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Fractional Hadamard and Fejér-Hadamard inequalities for exponentially m-convex function

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Abstract. Fractional integral operators play a vital role in the advancement of mathematical inequalities. The aim of this paper is to present the Hadamard and the Fejér-Hadamard inequalities for generalized fractional integral operators containing Mittag-Leffler function. Exponentially m-convexity is utilized to establish these inequalities. By fixing parameters involved in the Mittag-Leffler function Hadamard and the Fejér-Hadamard inequalities for various well known fractional integral operators can be obtained.

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1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex, if for all $x, y \in [a, b]$ and $z \in [0, 1]$, the following inequality holds:

$$f(zx + (1 - z)y) \leq zf(x) + (1 - z)f(y). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be concave.
Convex functions are equivalently defined by well known Hadamard inequality stated as follows:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function such that $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Fejér-Hadamard inequality is a weighted version of the Hadamard inequality established by Fejér [13].

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}$ be a non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.3}$$

Many researchers are continuously working on inequalities (1.2) and (1.3), and have produced very interesting results for convex and related functions (for example see, [1, 5, 6, 7, 8, 9, 12, 11, 14, 21, 22]).
Next we define exponentially convex function.

Definition 1.3. [4, 7] A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially convex, if for all $x, y \in [a,b]$ and $z \in [0,1]$, the following inequality holds:

$$e^{f(zx+(1-z)y)} \le ze^{f(x)} + (1-z)e^{f(y)}. \tag{1.4}$$

The concept of exponentially m-convex functions was introduced by Rashid et al. in [18]. It is defined as follows:

Definition 1.4. A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially m-convex, where $m \in (0,1]$, if for all $x, y \in [a,b]$ and $z \in [0,1]$, the following inequality holds:

$$e^{f(zx+m(1-z)y)} \le ze^{f(x)} + m(1-z)e^{f(y)}. \tag{1.5}$$

Remark 1.5. If we take $m = 1$ in (1.5), then (1.4) is achieved.

Mittag-Leffler function was introduced by the Swedish mathematician [15]. It is defined as follows:

$$E_\sigma(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\sigma n+1)},$$

where $\Gamma(.)$ is the gamma function and $t, \sigma \in \mathbb{C}$, $\Re(\sigma) > 0$.
In the solution of kinetic equations and fractional differential equations the Mittag-Leffler function arises naturally. It is generalized by many researchers due to it's importance. Recently in [3], Andrić et al. introduced generalized Mittag-Leffler function defined as follows:

Definition 1.6. [3] Let $\mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \ge 0$, $r > 0$ and $0 < q \le r + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\rho,r,q,c}_{\mu,\sigma,l}(t;p)$ is defined by:

$$E^{\rho,r,q,c}_{\mu,\sigma,l}(t;p) = \sum_{n=0}^{\infty}\frac{\beta_p(\rho+nq, c-\rho)}{\beta(\rho, c-\rho)}\frac{(c)_{nq}}{\Gamma(\mu n+\sigma)}\frac{t^n}{(l)_{nr}}, \tag{1.6}$$

where β_p is the generalized beta function defined by:

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

and $(c)_{nq}$ is the Pochhammer symbol defined as $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$.

Remark 1.7. (1.6) is a generalization of the following Mittag-Leffler functions defined by many authors:
(i) taking $p = 0$, it reduces to the Salim-Faraj function $E^{\rho,r,q,c}_{\mu,\sigma,l}(t)$ defined in [20],
(ii) taking $l = r = 1$, it reduces the function $E^{\rho,q,c}_{\mu,\sigma}(t;p)$ defined by Rahman et al. in [17],
(iii) taking $p = 0$ and $l = r = 1$, it reduces to the Shukla-Prajapati function $E^{\rho,q}_{\mu,\sigma}(t)$ defined in [23] see also [24],
(iv) taking $p = 0$ and $l = r = q = 1$, it reduces to the Prabhakar function $E^{\rho}_{\mu,\sigma}(t)$ defined in [16].

The left-sided and right-sided generalized fractional integral operators containing Mittag-Leffler function (1.6) are defined as follows:

Definition 1.8. [3] Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} f$ and $\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} f$ are defined by:

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p) f(t)dt, \tag{1.7}$$

and

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} f\right)(x;p) = \int_x^b (t-x)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega(t-x)^{\mu};p) f(t)dt. \tag{1.8}$$

Remark 1.9. (1.7) and (1.8) are the generalization of the following fractional integral operators defined by many authors:
(i) taking $p = 0$, it reduces to the fractional integral operators defined by Salim-Faraj in [20],
(ii) taking $l = r = 1$, it reduces to the fractional integral operators defined by Rahman et al. in [17],
(iii) taking $p = 0$ and $l = r = 1$, it reduces to the fractional integral operators defined by Srivastava-Tomovski in [24],
(iv) taking $p = 0$ and $l = r = q = 1$, it reduces to the fractional integral operators defined by Prabhakar in [16],
(v) taking $p = \omega = 0$, it reduces to the right-sided and left-sided Riemann-Liouville fractional integrals.

As shown in [2] also [10], for the constant function we have:

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} 1\right)(x;p) = (x-a)^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma+1,l}(\omega(x-a)^{\mu};p) := G_{\sigma,\omega,a^+}(x;p), \tag{1.9}$$

and

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} 1\right)(x;p) = (b-x)^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma+1,l}(\omega(b-x)^{\mu};p) := G_{\sigma,\omega,b^-}(x;p), \tag{1.10}$$

which we use in our results.

In the upcoming section, first we prove the Hadamard inequality for exponentially m-convex functions via fractional integral operators defined in (1.7) and (1.8). Also, the Fejér-Hadamard inequality for these operators is obtained. We mention results for particular fractional integral operators associated with (1.7) and (1.8).

2. Fractional Hadamard and Fejér-Hadamard inequalities for generalized fractional integral operators

First we give the fractional Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 2.1. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$\begin{aligned} & e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\bar{\omega},a^+}(mb;p) \\ & \leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega},a^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^f\right)\left(\frac{a}{m};p\right)}{2} \\ & \leq \frac{m^{\sigma+1}}{2(mb-a)}\left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\bar{\omega}m^\mu,b^-}\left(\frac{a}{m};p\right)\right. \\ & \left. +(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\bar{\omega}m^\mu,b^-}\left(\frac{a}{m};p\right)\right] \end{aligned} \tag{2.1}$$

where $m \in (0,1]$ and

$$\bar{\omega} = \frac{\omega}{(mb-a)^\mu}. \tag{2.2}$$

Proof. Since f is exponentially m-convex, we have

$$e^{f\left(\frac{x+my}{2}\right)} \leq \frac{e^{f(x)} + m e^{f(y)}}{2} \quad \forall\ x, y \in [a, mb] \ \ \textit{and} \ \ m \in (0,1]. \tag{2.3}$$

Putting $x = za + m(1-z)b$ and $y = (1-z)\frac{a}{m} + zb$ in (2.3), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f(za+m(1-z)b)} + m e^{f((1-z)\frac{a}{m}+zb)}. \tag{2.4}$$

Also from exponentially m-convexity of f, we have

$$\begin{aligned} & e^{f(za+m(1-z)b)} + m e^{f((1-z)\frac{a}{m}+zb)} \\ & \leq z e^{f(a)} + m(1-z)e^{f(b)} + m\left(m(1-z)e^{f\left(\frac{a}{m^2}\right)} + z e^{f(b)}\right) \\ & = z\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) + m\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right). \end{aligned} \tag{2.5}$$

Multiplying both sides of (2.4) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)$ and integrating over $[0,1]$, we have

$$\begin{aligned} & 2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)dz \\ & \leq \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f(za+m(1-z)b)}dz \\ & + m\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f((1-z)\frac{a}{m}+zb)}dz. \end{aligned} \tag{2.6}$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (2.6), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_a^{mb} (mb-u)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)du$$
$$\leq \int_a^{mb} (mb-u)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du$$
$$+ m^{\sigma+1} \int_{\frac{a}{m}}^{b} \left(v - \frac{a}{m}\right)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l} \left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right) e^{f(v)}dv.$$

By using (1.7), (1.8) and (1.9), first inequality of (2.1) is achieved.
Now multiplying both sides of (2.5) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f(za+m(1-z)b)}dz \tag{2.7}$$
$$+ m \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f((1-z)\frac{a}{m}+zb)}dz$$
$$\leq \left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right) \int_0^1 z^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz$$
$$+ m\left(e^{f(b)} + m e^{f(\frac{a}{m^2})}\right) \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz.$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (2.7), we get

$$\int_a^{mb} (mb-u)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du$$
$$+ m^{\sigma+1} \int_{\frac{a}{m}}^{b} \left(v-\frac{a}{m}\right)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right) e^{f(v)}dv$$
$$\leq \frac{m^{\sigma+1}}{(mb-a)} \Bigg[\left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right) \int_{\frac{a}{m}}^{b} \left(v-\frac{a}{m}\right)^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv$$
$$+(mb-a)\left(e^{f(b)} + m e^{f(\frac{a}{m^2})}\right) \int_{\frac{a}{m}}^{b} \left(v-\frac{a}{m}\right)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\Bigg].$$

By using (1.7), (1.8) and (1.10), second inequality of (2.1) is achieved. □

Corollary 2.2. *Suppose that assumptions of Theorem* 2.1 *hold and let* $m = 1$. *Then following inequalities for exponentially convex function hold:*

$$e^{f\left(\frac{a+b}{2}\right)} G_{\sigma,\omega^*,a^+}(b;p) \leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,a^+} e^f\right)(b;p) + \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,b^-} e^f\right)(a;p)}{2}$$
$$\leq \frac{e^{f(a)} + e^{f(b)}}{2} G_{\sigma,\omega^*,b^-}(a;p)$$

where

$$\omega^* = \frac{\omega}{(b-a)^{\mu}}. \tag{2.8}$$

In [19], S. Rashid et. al. prove the following Hadamard inequality for exponentially m-convex function which has several misprints.

Theorem 2.3. *Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$2e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\left(\frac{a+mb}{2}\right)^+}(mb;p) \tag{2.9}$$
$$\leq \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^{\mu},\left(\frac{a+mb}{2}\right)^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^{\mu},\left(\frac{a+mb}{2m}\right)^-} e^f\right)\left(\frac{a}{m};p\right)$$
$$\leq \frac{a}{(mb-a)}\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\left(\frac{a+mb}{2}\right)^+}(mb;p)$$
$$+ m^{\sigma+1}\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\left(\frac{a+mb}{2m}\right)^-}.$$

The correct form of the above theorem is stated and proved in the following theorem.

Theorem 2.4. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\bar{\omega}2^{\mu},\left(\frac{a+mb}{2}\right)^+}(mb;p) \tag{2.10}$$
$$\leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^{\mu},\left(\frac{a+mb}{2}\right)^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}(2m)^{\mu},\left(\frac{a+mb}{2m}\right)^-} e^f\right)\left(\frac{a}{m};p\right)}{2}$$
$$\leq \frac{m^{\sigma+1}}{2(mb-a)}\Bigg[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\bar{\omega}(2m)^{\mu},\left(\frac{a+mb}{2m}\right)^-}\left(\frac{a}{m};p\right)$$
$$+(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\bar{\omega}(2m)^{\mu},\left(\frac{a+mb}{2m}\right)^-}\left(\frac{a}{m};p\right)\Bigg]$$

where $m \in (0, 1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Putting $x = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $y = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (2.3), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f\left(\frac{z}{2}a + m\frac{(2-z)}{2}b\right)} + m e^{f\left(\frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}\right)}. \tag{2.11}$$

Multiplying both sides of (2.11) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)$ and integrating over $[0, 1]$, we have

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz \tag{2.12}$$
$$\leq \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p) e^{f\left(\frac{z}{2}a + \frac{(2-z)}{2}b\right)}dz$$
$$+ m\int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p) e^{f\left(\frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}\right)}dz.$$

Putting $u=\frac{z}{2}a+m\frac{(2-z)}{2}b$ and $v=\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m}$ in (2.12), we get

$$\begin{aligned}&2e^{f\left(\frac{a+mb}{2}\right)}\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)du\\&\leq\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du\\&+m^{\sigma+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}dv.\end{aligned}$$

By using (1.7), (1.8) and (1.9), first inequality of (2.10) is achieved.
From exponentially m-convexity of f, we have

$$\begin{aligned}&e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)}+me^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})}\\&\leq\frac{z}{2}e^{f(a)}+m\frac{(2-z)}{2}e^{f(b)}+m\left(\frac{z}{2}e^{f(b)}+m\frac{(2-z)}{2}e^{f(\frac{a}{m^2})}\right)\\&=\frac{z}{2}\left(e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right)+m\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right).\end{aligned}\tag{2.13}$$

Multiplying both sides of (2.13) with $z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)}dz\\&+m\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})}dz\\&\leq\frac{1}{2}\left(e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)dz\\&+m\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)dz.\end{aligned}\tag{2.14}$$

Putting $u=\frac{z}{2}a+m\frac{(2-z)}{2}b$ and $v=\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m}$ in (2.14), we get

$$\begin{aligned}&\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du\\&+m^{\sigma+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}dv\\&\leq\frac{m^{\sigma+1}}{mb-a}\left[\left(e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma}E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right.\\&\left.+(mb-a)\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right].\end{aligned}$$

By using (1.7), (1.8) and (1.10), second inequality of (2.10) is achieved. □

Corollary 2.5. *Suppose that assumptions of Theorem 2.4 hold and let $m = 1$. Then following inequalities for exponentially convex function hold:*

$$\begin{aligned}
&e^{f\left(\frac{a+b}{2}\right)} G_{\sigma,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^+}(b;p) \\
&\leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^+} e^f\right)(b;p) + \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^-} e^f\right)(a;p)}{2} \\
&\leq \frac{e^{f(a)} + e^{f(b)}}{2} G_{\sigma,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^-}(a;p)
\end{aligned}$$

where ω^* is defined in (2.8).

Remark 2.6. If we take $\omega = p = 0$ in (2.10), then [18, Theorem 3.3] is obtained.

In the following we give Fejér-Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 2.7. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. Also, let $g : [a, mb] \to \mathbb{R}$ be a function which is non-negative and integrable. If f is exponentially m-convex function and $f(v) = f(a + mb - mv)$, then the following inequalities hold:*

$$\begin{aligned}
&e^{f\left(\frac{a+mb}{2}\right)} \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right) \\
&\leq \frac{(1+m)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^f e^g\right)\left(\frac{a}{m};p\right)}{2} \\
&\leq \frac{m}{2(mb-a)} \left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma+1,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right)\right. \\
&\left. +(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right)\right]
\end{aligned} \tag{2.15}$$

where $m \in (0, 1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Multiplying both sides of (2.4) with $z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu; p) e^{g((1-z)\frac{a}{m}+zb)}$ and integrating over $[0, 1]$, we have

$$\begin{aligned}
&2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu; p) e^{g((1-z)\frac{a}{m}+zb)} dz \\
&\leq \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu; p) e^{f(za+m(1-z)b)} e^{g((1-z)\frac{a}{m}+zb)} dz \\
&+ m \int_0^1 z^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu; p) e^{f((1-z)\frac{a}{m}+zb)} e^{g((1-z)\frac{a}{m}+zb)} dz.
\end{aligned} \tag{2.16}$$

Putting $v = (1-z)\frac{a}{m} + zb$ in (2.16), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_{\frac{a}{m}}^{b} \left(v - \frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v - \frac{a}{m})^{\mu}; p\right) e^{g(v)} dv$$
$$\leq \int_{\frac{a}{m}}^{b} \left(v - \frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v - \frac{a}{m})^{\mu}; p\right) e^{f(a+mb-mv)} e^{g(v)} dv$$
$$+ m \int_{\frac{a}{m}}^{b} \left(v - \frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v - \frac{a}{m})^{\mu}; p\right) e^{f(v)} e^{g(v)} dv.$$

By using (1.7), (1.8), (1.9) and given condition $f(v) = f(a+mb-mv)$, first inequality of (2.15) is achieved.
Now multiplying both sides of (2.5) with $z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(za+m(1-z)b)}e^{g((1-z)\frac{a}{m}+zb)}dz$$
$$+ m\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f((1-z)\frac{a}{m}+zb)}e^{g((1-z)\frac{a}{m}+zb)}dz.$$
$$\leq \left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right) \int_0^1 z^{\sigma} E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}dz$$
$$+ m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right) \int_0^1 z^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}dz.$$

From above second inequality of (2.15) is achieved. □

Corollary 2.8. *Suppose that assumptions of Theorem 2.7 hold and let $m = 1$. Then following inequalities for exponentially convex function hold:*

$$e^{f\left(\frac{a+b}{2}\right)} \left(\epsilon_{\mu,\sigma,l,\omega^*,b^-}^{\rho,r,q,c} e^{g}\right)(a;p) \leq \left(\epsilon_{\mu,\sigma,l,\omega^*,b^-}^{\rho,r,q,c} e^{f} e^{g}\right)(a;p)$$
$$\leq \frac{e^{f(a)} + e^{f(b)}}{2} \left(\epsilon_{\mu,\sigma,l,\omega^*,b^-}^{\rho,r,q,c} e^{g}\right)(a;p)$$

where ω^* is defined in (2.8).

Theorem 2.9. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f, g : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be the functions such that $f, g \in L_1[a, mb]$ with $a < mb$. If f and g are exponentially m-convex functions, then the following inequality holds:*

$$\left(\epsilon_{\mu,\sigma,l,\bar{\omega},mb^-}^{\rho,r,q,c} e^{f}\right)(a;p) + \left(\epsilon_{\mu,\sigma,l,\bar{\omega},a^+}^{\rho,r,q,c} e^{g}\right)(mb;p) \tag{2.17}$$
$$\leq \frac{1}{(mb-a)} \Big[\left(e^{g(a)} + me^{f(b)}\right) G_{\sigma+1,\bar{\omega},a^+}(mb;p)$$
$$+ \left(e^{f(a)} + me^{g(b)}\right) \left\{(mb-a)G_{\sigma,\bar{\omega},a^+}(mb;p) - G_{\sigma+1,\bar{\omega},a^+}(mb;p)\right\}\Big]$$

where $m \in (0, 1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Since f and g are exponentially m-convex, we have

$$e^{f((1-z)a+mzb)}+e^{g(za+m(1-z)b)}\leq(1-z)\left(e^{f(a)}+me^{g(b)}\right)+z\left(e^{g(a)}+me^{f(b)}\right). \tag{2.18}$$

Multiplying both sides of (2.18) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f((1-z)a+mzb)}dz \\ &+\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{g(za+m(1-z)b)}dz \\ &\leq\left(e^{f(a)}+me^{g(b)}\right)\int_0^1(1-z)z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz \\ &+\left(e^{g(a)}+me^{f(b)}\right)\int_0^1 z^{\sigma}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz.\end{aligned} \tag{2.19}$$

Putting $u=(1-z)a+mzb$ and $v=za+m(1-z)b$ in (2.19), then by using (1.7), (1.8) and (1.9), inequality (2.17) is achieved. □

Corollary 2.10. *Suppose that assumptions of Theorem* 2.9 *hold and let* $m=1$. *Then following inequality for exponentially convex function holds:*

$$\begin{aligned}&\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,b^-}e^{f}\right)(a;p)+\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,a^+}e^{g}\right)(b;p) \\ &\leq\frac{1}{(b-a)}\left[\left(e^{g(a)}+e^{f(b)}\right)G_{\sigma+1,\omega^*,a^+}(b;p)\right. \\ &\left.+\left(e^{f(a)}+e^{g(b)}\right)\left\{(b-a)G_{\sigma,\omega^*,a^+}(b;p)-G_{\sigma+1,\omega^*,a^+}(b;p)\right\}\right].\end{aligned}$$

where ω^* *is defined in (*2.8*).*

Remark 2.11. If we take $\omega=p=0$ in (2.17), then [18, Theorem 3.2] is obtained.

Concluding remarks. The aim of this paper is to establish two versions of the fractional Hadamard inequalities for exponentially m-convex functions via generalized fractional integral operators. Further, a generalized version of the Hadamard inequality so called Fejér-Hadamard inequality is proved. The results of this paper are hold for various associated fractional integral operators.

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