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Majorization problems for certain starlike functions associated with the exponential function

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Abstract. Let $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ denote the class of analytic functions f in the open unit disc normalized by $f(0) = 0 = f'(0) - 1$ and satisfying, respectively, the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec e^z \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec e^{e^z - 1}.$$

In this article, we investigate majorization problems for the classes $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ without acting upon any linear or nonlinear operators.

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1. Introduction

Let $\mathcal{H}$ be the set of analytic functions f on the open unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

where $\mathbb{C}$ denotes the complex plane. Also let $\mathcal{A}$ be a subclass of $\mathcal{H}$ that whose members are normalized by the condition $f(0) = 0 = f'(0) - 1$. Let the functions f and g belong to the class $\mathcal{H}$ and there exists a Schwarz function $\phi : \Delta \to \Delta$ with the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f(z) = g(\phi(z))$. Then we say that f is subordinate to g, written as $f(z) \prec g(z)$ or $f \prec g$. It is clear that if $f \prec g$, then

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta). \tag{1.1}$$

Also, if g is univalent (one-to-one) in Δ, then $f(z) \prec g(z)$ iff the conditions (1.1) hold true. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ will be denoted by $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be starlike if f maps Δ onto a domain

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which is starlike with respect to origin. The class of starlike functions in $\mathcal{U}$ is denoted $\mathcal{S}^*$. Analytically, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*$ iff

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \Delta).$$

In 1992, Ma and Minda (see [15]) have introduced the class

$$\mathcal{S}^*(\varphi) := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$$

where φ is analytic univalent function with $\mathrm{Re}\{\varphi(z)\} > 0$ $(z \in \Delta)$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. For special choices of φ, the class $\mathcal{S}^*(\varphi)$ becomes to the well-known subclasses of the starlike functions. For example, the class

$$\mathcal{S}^*((1 + Az)/(1 + Bz)) =: \mathcal{S}^*[A, B] \quad (-1 \leq B < A \leq 1)$$

was introduced by Janowski, see [8]. If we also let $\varphi(z) := (1 + (1 - 2\alpha)z)/(1 - z)$, then the class $\mathcal{S}^*(\varphi)$ $(0 \leq \alpha < 1)$ gives the well-known class of the starlike functions of order α. We recall that a function $f \in \mathcal{A}$ is starlike of order α iff

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \Delta).$$

The family of all such functions is denoted by $\mathcal{S}^*(\alpha)$. We put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. The family $\mathcal{S}^*(\alpha)$ for $\alpha \in [0, 1)$ is a subfamily of the univalent functions (e.g., see [7]) and the function

$$\mathrm{K}_\alpha(z) := \frac{z}{(1 - z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} c_n(\alpha) z^n \quad (z \in \Delta, 0 \leq \alpha < 1),$$

where

$$c_n(\alpha) := \frac{\prod_{k=2}^{n}(k - 2\alpha)}{(n - 1)!} \quad (n \geq 2),$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$.
In 2015, Mendiratta et al. [17] introduced the class $\mathcal{S}^*_e$ as follows:

$$\mathcal{S}^*_e := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z =: \varphi_0(z)\right\}.$$

An extremal function for the class $\mathcal{S}^*_e$ is

$$f_1(z) := z \exp\left(\int_0^z \frac{e^\zeta - 1}{\zeta} \mathrm{d}\zeta\right) = z + z^2 + \frac{3}{4}z^3 + \frac{17}{36}z^4 + \cdots.$$

This function f_1 also plays the role extremal for many extremal problems. We notice that the exponential function $\varphi_0(z) = e^z$ has positive real part in Δ and

$$\varphi_0(\Delta) = \{\zeta \in \mathbb{C} : |\log \zeta| < 1\} =: \Omega.$$

It is easy to see that Ω is symmetric with respect to the real axis, starlike with respect to 1 and $\varphi_0'(0) > 0$ (see Figure 1(a)). Thus we have

$$f \in \mathcal{S}^*_e \Leftrightarrow \left|\log\left\{\frac{zf'(z)}{f(z)}\right\}\right| < 1 \quad (z \in \Delta).$$

For more details about the class $\mathcal{S}_e^*$ one can refer to [17].
Motivated by the above defined classes, Kumar et al. [12] (see also [6]) defined the class $\mathcal{S}_B^*$ associated with the Bell numbers where

$$\mathcal{S}_B^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^{e^z-1} =: Q(z) \right\} =: \mathcal{S}^*(Q).$$

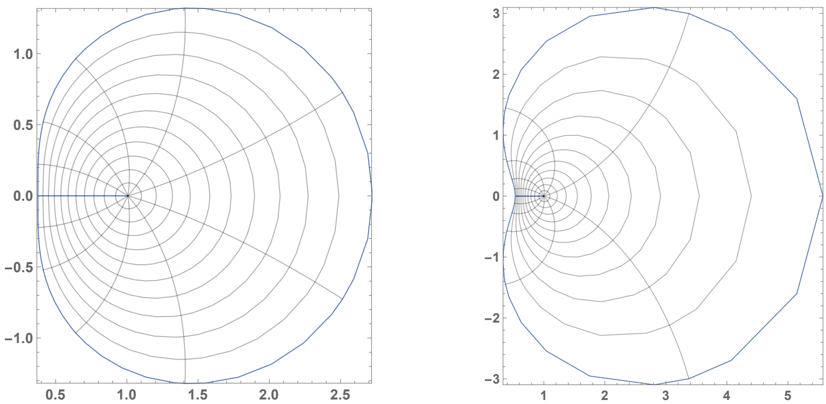
The function f_2 defined by

$$f_2(z) := z\exp\left(\int_0^z \frac{Q(\zeta)-1}{\zeta}\mathrm{d}\zeta\right) = z + z^2 + z^3 + \frac{17}{18}z^4 + \frac{245}{288}z^5 + \cdots,$$

belongs to the class $\mathcal{S}_B^*$ and serve as an extremal function in many problems. We also note that

$$Q(z) = e^{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \quad (z \in \Delta),$$

is starlike with respect to 1 (see Figure 1(b)) and its coefficients generate the Bell numbers. For a brief survey on these numbers, readers may refer to [4, 3].



(a) (b)

FIGURE 1. (a): The boundary curve of $\varphi_0(\Delta) = \exp(\Delta)$
(b): The boundary curve of $Q(\Delta) = \exp(\exp(\Delta) - 1)$

Also, for more details about some another subclasses of the starlike functions with various special cases of φ, see [10, 9, 11, 13, 14, 19, 20, 21].

The following theorem due to Carathéodory, see [5]:

Theorem A. *If the function $f \in \mathcal{H}$ satisfies the conditions*

$$|f(z)| \le 1 \quad \text{and} \quad f(0) = 0,$$

then $|f'(z)| \le 1$ for $|z| \le \sqrt{2} - 1$.

Theorem B (below) is a generalization of the Theorem A which was proved by MacGregor, see [16]. Indeed, by letting $g(z) = z$, Theorem B reduces to the Theorem A.

Theorem B. *If $f(z)$ is majorized by $g(z)$ in Δ and $g(0) = 0$, then*

$$\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)|$$

for each number r in the interval $[0, \sqrt{2} - 1]$.

We recall that a function $f \in \mathcal{H}$ is called to be majorized by $g \in \mathcal{H}$ written as

$$f(z) \ll g(z),$$

if there exists an analytic function ψ in Δ and satisfying the following conditions

$$|\psi(z)| \leq 1 \quad \text{and} \quad f(z) = \psi(z)g(z) \tag{1.2}$$

for all $z \in \Delta$. It should be noted that for the first time Mac-Gregor defined the concept of majorization. Indeed, he has been studied majorization problem for the class of starlike functions [16]. Recently, also many researchers have studied several majorization problems for certain subclasses of analytic functions which are defined by the concept of subordination, see for instance [1, 2, 25, 22, 23, 24].

The present paper aims to study majorization problems for the classes $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ without acting upon any linear or nonlinear operators to the above function classes.

2. Main Results

The following lemma (see [18]) will be needed in our investigation.

Lemma 2.1. *Let $\psi(z)$ be analytic in Δ and satisfying $|\psi(z)| \leq 1$ for all $z \in \Delta$. Then*

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}.$$

The first result of this section is continued in the following form.

Theorem 2.2. *Let the function f be in the class $\mathcal{A}$ and $g \in \mathcal{S}_e^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)|$$

for each number r in the interval $[0, 0.323784]$ where $r_1 \approx 0.323784$ is the positive root of the equation

$$1 - r^2 - 2re^r = 0. \tag{2.1}$$

Proof. Let $f \in \mathcal{A}$ and the function g belongs to the class $\mathcal{S}_e^*$. Then by definition of the class $\mathcal{S}_e^*$ we have

$$\frac{zg'(z)}{g(z)} \prec e^z,$$

or equivalently

$$\frac{zg'(z)}{g(z)} = e^{\phi(z)} \quad (z \in \Delta), \tag{2.2}$$

where ϕ is a Schwarz function. With a simple calculation and since $|\phi(z)| \leq |z|$ (see [7]), (2.2) implies that

$$\left| \frac{g(z)}{g'(z)} \right| \leq re^r \quad (|z| = r < 1). \tag{2.3}$$

By the assumption since $f(z) \ll g(z)$ in Δ, thus there exists an analytic function ψ in Δ satisfying $|\psi(z)| \le 1$ such that

$$f(z) = \psi(z)g(z) \quad (z \in \Delta). \tag{2.4}$$

Differentiating of both sides of (2.4) gives us

$$f'(z) = \psi'(z)g(z) + \psi(z)g'(z) = g'(z)\left(\psi'(z)\frac{g(z)}{g'(z)} + \psi(z)\right). \tag{2.5}$$

Now by (2.3), (2.5) and by Lemma 2.1 we get

$$\begin{aligned}|f'(z)| &\le \left(|\psi(z)| + \frac{1-|\psi(z)|^2}{1-r^2} \times re^r\right)|g'(z)| \\ &= \left(\gamma + \frac{1-\gamma^2}{1-r^2} \times re^r\right)|g'(z)|,\end{aligned}$$

where $|\psi(z)| =: \gamma \in [0,1]$. We now define the function $\mu(\gamma, r)$ as follows

$$\mu(\gamma, r) := \gamma + \frac{1-\gamma^2}{1-r^2} \times re^r.$$

It is enough to consider r_1 as follows

$$r_1 = \max\{r \in [0,1) : \mu(\gamma, r) \le 1, \forall \gamma \in [0,1]\}.$$

Therefore

$$\mu(\gamma, r) \le 1 \Leftrightarrow \lambda(\gamma, r) \ge 0,$$

where $\lambda(\gamma, r) := 1 - r^2 - (1+\gamma)re^r$. We see that $\lambda(\gamma, r)$ is decreasing function with respect to γ and gets its minimum value in $\gamma = 1$, namely

$$\min\{\lambda(\gamma, r) : \gamma \in [0,1]\} = \lambda(1, r) = \lambda(r),$$

where $\lambda(r) := 1 - r^2 - 2re^r$. On the other hand, since $\lambda(0) = 1 > 0$ and $\lambda(1) = -2e < 0$, thus there exists a r_1 such that $\lambda(r) \ge 0$ for all $r \in [0, r_1]$ where r_1 is the smallest positive root of the Eq. (2.1). □

Since the identity function $g(z) = z$ belongs to the class $\mathcal{S}_e^*$, therefore we have the following result.

Corollary 2.3. *If a function $f \in \mathcal{A}$ satisfies the condition*

$$|f(z)| < 1 \quad (z \in \Delta),$$

then $|f'(z)| \le 1$ for $|z| \le 0.323784$.

The next result gives a same result for the class $\mathcal{S}_B^*$.

Theorem 2.4. *Let the function f be in the class $\mathcal{A}$ and $g \in \mathcal{S}_B^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$\max_{|z|=r} |f'(z)| \le \max_{|z|=r} |g'(z)| \quad (0 \le r \le r_2) \tag{2.6}$$

where r_2 is the smallest positive root of the equation

$$(1 - r^2)e^{e^{-r}-1} - 2r = 0. \tag{2.7}$$

Proof. Let f belong to the class $\mathcal{A}$. If $g \in \mathcal{S}^*_B$ then the following subordination relation holds true:

$$\frac{zg'(z)}{g(z)} \prec e^{e^z-1},$$

or equivalently

$$\frac{zg'(z)}{g(z)} = e^{e^{\phi(z)}-1} \quad (z \in \Delta), \tag{2.8}$$

where ϕ is a Schwarz function. With a simple calculation and since $|\phi(z)| \leq |z|$, (2.8) yields that

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{r}{e^{e^{-r}-1}} \quad (|z| = r < 1). \tag{2.9}$$

On the other hand we have $f(z) \ll g(z)$ in Δ. Therefore by (2.4), (2.5), (2.9) and Lemma 2.1 we get

$$\begin{aligned}|f'(z)| &\leq \left(|\psi(z)| + \frac{1-|\psi(z)|^2}{1-r^2} \times \frac{r}{e^{e^{-r}-1}}\right)|g'(z)| \\ &= \left(\gamma + \frac{1-\gamma^2}{1-r^2} \times \frac{r}{e^{e^{-r}-1}}\right)|g'(z)|,\end{aligned}$$

where $|\psi(z)| =: \gamma \in [0,1]$. We define

$$\eta(\gamma, r) := \gamma + \frac{1-\gamma^2}{1-r^2} \times \frac{r}{e^{e^{-r}-1}}.$$

Therefore we are looking for r_2 such that (2.6) holds. It is sufficient to consider r_2 as follows:

$$r_2 = \max\{r \in [0,1) : \eta(\gamma, r) \leq 1, \forall \gamma \in [0,1]\}.$$

Thus

$$\eta(\gamma, r) \leq 1 \Leftrightarrow \theta(\gamma, r) \geq 0,$$

where $\theta(\gamma, r) := (1-r^2)(e^{e^{-r}-1}) - r(1+\gamma)$. We see that $\frac{\partial \theta}{\partial \gamma} = -r < 0$. In conclusion, $\theta(\gamma, r)$ gets its minimum value in $\gamma = 1$, namely

$$\min\{\theta(\gamma, r) : \gamma \in [0,1]\} = \theta(1, r) = \theta(r),$$

where $\theta(r) := (1-r^2)(e^{e^{-r}-1}) - 2r$. We have $\theta(0) = 1 > 0$ and $\theta(1) = -2 < 0$. So there exists a r_2 such that $\theta(r) \geq 0$ for all $r \in [0, r_2]$ where r_2 is the smallest positive root of the Eq. (2.7). This completes the proof. □

If we let $g(z) = z$ in the above Theorem 2.4, then we get the following.

Corollary 2.5. *If a function $f \in \mathcal{A}$ satisfies the condition*

$$|f(z)| < 1 \quad (z \in \Delta),$$

then $|f'(z)| \leq 1$ for all z which $|z| \leq r_2$, where r_2 is the smallest positive root of the Eq. (2.7).

Remark 2.6. Figure 2 shows the roots r_1 and r_2 in Theorem 2.2 and Theorem 2.4, respectively, are approximately equal.

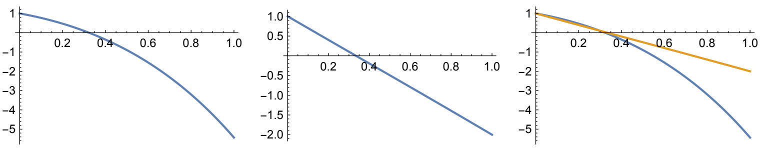


FIGURE 2. graph of Eq. (2.1) (left), graph of Eq. (2.7) (centre), graph of both Eqs. (2.1) and (2.7) (right)

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