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Local C-semigroups and complete second order abstract Cauchy problems

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Abstract. Let $C : X \to X$ be an injective bounded linear operator on a Banach space X over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ and $0 < T_0 \leq \infty$. Under some suitable assumptions, we deduce some relationship between the generation of a local (or an exponentially bounded) $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and one of the following cases: (i) the well-posedness of a complete second-order abstract Cauchy problem ACP(A, B, f, x, y): $w''(t) = Aw'(t)+Bw(t)+f(t)$ for a.e. $t \in (0, T_0)$ with $w(0) = x$ and $w'(0) = y$; (ii) a Miyadera-Feller-Phillips-Hille-Yosida type condition; (iii) B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X for which A may not be bounded; (iv) A is a subgenerator (resp., the generator) of a local C-semigroup on X for which B may not be bounded.

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1. Introduction

Let X be a Banach space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with norm $\|\cdot\|$, and let $\mathrm{L}(X)$ denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following two abstract Cauchy problems:

$$\mathrm{ACP}(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ u(0) = x \end{cases}$$

and

$$\mathrm{ACP}(A, B, f, x, y) \qquad \begin{cases} w''(t) = Aw'(t) + Bw(t) + f(t) & \text{for a.e. } t \in (0, T_0) \\ w(0) = x, w'(0) = y, \end{cases}$$

where $x, y \in X$, $A : \mathrm{D}(A) \subset X \to X$ and $B : \mathrm{D}(B) \subset X \to X$ are closed linear operators, and $f \in \mathrm{L}^1_{loc}([0, T_0), X)$ (the family of all locally Bochner integrable functions from $[0, T_0)$ into X). A function u is called a (strong) solution of ACP(A, f, x) if $u \in \mathrm{C}([0, T_0), X)$ satisfies ACP(A, f, x) (that is $u(0) = x$ and for a.e. $t \in (0, T_0)$, $u(t)$ is differentiable and $u(t) \in \mathrm{D}(A)$, and $u'(t) = Au(t) + f(t)$ for a.e. $t \in (0, T_0)$). For each $\alpha > 0$ and each injection $C \in L(X)$, a subfamily $S(\cdot)(= \{S(t) | 0 \le t < T_0\})$ of $L(X)$ is called a local α-times integrated C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.1) \quad S(t)S(s)x = \frac{1}{\Gamma(\alpha)} \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] (t + s - r)^{\alpha-1} S(r)Cx dr$$

for all $x \in X$ and $0 \le t, s \le t + s < T_0$ (see [1-2,12-14,18-21,28,30,32,35]) or called a local (0-times integrated) C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$(1.2) \quad S(t)S(s)x = S(t + s)Cx$$

for all $x \in X$ and $0 \le t, s \le t + s < T_0$ (see [4,6,13,23,29]), where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $S(\cdot)$ is

(1.3) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|S(t + h) - S(t)\| \le K_{t_0} h$ for all $0 \le t, h, t + h \le t_0$;
(1.4) exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \ge 0$ such that $\|S(t)\| \le Ke^{\omega t}$ for all $t \ge 0$;
(1.5) nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $0 \le t < T_0$.

A nondegenerate local α-times integrated C-semigroup $S(\cdot)$ on X implies that $S(0) = C$ if $\alpha = 0$, and $S(0) = 0$ (zero operator on X) otherwise, and the (integral) generator $A : \mathrm{D}(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in X defined by $\mathrm{D}(A) = \{x \in X \mid S(\cdot)x - j_\alpha(\cdot)Cx = \widetilde{S}(\cdot)y_x$ on $[0, T_0)$ for some $y_x \in X\}$ and $Ax = y_x$ for all $x \in \mathrm{D}(A)$ (see [6,13-14,23]), which is also equal to the linear operator A in X defined by $D(A) = \{x \in X \mid \lim_{h \to 0^+} (S(h)x - Cx)/h \in R(C)\}$ and $Ax = C^{-1} \lim_{h \to 0^+} (S(h)x - Cx)/h$ for $x \in D(A)$ when $\alpha = 0$ (see [4,23,27]). Here $j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$ and $\widetilde{S}(t)z = \int_0^t S(s)z ds$. In general, a local C-semigroup is called a C-semigroup if $T_0 = \infty$(see [2,4,14,26,32]) or a C_0-semigroup if $C = I$ (identity operator on X) (see [1,5]). It is known that the theory of local C-semigroup is related to another family in $\mathrm{L}(X)$ which is called a local C-cosine function (see [2,4,8-9,24,28-29,32]). Perturbation of local (integrated) C-semigroups has been extensively studied by many authors appearing in [1,6-7,10-12,15-16,22,30-32]. Some interesting applications of this topic are also illustrated there. The well-posedness of ACP(A, B, f, x, y) had been studied by many authors when $f = 0$ (see [3,6,9,17-18,20,25,32-34]). Some relationship between the well-posedness of ACP$(A, B, 0, x, y)$, a Miyadera-Feller-Phillips-Hille-Yosida type condition (see (1.6) below), and the generation of a C_0-semigroup on $X \times X$ with generator $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ have been established in [25] when A and B are commutable on $\mathrm{D}(B) \cap \mathrm{D}(A)$, in [20] and [32] for $A \in L(X)$, in [32] for $B \in L(X)$, and in [17] for the general case. In particular, Xiao and Liang [32, Theorems 2.6.1, 2.5.2 and 2.5.1] show that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ generates a C_0-semigroup on $X \times X$ (if and) only if $B \in L(X)$ (and A generates a C_0-semigroup on X), but the conclusion may not be true when C_0-semigroups are replaced by local C-semigroups; and the well-posedness of $\mathrm{ACP}(A, B, 0, x, y)$ is equivalent to A generates a C_0-semigroup on X if $B \in L(X)$, and equivalent to B generates a cosine function on X if $A \in L(X)$. Unfortunately, a local C-semigroup may not be exponentially bounded and is not necessarily extendable to the half line $[0, \infty)$, and $\begin{pmatrix} 0 & 0 \\ B & A \end{pmatrix}$ may not be the generator of a local $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$-semigroup on $X \times X$ whenever $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is. Moreover, $\lambda \in \rho_C(A, B)$ may not imply that $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}}$ are bounded even though $D(B) \cap D(A)$ is dense in X and $C = I$, and $\lambda \in \rho_{\mathcal{C}}(\mathcal{T})$ implies that $\lambda \in \rho_C(A, B)$, $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})}$ and $\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}}$ are bounded, but may not be bounded on X even though $C = I$. In particular, they are bounded on X when the assumption of $D(B) \cap D(A)$ is dense in X is added (see [17] for the case $C = I$). In this paper, we will extend the aforementioned results to the case of local C-semigroup by different methods (see Theorems 2.2 and 2.3 below). We show that for each $(x, y) \in \mathcal{D}$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0), X)$ if and only if $\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) and satisfies $Bw + Aw' \in C([0, T_0), X)$ (see Theorem 2.5 below). Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and $\mathcal{D}$ is a fixed subspace of $D(B) \times D(A)$ that is dense in $X \times X$. We then prove two important lemmas (see Lemmas 2.7 and 2.8 below) which can be used to show that there exist $M, \omega > 0$ so that for each pair $x, y \in D(B) \cap D(A)$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$ if and only if $\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ so that $\lambda \in \rho_C(A, B)$ and

$$(1.6) \quad \|[\lambda(\lambda^2 - \lambda A - B)^{-1}C]^{(k)}\|, \|[\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}}]^{(k)}\| \leq \frac{Mk!}{(\lambda - \omega)^{k+1}}$$

for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$ if and only if there exist $M, \omega > 0$ so that for each pair $x, y \in D(B) \cap D(A)$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, \infty), X)$ (see Corollary 2.6 and Theorem 2.9 below). Here $\rho_C(A, B) = \{\lambda \in \mathbb{F} \mid \lambda^2 - \lambda A - B$ is injective, $R(C) \subset R(\lambda^2 - \lambda A - B)$, and $(\lambda^2 - \lambda A - B)^{-1}C \in L(X)\}$. When $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty, we can combine Lemma 2.4 with [23, Corollary 3.6] to show that for each $(x, y) \in D(B) \times D(A)$ $\mathrm{ACP}(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$ if and only if $\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$ if and only if for each $(x, y) \in D(B) \times D(A)$ $\mathrm{ACP}(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$ (see

Theorem 2.11 below). We then apply the modifications of [12, Theorem 2.12 and Theorem 3.2] concerning the bounded and unbounded perturbations of a locally Lipschitz continuous local once integrated C-semigroup on X (see Theorem 2.12 below) and a basic property of local C-cosine function (see [6, Theorem 2.1.11]) to obtain two new equivalence relations concerning the generations of a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and either a locally Lipschitz continuous local C-cosine function on X with subgenerator (resp., the generator) B for which A may not be bounded (see Theorem 2.13 below) or a local C-semigroup on X with subgenerator (resp., the generator) A for which B may not be bounded (see Theorem 2.16 below). Under some suitable assumptions, which can be used to show those preceding equivalence conditions which are equivalent to B is the generator of a locally Lipschitz continuous local C-cosine function on X for which A may not be bounded (see Corollaries 2.14 and 2.15 below), and are also equivalent to A is the generator of a local C-semigroup on X for which B may not be bounded (see Corollaries 2.17 and 2.18 below).

2. Abstract Cauchy problems

In this section, we consider the existence of solutions of the abstract Cauchy problem ACP(A,B,f,x,y). A function u is called a (strong) solution of ACP(A,B,f,x,y) if $u \in \mathrm{C}^1([0,T_0),X)$ satisfies ACP(A,B,f,x,y) (that is $u(0)=x$, $u'(0)=y$, and for a.e. $t \in (0,T_0)$, $u'(t)$ is differentiable and $u'(t) \in \mathrm{D}(A)$, and $u''(t){=}Au'(t){+}Bu(t){+}f(t)$ for a.e. $t \in (0,T_0)$). A linear operator A in X is called a subgenerator of a local α-times integrated C-semigroup $S(\cdot)$ if $S(t)x - j_\alpha(t)Cx = \int_0^t S(r)Axdr$ for all $x \in \mathrm{D}(A)$ and $0 \le t < T_0$, and $\int_0^t S(r)xdr \in \mathrm{D}(A)$ and $A\int_0^t S(r)xdr{=}S(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $0 \le t < T_0$. Moreover, a subgenerator A of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$. We next note some basic properties of a local C-semigroup, and then deduce some results about connections between the unique existence of solutions of ACP$(A,B,CBx,0,Cy)$, ACP$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$, and ACP$(A,B,0,Cx,Cy)$.

Proposition 2.1. (see [4,13,23]) *Let A be the generator of a local C-semigroup $S(\cdot)$ on X. Then*

(2.1) $S(t)S(s) = S(s)S(t)$ *for* $0 \le t, s, t+s < T_0$;
(2.2) *A is closed and $C^{-1}AC = A$;*
(2.3) $S(t)x \in D(A)$ *and* $S(t)Ax = AS(t)x$ *for* $x \in D(A)$ *and* $0 \le t < T_0$;
(2.4) $\int_0^t S(r)xdr \in D(A)$ *and* $A\int_0^t S(r)xdr = S(t)x - Cx$ *for* $x \in X$ *and* $0 \le t < T_0$;
(2.5) $R(S(t)) \subset \overline{D(A)}$ *for* $0 \le t < T_0$;
(2.6) *A is the maximal subgenerator of $S(\cdot)$;*
(2.7) *$C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$.*

Theorem 2.2. (see [13,23]) *Let A be a closed linear operator in X such that $CA \subset AC$. Then A is a subgenerator of a local C-semigroup $S(\cdot)$ on X if and only if for each*

$x \in X$ ACP$(A, Cx, 0)$ has a unique (strong) solution $u(\cdot, x)$ in $C^1([0, T_0), X)$. In this case, we have $u(t, x) = j_0 * S(t)x (= \int_0^t S(s)x ds)$ for all $x \in X$. By slightly modifying the proof of [23, Corollary 3.5], the next theorem concerning the well-posedness of ACP$(A, f, x)$is attained, and so its proof is omitted.

Theorem 2.3. *Let A be a closed linear operator in X such that $CA \subset AC$ and D dense in X for some subspace D of $D(A)$. Then the following are equivalent:*

(i) *A is a subgenerator of a nondegenerate local C-semigroup $S(\cdot)$ on X;*
(ii) *For each $x \in D$ ACP$(A, 0, Cx)$ has a unique solution $u(\cdot; Cx)$ in $C([0, T_0), [D(A)])$ which depends continuously on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D, \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $[0, T_0)$.*

In this case, $u(\cdot, Cx) = S(\cdot)x$.

In the following, we always assume that A and B are biclosed linear operators in X such that $CA \subset AC$ and $CB \subset BC$.

Lemma 2.4. *Assume that $\mathcal{D}$ is a subspace of $D(B) \times D(A)$. Then the following are equivalent:*

(i) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
(ii) *For each $(x, y) \in \mathcal{D}$ ACP$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ has a unique solution $\begin{pmatrix} u \\ v \end{pmatrix}$ in $C([0, T_0), [\mathcal{T}])$;*
(iii) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

In this case, $w = j_0 * v$ and $z = u$.
In particular, $z, w \in C^1([0, T_0), [D(A)]) \cap C([0, T_0), [D(B)])$ if either A or B is bounded. Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since the biclosedness of A and B with $CA \subset AC$ and $CB \subset BC$ implies that $\mathcal{T}$ is a closed linear operator in $X \times X$ so that $\mathcal{CT} \subset \mathcal{TC}$. Suppose that $(x, y) \in \mathcal{D}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ denotes the unique solution of ACP$\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ in C$([0, T_0), [(\mathcal{T}])$. Then v and $Bu + Av$ are continuous on $[0, T_0)$, and $u' = v$ and $v' = Bu + Av+$ a.e. on $[0, T_0)$, so that $u = j_0 * v + Cx$ on $[0, T_0)$, $j_0 * v(t) \in$ D(B) for all $t \in [0, T_0)$, and $v' = Bj_0 * v + CBx$ a.e. on $[0, T_0)$. Hence, $w = j_0 * v$ is a solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in$ C$([0, T_0), X)$. The uniqueness of solutions of ACP$(A, B, CBx, 0, Cy)$ follows from the fact that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution of ACP $\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in C$([0, T_0), [\mathcal{T}])$. Conversely, suppose that $(x, y) \in \mathcal{D}$ and w denotes the unique solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in$ C$([0, T_0), X)$. We set $u = w + Cx$ and $v = w'$ on $[0, T_0)$. Then

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathrm{D}(B) \times \mathrm{D}(A) = \mathrm{D}(\mathcal{T})$$

for all $t \in [0, T_0)$ and $\mathcal{T}\begin{pmatrix} u \\ v \end{pmatrix}$ is continuous on $[0, T_0)$, and for a.e. $t \in (0, T_0)$ $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is differentiable and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} w'(t) \\ w''(t) \end{pmatrix} = \begin{pmatrix} w'(t) \\ Aw'(t) + Bw(t) + CBx \end{pmatrix}$$

$$= \begin{pmatrix} v(t) \\ Av(t) + Bu(t) \end{pmatrix} = \mathcal{T}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix},$$

and so $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of ACP $\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$ in $\mathrm{C}([0, T_0), [\mathcal{T}])$. The uniqueness of solutions follows from the fact that 0 is the unique solution of ACP$(A, B, 0, 0, 0)$. Similarly, we can show that (ii) and (iii) are equivalent. □

Just as an application of Theorem 2.3, the next theorem concerning the well-posedness of ACP(A, B, f, x, y) is also attained.

Theorem 2.5. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$. Then the following are equivalent:*

(i) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y) (that is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(B), \|\cdot\|)$ and $\{y_n\}_{n=1}^{\infty}$ a Cauchy sequence in $(D(A), \|\cdot\|)$, and w_n denotes the unique solution of ACP$(A, B, CBx_n, 0, Cy_n)$, then $\{w_n(\cdot)\}_{n=1}^{\infty}$ and $\{w_n'(\cdot)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$) and satisfies $Bw + Aw' \in C([0, T_0), X)$;*

(ii) *$\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*

(iii) *For each $(x, y) \in \mathcal{D}$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z which depends continuously differentiable on (x, y) and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Here $\mathcal{T} = \begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.

Proof. Since for each $(x, y) \in \mathcal{D}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ is the unique solution of

$$\mathrm{ACP}\left(\mathcal{T}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}\right)$$

in $\mathrm{C}([0, T_0), [\mathcal{T}])$ if and only if for each $(x, y) \in \mathcal{D}$ $u = w + Cx$ and $v = w'$ on $[0, T_0)$, and w is the unique solution of ACP$(A, B, CBx, 0, Cy)$ such that $Bw + Aw' \in \mathrm{C}([0, T_0), X)$. By Theorem 2.3, we also have $\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{S}(\cdot)\begin{pmatrix} x \\ y \end{pmatrix}$. Consequently, $\mathcal{T}$ is a subgenerator of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if for each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w which depends continuously differentiable on (x, y). Similarly, we can show that (ii) and (iii) are equivalent. □

Corollary 2.6. *Assume that $\mathcal{D}$ is dense in $X \times X$ for some subspace $\mathcal{D}$ of $D(B) \times D(A)$. Then the following are equivalent:*

(i) *There exist $M, \omega > 0$ such that for each $(x, y) \in \mathcal{D}$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and $Bw + Aw' \in C([0, \infty), X)$;*

(ii) *$\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *There exist $M, \omega > 0$ such that for each $(x, y) \in \mathcal{D}$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \leq Me^{\omega t}(\|x\| + \|y\|)$ for all $t \geq 0$ and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Lemma 2.7. *Assume that $\lambda \in \rho_{\mathcal{C}}(\mathcal{T})$ ($\mathcal{C}$-resolvent set of $\mathcal{T}$). Then*

(i) *$\lambda \in \rho_C(A, B)$;*

(ii) *$(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}$ are closable, and their closures are bounded and have the same domain;*

(iii) $$(\lambda - \mathcal{T})^{-1}\mathcal{C} = \begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix}$$
on $D(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{D(B)\cap D(A)})}) \times X$, and on $X \times X$ if $D(B) \cap D(A)$ is dense in X.

Proof. To show that $\lambda^2 - \lambda A - B$ is closed. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathrm{D}(B) \cap \mathrm{D}(A)$ which converges to x in X and $\{(\lambda^2 - \lambda A - B)x_n\}_{n=1}^{\infty}$ converges to y in X. Then $\begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} \in \mathrm{D}(\mathcal{T})$, $\begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} \to \begin{pmatrix} x \\ \lambda x \end{pmatrix}$, and

$$(\lambda - \mathcal{T})\begin{pmatrix} x_n \\ \lambda x_n \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x_n \end{pmatrix} \to \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

By the closedness of $\lambda - \mathcal{T}$, we have $\begin{pmatrix} x \\ \lambda x \end{pmatrix} \in \mathrm{D}(\mathcal{T})$ and

$$\begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x \end{pmatrix} = (\lambda - \mathcal{T})\begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix},$$

and so $(\lambda^2 - \lambda A - B)x = y$. Hence, $\lambda^2 - \lambda A - B$ is closed. To show that $\lambda^2 - \lambda A - B$ is injective. Suppose that $(\lambda^2 - \lambda A - B)x = 0$. Then

$$(\lambda - \mathcal{T})\begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^2 - \lambda A - B)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so $\begin{pmatrix} x \\ \lambda x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, $x = 0$, which implies that $\lambda^2 - \lambda A - B$ is injective. To show that $\mathrm{R}(C) \subset \mathrm{R}(\lambda^2 - \lambda A - B)$. Suppose that $z \in X$ is given. Then

$$(\lambda - \mathcal{T})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ Cz \end{pmatrix}$$

for some $(x, y) \in \mathrm{D}(\mathcal{T}) = \mathrm{D}(B) \times \mathrm{D}(A)$, so that $\lambda x - y = 0$ and $-Bx + (\lambda - A)y = Cz$. Hence, $x \in \mathrm{D}(B) \cap \mathrm{D}(A) (= \mathrm{D}(\lambda^2 - \lambda A - B))$ and $(\lambda^2 - \lambda A - B)x = Cz$, which implies that $\mathrm{R}(C) \subset \mathrm{R}(\lambda^2 - \lambda A - B)$. Consequently, $\lambda \in \rho_C(A, B)$.

To show that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ and $(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ are closable, we need only to show that $(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ or $(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}$ is closable. We will show that

$$(2.8)\quad (\lambda-\mathcal{T})^{-1}\mathcal{C}=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}$$

on $\mathrm{D}(B)\cap\mathrm{D}(A)$ first or equivalently,

$$(\lambda-\mathcal{T})\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} Cx \\ Cy \end{pmatrix}$$
$$=\mathcal{C}\begin{pmatrix} x \\ y \end{pmatrix}$$

for all $x,y\in\mathrm{D}(B)\cap\mathrm{D}(A)$. Suppose that $x,y\in\mathrm{D}(B)\cap\mathrm{D}(A)$ are given. Then by the fact $B(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x{=}(\lambda-A)(\lambda^2-\lambda A-B)^{-1}CBx$ that we have

$$\begin{aligned}
&(\lambda-\mathcal{T})\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\
&=\begin{pmatrix} \lambda & -I \\ -B & \lambda-A \end{pmatrix}\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\\
&=\begin{pmatrix} \lambda & -I \\ -B & \lambda-A \end{pmatrix}\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x+(\lambda^2-\lambda A-B)^{-1}Cy \\ (\lambda^2-\lambda A-B)^{-1}CBx+\lambda(\lambda^2-\lambda A-B)^{-1}Cy \end{pmatrix}\\
&=\begin{pmatrix} Cx \\ Cy \end{pmatrix}.
\end{aligned}$$

Suppose that $x_n\in\mathrm{D}(B)\cap\mathrm{D}(A)$, $x_n\to 0$ in X, and $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to y$ in X. Then

$$\begin{aligned}
(\lambda^2-\lambda A-B)^{-1}CB)x_n &= (\lambda^2-\lambda A-B)^{-1}C(B+\lambda A-\lambda^2))x_n\\
&\quad+(\lambda^2-\lambda A-B)^{-1}C(\lambda^2-\lambda A)x_n\\
&= Cx_n+(\lambda^2-\lambda A-B)^{-1}C(\lambda^2-\lambda A)x_n\\
&\to \lambda y,
\end{aligned}$$

and so

$$\begin{aligned}
(\lambda-\mathcal{T})^{-1}C\begin{pmatrix} x_n \\ 0 \end{pmatrix} &= \begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A) & (\lambda^2-\lambda A-B)^{-1}C \\ (\lambda^2-\lambda A-B)^{-1}CB & \lambda(\lambda^2-\lambda A-B)^{-1}C \end{pmatrix}\begin{pmatrix} x_n \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n \\ (\lambda^2-\lambda A-B)^{-1}CBx_n \end{pmatrix}\\
&\to \begin{pmatrix} y \\ \lambda y \end{pmatrix}=(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Hence, $y=0$, which implies that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{\mathrm{D}(B)\cap\mathrm{D}(A)})$ is closable.
To show that $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}$ is bounded.
Let $x\in\mathrm{D}(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}})$ be given.
Then $(x_n,(\lambda^2-\lambda A-B)^{-1}CBx_n)\to(x,\overline{(\lambda^2-\lambda A-B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}x)$ for some $x_n\in\mathrm{D}(B)\cap\mathrm{D}(A)$, and so

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x_n \\ 0 \end{pmatrix}=\begin{pmatrix} (\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n \\ (\lambda^2-\lambda A-B)^{-1}CBx_n \end{pmatrix}\to(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Hence, $\{(\lambda^2-\lambda A-B)^{-1}(\lambda-A)x_n\}_{n=1}^{\infty}$ and $\{(\lambda^2-\lambda A-B)^{-1}Bx_n\}_{n=1}^{\infty}$ both converge. By the closedness of $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}$ and $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$, we have $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})$ and

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0\end{pmatrix}=\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}x\end{pmatrix},$$

which implies that $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$ is bounded and

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}).$$

Similarly, we can show that $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}$ is bounded and

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}),$$

which implies that

$$\begin{aligned}(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ y\end{pmatrix}&=(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} x \\ 0\end{pmatrix}+(\lambda-\mathcal{T})^{-1}\mathcal{C}\begin{pmatrix} 0 \\ y\end{pmatrix}\\ &=\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}\end{aligned}$$

for all $(x,y)\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\times X$. Combining this with the closedness of $\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}$ and the denseness of $D(B)\cap D(A)$ in X, we have

$$\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}\in L(X).$$ □

Lemma 2.8. *Assume that $\lambda\in\rho_C(A,B)$. Then*

(i) *$\lambda-\mathcal{T}$ is injective;*

(ii) *$(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ and $(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}$ are closable and their closures have the same domain, and*

$$(\lambda-\mathcal{T})\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C\end{pmatrix}=\mathcal{C}$$

on $D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\times X$;

(iii) *$\lambda\in\rho_{\mathcal{C}}(\mathcal{T})$ and*

$$(\lambda-\mathcal{T})^{-1}\mathcal{C}=\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})} & (\lambda^2-\lambda A-B)^{-1}C \\ \overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}} & \lambda(\lambda^2-\lambda A-B)^{-1}C\end{pmatrix},$$

if $\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}\in L(X)$.

In particular, the conclusion of (iii) holds when A or B in $L(X)$, or $D(B)\cap D(A)$ is dense in X with $AB=BA$ on $D(B)\cap D(A)$.

Proof. To show that $\lambda-\mathcal{T}$ is injective. Suppose that $(\lambda-\mathcal{T})\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$. Then $\lambda x-y=0$ and $-Bx+(\lambda-A)y=0$, so that $\lambda x=y$ and $-Bx+(\lambda^2-\lambda A)x=0$. Hence, $x=0=y$, which implies that $\lambda-\mathcal{T}$ is injective. Just as in the proof of Lemma 2.7, we will apply (2.8) to show that $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ and

$(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}$ are closable. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence in $D(B)\cap D(A)$ which converges to 0 in X and $\{(\lambda^2-\lambda A-B)(\lambda-A)x_n\}_{n=1}^\infty$ converges to y in X. Then

$$(\lambda^2-\lambda A-B)^{-1}CBx_n=-Cx_n+(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to\lambda y,$$

and so $\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}\to\begin{pmatrix}y\\\lambda y\end{pmatrix}$. Hence,

$$(\lambda-\mathcal{T})\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}=\begin{pmatrix}Cx_n\\0\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}.$$

By the closedness of $\mathcal{T}$, we have $\begin{pmatrix}y\\\lambda y\end{pmatrix}\in D(\mathcal{T})$ and $(\lambda-\mathcal{T})\begin{pmatrix}y\\\lambda y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, which together with the injectivity of $\lambda-\mathcal{T}$ implies that $y=0$.
Consequently, $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})$ is closable. Similarly, we can show that $(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}$ is closable. Just as in the proof of Lemma 2.7, we will show that

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})=D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}),$$

and for each $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})$

$$\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}x\end{pmatrix}\in D(\mathcal{T}).$$

Suppose that $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})$ is given. Then $x_n\to x$ and $(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\to\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x$ for some sequence $\{x_n\}_{n=1}^\infty$ in $D(B)\cap D(A)$, and so

$$(\lambda^2-\lambda A-B)^{-1}CBx_n\to-Cx+\lambda\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x.$$

Hence, $x\in D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}})$, which implies that

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}).$$

Similarly, we can show that

$$D(\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}})\subset D(\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}).$$

Since

$$\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}\to\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}x\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}x\end{pmatrix}$$

and

$$(\lambda-\mathcal{T})\begin{pmatrix}(\lambda^2-\lambda A-B)^{-1}C(\lambda-A)x_n\\(\lambda^2-\lambda A-B)^{-1}CBx_n\end{pmatrix}=\begin{pmatrix}Cx_n\\0\end{pmatrix}\to\begin{pmatrix}Cx\\0\end{pmatrix}.$$

By the closedness of $\lambda-\mathcal{T}$, we have

$$(\lambda-\mathcal{T})\begin{pmatrix}\overline{(\lambda^2-\lambda A-B)^{-1}C(\lambda-A_{D(B)\cap D(A)})}&(\lambda^2-\lambda A-B)^{-1}C\\\overline{(\lambda^2-\lambda A-B)^{-1}CB_{D(B)\cap D(A)}}&\lambda(\lambda^2-\lambda A-B)^{-1}C\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix}=\mathcal{C}\begin{pmatrix}x\\0\end{pmatrix}.$$

Consequently,

$$(\lambda - \mathcal{T})\begin{pmatrix} \overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})} & (\lambda^2 - \lambda A - B)^{-1}C \\ \overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}} & \lambda(\lambda^2 - \lambda A - B)^{-1}C \end{pmatrix} = \mathcal{C}$$

on $\mathrm{D}(\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})}) \times X$. □

Since $\overline{(\lambda^2 - \lambda A - B)^{-1}C(\lambda - A_{\mathrm{D}(B)\cap\mathrm{D}(A)})} = [\overline{(\lambda^2 - \lambda A - B)^{-1}CB_{\mathrm{D}(B)\cap\mathrm{D}(A)}}]\frac{1}{\lambda} + \frac{1}{\lambda}C$ and $(\lambda^2 - \lambda A - B)^{-1}C = [\lambda(\lambda^2 - \lambda A - B)^{-1}C]\frac{1}{\lambda}$, we can combine Lemma 2.7 with Lemma 2.8 and [1, Theorem 2.4.1] or [32, Theorem 1.2.1] to obtain the next new Miyadera-Feller-Phillips-Hille-Yosida type theorem concerning the generation of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$.

Theorem 2.9. *Assume that $D(B) \cap D(A)$ is dense in X. Then $\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$ if and only if there exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$.*

Just as a result in [17, Theorem 2] for the case of C_0-semigroup, we can combine Corollary 2.6 with Theorem 2.9 to obtain the next corollary.

Corollary 2.10. *Assume that $D(B) \cap D(A)$ is dense in X. Then the following statements are equivalent:*

(i) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, CBx, 0, Cy)$ has a unique solution w with $\|w(t)\|, \|w'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and $Bw + Aw' \in C([0, \infty), X)$;*
(ii) *$\mathcal{T}$ is a subgenerator of an exponentially bounded $\mathcal{C}$-semigroup on $X \times X$;*
(iii) *There exist $M, \omega > 0$ such that $\lambda \in \rho_C(A, B)$ and (1.6) holds for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$;*
(iv) *There exist $M, \omega > 0$ such that for each pair $x, y \in D(B) \cap D(A)$, $ACP(A, B, 0, Cx, Cy)$ has a unique solution z with $\|z(t)\|, \|z'(t)\| \le Me^{\omega t}(\|x\| + \|y\|)$ for all $t \ge 0$ and satisfies $Bz + Az' \in C([0, T_0), X)$.*

Combining Lemma 2.4 with [23, Corollary 3.6], the next theorem is also attained.

Theorem 2.11. *Assume that $\rho(\mathcal{T})$ (resolvent set of $\mathcal{T}$) is nonempty. Then the following are equivalent:*

(i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$;*
(iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

By modifying slightly the proofs of [12, Theorem 2.12 and Theorem 3.2], the next theorem is also attained, and so its proof is omitted.

Theorem 2.12. *Let B be a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated C-semigroup on X. Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$ and $A + B$ is closed. Then $A + B$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated C-semigroup on X.*

Since B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X if and only if $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$ (see [6, Theorem 2.1.11]); and A is a bounded linear operator from $[D(B)]$ into $\mathrm{R}(C)$ so that $\mathrm{R}(C^{-1}A) \subset D(B)$ implies that

$$\mathrm{R}\left(\mathcal{C}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\right) = \mathrm{R}\left(\begin{pmatrix} 0 & 0 \\ 0 & C^{-1}A \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can apply Theorem 2.12 to obtain the next new result concerning the generations of a locally Lipschitz continuous local C-cosine function on X with subgenerator (resp., the generator) B and a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which A may not be bounded.

Theorem 2.13. *Assume that A is a bounded linear operator from $\overline{D(B)}$ into $R(C)$ or a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$ only if B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. The "if part" is also true when the assumption of $D(B)$ is dense in X is added.*

Proof. Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. Then it is also a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$. Hence, B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. Conversely, suppose that $\mathrm{D}(B)$ is dense in X and B is a subgenerator (resp., the generator) of a locally Lipschitz continuous local C-cosine function on X. Then $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$, and so $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a locally Lipschitz continuous local once integrated $\mathcal{C}$-semigroup on $X \times X$. Hence, it is also a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. $\square$

Combining Theorem 2.11 with Theorem 2.13, we can obtain the next two corollaries.

Corollary 2.14. *Assume that $\rho(A, B)$ is nonempty and $A \in L(X)$. Then the following are equivalent:*

- (i) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, CBx, 0, Cy)$ has a unique solution w in $C([0, T_0), [D(B)])$;*
- (ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*
- (iii) *For each $(x, y) \in D(B) \times D(A)$ ACP$(A, B, 0, Cx, Cy)$ has a unique solution z in $C([0, T_0), [D(B)])$.*

Moreover, (i)-(iii) imply

(iv) *B is the generator of a locally Lipschitz continuous local C-cosine function on X*

if $R(A) \subset R(C)$, and (i)-(iv) are equivalent if the assumption of $D(B)$ is dense in X is also added. Here $[D(B)]$ denotes the Banach space $D(B)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Bx\|$ for $x \in D(B)$.

Corollary 2.15. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A, B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:*

(i) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, CBx, 0, Cy)$ has a unique solution w such that $Bw + Aw' \in C([0, T_0), X)$;*
(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*
(iii) *For each $(x, y) \in D(B) \times D(A)$ $ACP(A, B, 0, Cx, Cy)$ has a unique solution z such that $Bz + Az' \in C([0, T_0), X)$.*

Moreover, (i)-(iii) are equivalent to

(iv) *B is the generator of a locally Lipschitz continuous local C-cosine function on X*

if A is a bounded linear operator from $[D(B)]$ into $R(C)$ so that $R(C^{-1}A) \subset D(B)$.

Since B is a bounded linear operator from $[D(A)]$ into $\mathrm{R}(C)$ so that $\mathrm{R}(C^{-1}B) \subset D(A)$ implies that

$$\mathrm{R}\left(\mathcal{C}^{-1}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right) = \mathrm{R}\left(\begin{pmatrix} 0 & 0 \\ C^{-1}B & 0 \end{pmatrix}\right) \subset D\left(\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}\right) = D(B) \times D(A),$$

we can combine Theorem 2.11 with Theorem 2.13 to obtain the next new result concerning the generations of a local C-semigroup on X with subgenerator (resp., the generator) A and a local $\mathcal{C}$-semigroup on $X \times X$ with subgenerator (resp., the generator) $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ for which B may not be bounded.

Theorem 2.16. *Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ or a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$. Then $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$ if and only if A is a subgenerator (resp., the generator) of a local C-semigroup on X.*

Proof. Clearly, $\mathcal{C}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\mathcal{C}$ on $X \times \mathrm{D}(A)$ (resp., $\mathcal{C}^{-1}\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\mathcal{C} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$) is equivalent to $CA = AC$ on $\mathrm{D}(A)$ (resp., $C^{-1}AC = A$). Suppose that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. Then $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup $\mathcal{S}(\cdot)$ on $X \times X$. For each pair $x, y \in X$, we set

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = j_0 * \mathcal{S}(t)\begin{pmatrix} x \\ y \end{pmatrix}$$

for all $0 \le t < T_0$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix} \in C^1([0,T_0), X\times X) \cap C([0,T_0),[\mathcal{T}]), \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix} = \begin{pmatrix} v(t) \\ Av(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \le t < T_0$, so that $u(0) = 0 = v(0)$, $u'(t) = v(t) + Cx$ and $v'(t) = Av(t) + Cy$ for all $0 \le t < T_0$. Hence, v is a solution of ACP$(A, Cy, 0)$ in $C^1([0,T_0), X) \cap C([0,T_0),[D(A)])$, $u(0) = 0$, and $u' = v$ on $[0,T_0)$. To show that A is a subgenerator (resp., the generator) of a local C-semigroup on X, we remain only to show that 0 is the unique solution of ACP$(A,0,0)$ in $C^1([0,T_0),X)\cap C([0,T_0),[D(A)])$ (see Theorem 2.2). To this end. Suppose that v is a solution of ACP$(A,0,0)$ in $C^1([0,T_0),X)\cap C([0,T_0),[D(A)])$. We set $u = j_0 * v$, then $u(0) = 0 = v(0)$ and

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ Av(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

for all $0 \le t < T_0$. The uniqueness of solutions of ACP$(A,0,0)$ follows from the uniqueness of solutions of ACP $\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. Conversely, suppose that A is a subgenerator (resp., the generator) of a local C-semigroup $S(\cdot)$ on X. To show that $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X\times X$, we need only to show that for each pair $x, y \in X$, ACP $\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ has a unique solution in $C^1([0,T_0), X\times X) \cap C\left([0,T_0), \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. To do this. For each pair $x, y \in X$, we set $v(t) = j_0 * S(t)y$ and $u(t) = j_0 * v(t) + tCx$ for all $0 \le t < T_0$. Then $u(0) = 0 = v(0)$, and $v'(t){=}S(t)y{=}Av(t){+}Cy$ and $u'(t) = v(t){+}Cx$ for all $0 \le t < T_0$, so that

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) + Cx \\ Av(t) + Cy \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} Cx \\ Cy \end{pmatrix}$$

for all $0 \le t < T_0$. Hence, $\begin{pmatrix} u \\ v \end{pmatrix}$ is a solution of ACP $\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} Cx \\ Cy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $C^1([0,T_0), X\times X) \cap C\left([0,T_0), \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$. The uniqueness of solutions of ACP $\left(\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ in $C^1([0,T_0), X\times X)\cap C\left([0,T_0), \left[\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}\right]\right)$ follows from the uniqueness of solutions of ACP$(A,0,0)$. Consequently, $\begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X\times X$, which implies that $\mathcal{T}$ is a subgenerator (resp., the generator) of a local $\mathcal{C}$-semigroup on $X \times X$. $\square$

Corollary 2.17. *Assume that $\rho(A,B)$ is nonempty and $B \in L(X)$. Then the following are equivalent:*

(i) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,CBx,0,Cy)$ has a unique solution w in $C^1([0,T_0),[D(A)])$;*

(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,0,Cx,Cy)$ has a unique solution z in $C^1([0,T_0),[D(A)])$.*

Moreover, (i)-(iii) are equivalent to

(vi) *A is the generator of a local C-semigroup on X,*

if $R(B) \subset R(C)$.

Corollary 2.18. *Assume that $D(B) \cap D(A)$ is dense in X, $\rho(A,B)$ nonempty, and $AB = BA$ on $D(B) \cap D(A)$. Then the following are equivalent:*

(i) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,CBx,0,Cy)$ has a unique solution w such that $Bw + Aw' \in C([0,T_0),X)$;*

(ii) *$\mathcal{T}$ is the generator of a local $\mathcal{C}$-semigroup on $X \times X$;*

(iii) *For each $(x,y) \in D(B) \times D(A)$ $ACP(A,B,0,Cx,Cy)$ has a unique solution z such that $Bz + Az' \in C([0,T_0),X)$.*

Moreover, (i)-(iii) are equivalent to

(iv) *A is the generator of a local C-semigroup on X,*

if B is a bounded linear operator from $[D(A)]$ into $R(C)$ so that $R(C^{-1}B) \subset D(A)$.

We end this paper with a simple illustrative example. Let $S(\cdot)(=\{S(t)|0 \le t < 1\})$ be a family of bounded linear operators on c_0(, family of all convergent sequences in $\mathbb{F}$ with limit 0,) defined by $S(t)x = \{e^{-n}e^{nt}x_n\}_{n=1}^{\infty}$, then $S(\cdot)$ is a local C-semigroup on c_0 with generator A defined by $Ax = \{nx_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{nx_n\}_{n=1}^{\infty} \in c_0$. Here $C = S(0)$. Let $\{p_n\}_{n=1}^{\infty} \in l^{\infty}$ with $\{e^n p_n\}_{n=1}^{\infty} \in l^{\infty}$, and B be a bounded linear operator from $[D(A)]$ into $R(C)$ defined by $Bx = \{nx_n p_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in D(A)$, then $R(C^{-1}B) \subseteq D(A)$, $CB = BC$ on $D(A)$, and $B : D(A) \subset c_0 \to c_0$ can be extended to a bounded linear operator on $\overline{D(A)} = c_0$. Applying Corollary 2.17, we get that $\begin{pmatrix} 0 & I \\ B & A \end{pmatrix}$ is the generator of a local $\mathcal{C}$-semigroup on $c_0 \times c_0$.

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