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Bounds for blow-up time in a semilinear parabolic problem with variable exponents

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Abstract. This report deals with a blow-up of the solutions to a class of semilinear parabolic equations with variable exponents nonlinearities. Under some appropriate assumptions on the given data, a more general lower bound for a blow-up time is obtained if the solutions blow up. This result extends the recent results given by Baghaei Khadijeh et al. [8], which ensures the lower bounds for the blow-up time of solutions with initial data $\varphi(0) = \int_\Omega {u_0}^k dx$, $k =$ constant.

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1. Introduction

In this paper, we are concerned with the following semilinear parabolic equation

$$\begin{cases} u_t - \Delta u = u^{p(x)}, \quad x \in \Omega,\ t > 0 \\ u = 0 \text{ on } \Gamma,\quad t \geq 0 \\ u(x,0) = u_0(x) \geq 0,\ x \in \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^n$, with a smooth boundary $\Gamma = \partial\Omega$, $T \in (0, +\infty]$, and the initial value $u_0 \in H_0^1(\Omega)$, the exponent $p(.)$ is given measurable function on $\overline{\Omega}$ such that:

$$1 < p_1 = ess \inf_{x\in\Omega} p(x) \leq p(x) \leq p_2 = ess \sup_{x\in\Omega} p(x) < \infty, \tag{1.2}$$

and satisfy the following Zhikov-Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0. \tag{1.3}$$

The problem (1.1) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer,

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population dynamics, biological sciences, etc., and have interested a great deal of attention in the research, see [4, 7, 9] and the references therein. For problem (1.1), Hua Wang et al. [10] established a blow-up result with positive initial energy under some suitable assumptions on the parameters $p(.)$ and u_0. In [9], the authors proved that there are non-negative solutions with a blow-up in finite time if and only if $p_2 > 1$. The authors in [11] obtained the solution of problem (1.1) blows up in finite time when the initial energy is positive. The following problem was considered by R. Abita in [3]

$$u_t - \Delta u_t - \Delta u = u^{p(x)}, \quad x \in \Omega,\ t > 0.$$

The author proved that the nonnegative classical solutions blow-up in finite time with arbitrary positive initial energy and suitable large initial values. Also, he employed a differential inequality technique to obtain an upper bound for blow-up time if $p(.)$ and the initial value satisfies some conditions. In [8], the authors based exactly on the idea on the one in [6], derived the lower bounds for the time of blow-up, if the solutions blow-up. In order to declare the main results of this paper, we need to add the following energy functional corresponding to the problem (1.1) (see [2])

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u(t)|^2\, dx - \int_\Omega \frac{1}{p(x)+1} u(x,t)^{p(x)+1}\, dx. \tag{1.4}$$

2. Lower bounds of the blow-up time

In this section, we investigate the lower bound for the blow-up time T in some suitable measure. The idea of the proof of the following theorem is inspired by on the one in [6]. For this goal, we start by the following lemma concerning the energy of the solution.

Lemma 2.1. *Let $u(x,t)$ be a weak solution of (1.1), then $E(t)$ is a nonincreasing function on $[0,T]$, that is*

$$\frac{dE(t)}{dt} = -\int_\Omega u_t^2(x,t)\, dx \le 0 \tag{2.1}$$

and the inequality $E(t) \le E(0)$ is satisfied.

We consider the following partition of Ω,

$$\Omega^- = \{x \in \Omega \mid 1 > (k(x)-1)\ln|u|\},\quad \Omega^+ = \{x \in \Omega \mid 1 \le (k(x)-1)\ln|u|\},\ \forall t > 0$$

where each $\Omega^\pm$ depends on t, and setting

$$\widetilde{E}(0) = \frac{1}{2}\int_{\Omega^-} |\nabla u_0|^2\, dx - \int_{\Omega^-} \frac{1}{p(x)+1} u_0^{p(x)+1} dx.$$

Now, we are in a position to affirm our principal theorem results.

Theorem 2.2. *Assume $u_0 \in L^{k(\cdot)}(\Omega)$, and the nonnegative weak solution $u(x,t)$ of problem (1.1) blows up in finite time T, then T has a lower bound by:*

$$\int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2\gamma^{\frac{3n-6}{3n-8}}}, \tag{2.2}$$

where

$$\varphi(0) = \int_\Omega \frac{1}{k(x)(k(x)-1)} {u_0}^{k(x)} dx, \tag{2.3}$$

where $k(.)$ *is a measurable function on* $\overline{\Omega}$ *such that*

$$\begin{aligned} \max(1, 2(n-2)(p_2-1)) < k_1 = ess \inf_{x\in\Omega} k(x) \leq k(x) \leq k_2 \\ = ess \sup_{x\in\Omega} k(x) < \infty, \end{aligned} \tag{2.4}$$

and

$$\sqrt{C_k} = \sup_{x\in\overline{\Omega}} |\nabla k(x)| \in L^2(\Omega), \ C_k > 0 \tag{2.5}$$

and C_i $(i = 1, \ 2)$ *are positive constants will be described later.*

Notation 2.3. We note that the presence of the variable-exponent nonlinearities in (2.6) below, makes analysis in the paper somewhat harder than that in the related ones, we will establish and give a precise estimate for the lifespan T of the solution in this case. The method used here is the differential inequality technique. However, our argument is considerably different and it is more abbreviated.

Proof of Theorem (2.2). Set

$$\varphi(t) = \int_\Omega \frac{1}{k(x)(k(x)-1)} u(x,t)^{k(x)} dx. \tag{2.6}$$

Multiplying the equation Eq. (1.1) by u and integrating by parts, we see

$$\begin{aligned} \varphi'(t) &= \int_\Omega \frac{1}{k(x)-1} u^{k(x)-1} u_t dx = \int_\Omega \frac{1}{k(x)-1} u^{k(x)-1} \left(\Delta u + u^{p(x)}\right) dx \\ &= \int_\Omega \frac{1}{k(x)-1} u^{k(x)-1} \Delta u dx + \int_\Omega \frac{1}{k(x)-1} u^{k(x)+p(x)-1} dx \\ &= -\int_\Omega \nabla\left(\frac{1}{k(x)-1} u^{k(x)-1}\right) \nabla u dx + \int_\Omega \frac{1}{k(x)-1} |u|^{k(x)+p(x)-1} dx \end{aligned}$$

where we have used the divergence theorem, the boundary condition on u.
It is straightforward to check that

$$\nabla\left(\frac{1}{k(x)-1} u^{k(x)-1}\right) = u^{k(x)} |u|^{-2} \nabla u + \frac{\nabla k(x)}{k(x)-1} u^{k(x)-1} \left(\ln|u| - \frac{1}{k(x)-1}\right)$$

then, we get

$$\varphi'(t) = -\int_\Omega u^{k(x)} |u|^{-2} |\nabla u|^2 dx + \int_\Omega \frac{1}{k(x)-1} u^{k(x)+p(x)-1} dx + \mathcal{Q} \tag{2.7}$$

where

$$\mathcal{Q} = \int_\Omega u^{k(x)-1} \left(\frac{1}{(k(x)-1)^2} - \frac{1}{(k(x)-1)} \ln|u|\right) \nabla k(x) . \nabla u dx$$

Considering the following properties of the function $\mathcal{G}$,

$$\mathcal{G}(\lambda) = \frac{\lambda^{\gamma}}{\gamma^2}(1-\gamma\ln\lambda), \quad 0 \leq \lambda \leq e^{\frac{1}{\gamma}};$$

$$\mathcal{G}(0) = \mathcal{G}\left(e^{\frac{1}{\gamma}}\right) = 0,\ \mathcal{G}'(\lambda) = -\lambda^{\gamma-1}\ln\lambda, \quad \max_{0\leq\lambda\leq e^{\frac{1}{\gamma}}} \mathcal{G}(\lambda) = \mathcal{G}(1) = \frac{1}{\gamma^2},$$

and using the fact that

$$\int_{\Omega^-} |\nabla u|^2\,dx \leq 2\widetilde{E}(0) + 2\int_{\Omega^-} \frac{1}{p(x)+1} u(x,t)^{p(x)+1}\,dx, \text{ (by (1.4) and (2.1))}$$

applying the Hölder, Young inequalities and (2.5), $\mathcal{Q}$ is evaluated as follows:

$$\mathcal{Q} = \int_{\Omega} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$= \int_{\Omega\cap(1>(k(x)-1)\ln|u(x,t)|)} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$\int_{\Omega\cap(1\leq(k(x)-1)\ln|u(x,t)|)} u^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)\nabla k(x).\nabla u dx$$

$$\leq \int_{\Omega^-} \frac{1}{(k(x)-1)^2}|u|^{k(x)-1}(1-(k(x)-1)\ln|u|)|\nabla u||\nabla k(x)|\,dx$$

$$\leq \int_{\Omega^-} \frac{1}{(k(x)-1)^2}|\nabla k(x)||\nabla u|\,dx \leq \frac{1}{2(k_1-1)^2}\left(C_k + \int_{\Omega^-}|\nabla u|^2\,dx\right)$$

$$\leq \frac{1}{2(k_1-1)^2}\left(C_k + 2E(0) + 2\int_{\Omega^-}\frac{1}{p(x)+1}u(x,t)^{p(x)+1}\,dx\right)$$

$$\leq \frac{1}{2(k_1-1)^2}\left(C_k + 2E(0) + \frac{2}{p_1+1}\max\left(\int_{\Omega^-}|u|^{p_2+1}\,dx, \int_{\Omega^-}|u|^{p_1+1}\,dx\right)\right)$$

$$\leq \frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k + E(0) + \frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right). \tag{2.8}$$

Because in Ω^+, we have

$$\int_{\Omega^+} |u|^{k(x)-1}\left(\frac{1}{(k(x)-1)^2} - \frac{1}{k(x)-1}\ln|u|\right)|\nabla k(x)|\,dx \leq 0,$$

while that of the first term in the right-hand side of (2.7) was estimated as follows

$$-\int_{\Omega}|u|^{k(x)-2}|\nabla u|^2\,dx \leq -\min\left(\int_{\Omega}|u|^{k_2-2}|\nabla u|^2\,dx, \int_{\Omega}|u|^{k_1-2}|\nabla u|^2\,dx\right).$$

Using the fact

$$|\nabla u^{\gamma}| = \gamma u^{\gamma-1}|\nabla u|$$

to get

$$-\int_{\Omega}|u|^{k(x)-2}|\nabla u|^{2}dx\leq-\min\left(\frac{4}{(k_2)^2}\int_{\Omega}\left|\nabla u^{\frac{k_2}{2}}\right|^{2}dx,\frac{4}{(k_1)^2}\int_{\Omega}\left|\nabla u^{\frac{k_1}{2}}\right|^{2}dx\right) \tag{2.9}$$

Plugging this estimate (2.8) and (2.9) into (2.7), we obtain

$$\begin{aligned}\varphi'(t)\leq{}&\min\left(\frac{-4}{(k_2)^2}\int_{\Omega}\left|\nabla u^{\frac{k_2}{2}}\right|^{2}dx,\frac{-4}{(k_1)^2}\int_{\Omega}\left|\nabla u^{\frac{k_1}{2}}\right|^{2}dx\right)\\&+\frac{1}{k_1-1}\int_{\Omega}u^{k(x)+p_2-1}dx+\frac{1}{k_1-1}\int_{\Omega}u^{k(x)+p_1-1}dx\\&+\frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k+E(0)+\frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right)\end{aligned} \tag{2.10}$$

By using (2.4), we can apply the Hölder and Young inequalities to get

$$\begin{aligned}\int_{\Omega}u^{k(x)+p_2-1}dx&\leq\int_{\Omega}1.\alpha_1dx+\int_{\Omega}\alpha_2.u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\&\leq(\sup\alpha_1)|\Omega|+(\sup\alpha_2)\left(\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\right),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\int_{\Omega}u^{k(x)+p_1-1}dx&\leq\int_{\Omega}1.\alpha_3dx+\int_{\Omega}\alpha_4.u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\&\leq\left(\sup_{\Omega}\alpha_3\right)|\Omega|+\left(\sup_{\Omega}\alpha_4\right)\left(\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\right),\end{aligned} \tag{2.12}$$

where

$$\alpha_1=1-\frac{2(n-2)(k(x)+p_2-1)}{(2n-3)k(x)},\quad\alpha_2=\frac{2(n-2)(k(x)+p_2-1)}{(2n-3)k(x)},$$
$$\alpha_3=1-\frac{2(n-2)(k(x)+p_1-1)}{(2n-3)k(x)},\quad\alpha_4=\frac{2(n-2)(k(x)+p_1-1)}{(2n-3)k(x)};$$

observe that $\alpha_2\geq\alpha_4$ and $\alpha_1\leq\alpha_3$.

Combining (2.11) and (2.12) with (2.10) give

$$\begin{aligned}\varphi'(t)\leq{}&\frac{-1}{2}\frac{4}{(k_2)^2}\left(\int_{\Omega}\left|\nabla u^{\frac{k_2}{2}}\right|^{2}dx+\int_{\Omega}\left|\nabla u^{\frac{k_1}{2}}\right|^{2}dx\right)\\&+\frac{2}{k_1-1}\left(\sup_{\Omega}\alpha_2\right)\int_{\Omega}u^{\frac{k(x)(2n-3)}{2(n-2)}}dx\\&+\frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k+E(0)+\frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right)+\frac{|\Omega|}{k_1-1}\sup_{\Omega}(\alpha_3+\alpha_1)\end{aligned} \tag{2.13}$$

We now make use of Schwarz's inequality to the second term on the right-hand side of (2.13) as follows

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \le \left(\int_\Omega u^{k(x)}dx\right)^{\frac{1}{2}}\left(\int_\Omega u^{\frac{k(x)(n-1)}{n-2}}dx\right)^{\frac{1}{2}} \tag{2.14}$$
$$\le \left(\int_\Omega u^{k(x)}dx\right)^{\frac{3}{4}}\left(\int_\Omega \left(u^{\frac{k(x)}{2}}\right)^{\frac{2n}{n-2}}dx\right)^{\frac{1}{4}},$$

Next, by using the Sobolev inequality (see [5]), for $n \ge 3$, we get

$$\left\|u^{\frac{k(x)}{2}}\right\|_{\frac{2n}{n-2}}^{\frac{n}{2(n-2)}} \le B^{\frac{n}{2(n-2)}}\max\left(\left\|\nabla u^{\frac{k_2}{2}}\right\|_2^{\frac{n}{2(n-2)}},\left\|\nabla u^{\frac{k_1}{2}}\right\|_2^{\frac{n}{2(n-2)}}\right) \tag{2.15}$$
$$\le B^{\frac{n}{2(n-2)}}\left(\left\|\nabla u^{\frac{k_2}{2}}\right\|_2^{\frac{n}{2(n-2)}}+\left\|\nabla u^{\frac{k_1}{2}}\right\|_2^{\frac{n}{2(n-2)}}\right),$$

where B is the best constant in the Sobolev inequality.
By inserting the last inequality in (2.14) and (2.15), we have

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \le$$
$$\le B^{\frac{n}{2(n-2)}}\left(\int_\Omega u^{k(x)}dx\right)^{\frac{3}{4}}\left(\left(\int_\Omega\left|\nabla u^{\frac{k_2}{2}}\right|^2dx\right)^{\frac{n}{4(n-2)}}+\left(\int_\Omega\left|\nabla u^{\frac{k_1}{2}}\right|^2dx\right)^{\frac{n}{4(n-2)}}\right),$$

Now, we can use the Young inequality to get

$$\int_\Omega u^{\frac{k(x)(2n-3)}{2(n-2)}}dx \le 2B^{\frac{2n}{3n-8}}\frac{3n-8}{4(n-2)\varepsilon^{\frac{n}{3n-8}}}\left(\int_\Omega u^{k(x)}dx\right)^{\frac{3(n-2)}{3n-8}} \tag{2.16}$$
$$+\frac{\varepsilon n}{4(n-2)}\left(\int_\Omega\left|\nabla u^{\frac{k_1}{2}}\right|^2dx+\int_\Omega\left|\nabla u^{\frac{k_2}{2}}\right|^2dx\right)$$

where ε is a positive constant to be determined later. Combining (2.16) with (2.13), we obtain

$$\varphi'(t) \le C_1 + C_2\varphi(t)^{\frac{3(n-2)}{3n-8}} + C_3\left(\int_\Omega\left|\nabla u^{\frac{k_2}{2}}\right|^2dx+\int_\Omega\left|\nabla u^{\frac{k_1}{2}}\right|^2dx\right),$$

where

$$C_1 = \frac{1}{(k_1-1)^2}\left(\frac{1}{2}C_k + E(0) + \frac{1}{p_1+1}e^{\frac{p_2+1}{k_1-1}}|\Omega|\right)+\frac{|\Omega|}{k_1-1}\sup_\Omega(\alpha_3+\alpha_1)$$
$$C_2 = \frac{4}{k_1-1}\left(\sup_\Omega\alpha_2\right)B^{\frac{2n}{3n-8}}\frac{3n-8}{4(n-2)\varepsilon^{\frac{n}{3n-8}}},$$
$$C_3 = \frac{2}{k_1-1}\frac{\varepsilon n}{4(n-2)}\left(\sup_\Omega\alpha_2\right)-\frac{2}{(k_2)^2}$$

If we choose $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \frac{4(n-2)(k_1-1)}{\left(\sup_{\Omega}\alpha_2\right) n (k_2)^2}$$

then, we obtain the differential inequality

$$\varphi'(t) \leq C_1 + C_2 \varphi(t)^{\frac{3(n-2)}{3n-8}} \tag{2.17}$$

Integration of the differential inequality (2.17) from 0 to t leads to

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\gamma}{C_1 + C_2 \gamma^{\frac{3(n-2)}{3n-8}}} \leq t \tag{2.18}$$

In fact, let $t \to T^-$, (2.18) leads to

$$\int_{\varphi(0)}^{+\infty} \frac{d\gamma}{C_1 + C_2 \gamma^{\frac{3(n-2)}{3n-8}}} \leq T.$$

where

$$\varphi(0) = \int_\Omega \frac{1}{k(x)(k(x)-1)} {u_0}^{k(x)} dx.$$

Thus, the proof is achieved. □

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