all $z\in\mathbb{U}$ and by (2.12), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|, \tag{2.15}$$

for all $z\in\mathbb{U}$ and $c\geq 1$.

From (2.11) and (2.15) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \tag{2.16}$$

for all $z\in\mathbb{U}$ and $c\geq 1$.

And by (2.14), (2.16) and Lemma 1.2, it results that the integral operator $\mathcal{T}_n$, defined by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta=1$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.9. *Let $\gamma,\alpha_i,\beta_i,\gamma_i,\delta_i$ be complex numbers, $0<Re\gamma\leq 1$ and $f_i,h_i,k_i\in\mathcal{S}$, ${g_i}',{h_i}',{k_i}'\in\mathcal{P}$. If*

$$4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|\leq\frac{Re\gamma}{2},\quad for\ \ 0<c<1$$

then the integral operator $\mathcal{F}_n$ defined by (2.4) belongs to the class $\mathcal{S}$.

If we consider $\delta=1$ and $\beta_1=\beta_2=...=\beta_n=0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.10. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, ${h_i}', {k_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{X}_n$ defined by (2.6) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.11. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 2\sum_{i=1}^{n}|\beta_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{Y}_n$ defined by (2.8) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.12. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| + 4\sum_{i=1}^{n}|\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{D}_n$ defined by (2.7) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.13. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + 2\sum_{i=1}^{n}|\beta_i| + 8\sum_{i=1}^{n}|\gamma_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{S}_n$ defined by (2.5) belongs to the class $\mathcal{S}$.

Theorem 2.14. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $Re\gamma > 0$, M_i, N_i, P_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq N_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$|c| \leq 1 - \frac{1}{|\delta|}\left[(2+M_i)\sum_{i-1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i|\right] - \frac{1}{|\delta|}\left[(N_i+P_i+4)\sum_{i=}^{n}|\gamma_i| + 2\sum_{i=1}^{n}|\delta_i|\right], \tag{2.17}$$

where $c \in \mathbb{C}$, $c \neq -1$, *then the integral operator* $\mathcal{T}_n$, *defined by (1.1) is in the class* $\mathcal{S}$.

Proof. Also, a simple computation yields

$$\begin{aligned}\left||c\,|z|^{2\delta} + \left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| &\leq |c| + \frac{1}{|\delta|}\sum_{i-1}^{n}|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right) + \frac{1}{|\delta|}\sum_{i-1}^{n}|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right| \\ &+\frac{1}{|\delta|}\sum_{i=1}^{n}|\gamma_i|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}\right|+1\right)+\left(\left|\frac{zk_i'(z)}{k_i(z)}\right|+1\right)\right] \\ &+\frac{1}{|\delta|}\sum_{i=1}^{n}|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right),\end{aligned} \tag{2.18}$$

for all $z \in \mathbb{U}$.

Using these inequalities from hypothesis we have

$$\begin{aligned}\left||c\,|z|^{2\delta} + \left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| &\leq |c| + \frac{1}{|\delta|}\left[(2+M_i)\sum_{i-1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i|\right] \\ &+ \frac{1}{|\delta|}\left[(N_i+P_i+4)\sum_{i=}^{n}|\gamma_i| + 2\sum_{i=1}^{n}|\delta_i|\right],\end{aligned}$$

for all $z \in \mathbb{U}$. and hence, by inequality (2.17) we have

$$\left||c\,|z|^{2\delta} + \left(1-|z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq 1, \tag{2.19}$$

for all $z \in \mathbb{U}$.

Applying Lemma 1.3, we conclude that the integral operator $\mathcal{T}_n$, given by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ and $n = 1$ in Theorem 2.14, we obtain the next corollary:

Corollary 2.15. *Let* α *be complex number,* $Re\alpha > 0$ M, N, P *real positive numbers, and* $f, g, h, k \in \mathcal{A}$. *If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq N,$$

$$\left|\frac{zk'(z)}{k(z)} - 1\right| \leq P, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$ *and*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (M_i + N_i + P_i + 8), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then the integral operator $\mathcal{T}$*, given by (2.9) is in the class* $\mathcal{S}$*.*

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