Stud. Univ. Babeş-Bolyai Math. 67(2022), No. 1, 151–166
DOI: 10.24193/subbmath.2022.1.11

Hermite–Hadamard type inequalities for F–convex functions involving generalized fractional integrals

Hüseyin Budak, Muhammad Aamir Ali and Artion Kashuri

Abstract. In this paper, we firstly summarize some properties of the family $\mathcal{F}$ and F–convex functions which are defined by B. Samet. Utilizing generalized fractional integrals new Hermite–Hadamard type inequalities for F–convex functions have been provided. Some results given earlier works are also as special cases of our results.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D07, 26D10, 26D15.

Keywords: Hermite–Hadamard inequality, F–convex, general fractional integral.

1. Introduction

Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality holds [17]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities in (1.1) hold in the reversed direction if f is concave.

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, see [7, 6, 12, 16], [24]-[23] and the references therein. Also, many type of convexity have been defined, such as quasi-convex in [5], pseudo-convex in [13], strongly convex in [19], ε-convex in [10], s-convex in [9], h-convex

Received 15 August 2019; Accepted 17 January 2020.

in [28], etc. Recently, Samet in [20], have defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity, including ε-convex functions, α-convex functions, h-convex functions, and many others.
Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ satisfying the following axioms:
(A1) If $u_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda)dt = F\left(\int_0^1 e_1(t)dt, \int_0^1 e_2(t)dt, \int_0^1 e_3(t)dt, \lambda\right).$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)dt = T_{F,w}\left(\int_0^1 w(t)u(t)dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable.
(A3) For any $(w, e_1, e_2, e_3) \in \mathbb{R}^4$, $e_4 \in [0,1]$, we have

$$wF(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F-convex function), iff

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a,b] \times [a,b] \times [0,1].$$

Remark 1.2. 1) Let $\varepsilon \geq 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, see [10], that is

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \ \ (x, y, t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1 - e_4)e_3 - \varepsilon \tag{1.2}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 tw(t)dt\right) e_2 - \left(\int_0^1 (1-t)w(t)dt\right) e_3 - \varepsilon. \tag{1.3}$$

For

$$L_w = (1 - w)\varepsilon, \tag{1.4}$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,$$

that is f is an $F-$convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an $F-$convex function with respect to F defined above.

2) Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an α-convex function, $\alpha \in (0, 1]$, that is

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t^\alpha)f(y), \ \ (x, y, t) \in [a, b] \times [a, b] \times [0, 1] .$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4^\alpha e_2 - (1 - e_4^\alpha)e_3 \tag{1.5}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 t^\alpha w(t)dt \right) e_2 - \left(\int_0^1 (1-t^\alpha)w(t)dt \right) e_3. \tag{1.6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - t^\alpha f(x) - (1-t^\alpha)f(y) \leq 0,$$

that is, f is an F-convex function.
3) Let $h : J \to [0, +\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \to [0, +\infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an h-convex function, see [28], that is

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \ \ (x, y, t) \in [a, b] \times [a, b] \times [0, 1] .$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1 - e_4)e_3 \tag{1.7}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 h(t)w(t)dt \right) e_2 - \left(\int_0^1 h(1-t)w(t)dt \right) e_3. \tag{1.8}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is, f is an F-convex function.

In [20], author established the following Hermite-Hadamard type inequalities using the new convexity concept:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a, b]$. Then*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a}\int_a^b f(x)dx, \frac{1}{b-a}\int_a^b f(x)dx, \frac{1}{2} \right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b-a}\int_a^b f(x)dx, f(a), f(b) \right) \leq 0.$$

Theorem 1.4. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^{\circ} \times I^{\circ}$, $a < b$. Suppose that:*
(i) $|f'|$ is F-convex on $[a,b]$ for some $F \in \mathcal{F}$;
(ii) the function $t \in (0,1) \to L_{w(t)}$ belongs to $L^1(0,1)$, where $w(t) = |1-2t|$. Then

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|, |f'(a)|, |f'(b)|\right) + \int_0^1 L_{w(t)}dt \leq 0.$$

Theorem 1.5. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^{\circ} \times I^{\circ}$, $a < b$ and let $p > 1$. Suppose that $|f'|^{p/(p-1)}$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$ and $|f'| \in L^{p/(p-1)}(a,b)$. Then*

$$T_{F,1}\left(A(p,f), |f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\right) \leq 0,$$

where

$$A(p,f) = \sqrt[p-1]{p+1}\left(\frac{2}{b-a}\right)^{\frac{p}{p-1}}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|^{\frac{p}{p-1}}.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, see [8, 11, 14, 18].

Definition 1.6. Let $f \in L^1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al. in [25], first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.9}$$

with $\alpha > 0$.

Meanwhile, Sarikaya et al. in [25], presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]$$

$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt. \tag{1.10}$$

Budak et al. in [3], prove the following Hermite-Hadamard type inequalities for F-convex functions via fractional integrals:

Theorem 1.9. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a,b \in I^{\circ}$, $a < b$. If f is F-convex on $[a,b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{a+}f(b), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{b-}f(a), \frac{1}{2}\right)$$

$$+\int_0^1 L_{w(t)}dt \le 0 \tag{1.11}$$

and

$$T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right], f(a)+f(b), f(a)+f(b)\right)$$

$$+\int_0^1 L_{w(t)}dt \le 0, \tag{1.12}$$

where $w(t) = \alpha t^{\alpha-1}$.

Theorem 1.10. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^{\circ}$, $a < b$. Suppose that $|f'|$ is F-convex on $[a,b]$ for some $F \in \mathcal{F}$ and the function $t \in [0,1] \to L_{w(t)}$ belongs to $L^1[0,1]$, where $w(t) = |(1-t)^{\alpha} - t^{\alpha}|$. Then*

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right|, |f'(a)|, |f'(b)|, t\right)$$

$$+\int_0^1 L_{w(t)}dt \le 0. \tag{1.13}$$

For the other papers on inequalities for F-convex functions, see [2, 4, 15, 26, 27].
Now we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [22].
Let's define a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t}dt < +\infty, \tag{1.14}$$

$$\frac{1}{A} \le \frac{\varphi(s)}{\phi(r)} \le A \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.15}$$

$$\frac{\varphi(r)}{r^2} \le B\frac{\varphi(s)}{s^2} \text{ for } s \le r, \tag{1.16}$$

$$\left|\frac{\varphi(r)}{r^2}-\frac{\varphi(s)}{s^2}\right| \le C|r-s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.17}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies the conditions (1.14)–(1.17).
The following left-sided and right-sided generalized fractional integral operators are defined respectively, as follows:

$$_{a^+}I_{\varphi}f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a, \tag{1.18}$$

$$_{b^-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b. \tag{1.19}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.
Sarikaya and Ertuğral in [22], establish the following Hermite-Hadamard inequality and lemmas for the generalized fractional integral operators:

Theorem 1.11. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2\Lambda(1)}\left[_{a+}I_{\varphi}f(b) +_{b-} I_{\varphi}f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{1.20}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt. \tag{1.21}$$

Lemma 1.12. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for generalized fractional integrals hold:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right] \\ = &\frac{(b-a)}{2\Lambda(1)} \int_0^1 \left[\Lambda(1-t) - \Lambda(t)\right] f'(ta+(1-t)b)\, dt. \end{aligned} \tag{1.22}$$

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities for F–convex functions via generalized fractional integrals. Some special cases will be obtain from main results. At the end, a briefly conclusion will be given as well.

2. Hermite–Hadamard type inequality via generalized fractional integrals

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via F–convex functions.

Theorem 2.1. *Let $I \subseteq R$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^\circ$, $a < b$. If f is F-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_\varphi f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_\varphi f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}dt \le 0 \tag{2.1}$$

and

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)}dt \le 0, \tag{2.2}$$

where $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$.

Proof. Since f is F–convex, we have

$$F\left(f\left(\frac{x+y}{2}\right), f(x), f(y), \frac{1}{2}\right) \le 0, \ \forall\, x, y \in [a, b].$$

For

$$x = ta + (1-t)b \ \text{ and } \ y = tb + (1-t)a,$$

we have

$$F\left(f\left(\frac{a+b}{2}\right), f(ta+(1-t)b), f(tb+(1-t)a), \frac{1}{2}\right) \le 0, \ \forall\, t \in [0, 1].$$

Multiplying this inequality by $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$ and using axiom (A3), we get

$$F\Bigg(\frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f\left(\frac{a+b}{2}\right), \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f(ta+(1-t)b), \frac{\varphi\left((b-a)\,t\right)}{t\Lambda(1)} f(tb+(1-t)a), \frac{1}{2}\Bigg) + L_{w(t)} \le 0,$$

for all $t \in [0, 1]$. Integrating over $[0, 1]$ with respect to the variable t and using axiom (A1), we obtain

$$F\Bigg(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} dt, \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(ta+(1-t)b)dt, \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left((b-a)\,t\right)}{t} f(tb+(1-t)a)dt, \frac{1}{2}\Bigg) + \int_0^1 L_{w(t)}dt \le 0.$$

Using the facts that

$$\int_0^1 \frac{\varphi((b-a)t)}{t} f(ta+(1-t)b)dt = \int_a^b \frac{\varphi(b-x)}{b-x} f(x)dx = {}_{a+}I_\varphi f(b)$$

and

$$\int_0^1 \frac{\varphi((b-a)t)}{t} f(tb+(1-t)a)dt = \int_a^b \frac{\varphi(x-a)}{x-a} f(x)dx = {}_{b-}I_\varphi f(a),$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_\varphi f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_\varphi f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}dt \le 0,$$

which gives (2.1).
On the other hand, since f is F-convex, we have

$$F\left(f\left(ta+(1-t)b\right), f(a), f(b), t\right) \le 0, \ \forall t \in [0,1]$$

and

$$F\left(f\left(tb+(1-t)a\right), f(b), f(a), t\right) \le 0, \ \forall t \in [0,1].$$

Using the linearity of F, we get

$$F\left(f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right), f(a)+f(b), f(a)+f(b), t\right) \le 0,$$

$\forall t \in [0,1]$. Applying the axiom (A3) for $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we obtain

$$F\left(\frac{\varphi((b-a)t)}{t\Lambda(1)}\left[f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right)\right],\right.$$

$$\left.\frac{\varphi((b-a)t)}{t\Lambda(1)}\left[f(a)+f(b)\right], \frac{\varphi((b-a)t)}{t\Lambda(1)}\left[f(a)+f(b)\right], t\right) + L_{w(t)} \le 0,$$

for all $t \in [0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$T_{F,w}\left(\int_0^1 \frac{\varphi((b-a)t)}{t\Lambda(1)}\left[f\left(ta+(1-t)b\right) + f\left(tb+(1-t)a\right)\right]dt,\right.$$

$$\left. f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)}dt \le 0,$$

that is

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)\right], f(a)+f(b), f(a)+f(b)\right)$$

$$+\int_0^1 L_{w(t)}dt \le 0.$$

The proof of Theorem 2.1 is completed. □

Remark 2.2. If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, then the inequalities (2.1) and (2.2) reduce to the inequalities (1.11) and (1.12).

Corollary 2.3. *If we take* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem* 2.1*, then we have the following inequalities for k-Riemann-Liouville fractional integrals*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^{\alpha}_{a+,\ k} f(b), \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I^{\alpha}_{b-,\ k} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0$$

and

$$T_{F,w}\left(\frac{\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k} f(b) + I^{\alpha}_{b-,\ k} f(a)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

where $w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1}$.

Corollary 2.4. *If we choose* $F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1-e_4)e_3 - \varepsilon$ *in Theorem* 2.1*, then the function* f *is* ε*-convex on* $[a,b]$*, where* $\varepsilon \geq 0$ *and we have the following new double inequality:*

$$f\left(\frac{a+b}{2}\right) + \varepsilon \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right] \leq \frac{f(a)+f(b)}{2} + \frac{\varepsilon}{2}. \tag{2.3}$$

Proof. Using (1.4) with $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we have

$$\int_0^1 L_{w(t)} dt = \varepsilon \int_0^1 \left(1 - \frac{\varphi((b-a)t)}{t\Lambda(1)}\right) dt = 0. \tag{2.4}$$

Using (1.2), (2.1) and (2.4), we get

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\ {}_{a+}I_{\varphi} f(b), \frac{1}{\Lambda(1)}\ {}_{b-}I_{\varphi} f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

So

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right] - \varepsilon \leq 0,$$

that is

$$f\left(\frac{a+b}{2}\right) + \varepsilon \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi} f(b) + {}_{b-}I_{\varphi} f(a)\right].$$

On the other hand, using (1.3) with $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we have

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 t \frac{\varphi((b-a)t)}{t\Lambda(1)} dt\right) e_2 - \left(\int_0^1 (1-t) \frac{\varphi((b-a)t)}{t\Lambda(1)} dt\right) e_3 - \varepsilon \tag{2.5}$$

for $e_1, e_2, e_3 \in \mathbb{R}$. Hence, from (2.2) and (2.5), we obtain

$$\begin{aligned}
0 &\geq T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right], f(a)+f(b), f(a)+f(b)\right) \\
&\quad + \int_0^1 L_{w(t)}dt \\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] - \left(\int_0^1 t\frac{\varphi((b-a)t)}{t\Lambda(1)}dt\right)[f(a)+f(b)] \\
&\quad - \left(\int_0^1 (1-t)\frac{\varphi((b-a)t)}{t\Lambda(1)}dt\right)[f(a)+f(b)] - \varepsilon \\
&= \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] - [f(a)+f(b)] - \varepsilon.
\end{aligned}$$

This implies that

$$\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \leq f(a)+f(b)+\varepsilon$$

and thus the proof is completed. □

Remark 2.5. If we take $\varepsilon = 0$ in Corollary 2.4, then f is convex and we have the inequality (1.20).

Corollary 2.6. *If we choose $F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3$ in Theorem 2.1, then the function f is h-convex on $[a, b]$ and we have the following new double inequality:*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right] \\
&\leq \frac{[f(a)+f(b)]}{2\Lambda(1)} \\
&\times \int_0^1 \frac{\varphi((b-a)t)}{t}[h(t)+h(1-t)]dt. \qquad (2.6)
\end{aligned}$$

Proof. Using (1.7) and (2.1) with $L_{w(t)} = 0$, we have

$$\begin{aligned}
0 &\geq F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\,{}_{a+}I_{\varphi}f(b), \frac{1}{\Lambda(1)}\,{}_{b-}I_{\varphi}f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)}dt \\
&= f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right)\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right],
\end{aligned}$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right].$$

On the other hand, using (1.8) and (2.2) with $w(t) = \frac{\varphi((b-a)t)}{t\Lambda(1)}$, we obtain

$$\begin{aligned} 0 \geq\ & T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) + {}_{b-}I_{\varphi}f(a)\right], f(a)+f(b), f(a)+f(b)\right) \\ & + \int_0^1 L_{w(t)}dt \\ =\ & \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) +{}_{b-} I_{\varphi}f(a)\right] \\ & - \left[\int_0^1 h(t)\frac{\varphi((b-a)t)}{t\Lambda(1)}dt + \int_0^1 h(1-t)\frac{\varphi((b-a)t)}{t\Lambda(1)}dt\right][f(a)+f(b)] \\ =\ & \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) +{}_{b-} I_{\varphi}f(a)\right] \\ & - \frac{1}{\Lambda(1)}\left(\int_0^1 [h(t)+h(1-t)]\frac{\varphi((b-a)t)}{t}dt\right)[f(a)+f(b)], \end{aligned}$$

that is

$$\begin{aligned} & \frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b) +{}_{b-} I_{\varphi}f(a)\right] \\ \leq\ & \frac{[f(a)+f(b)]}{\Lambda(1)}\left(\int_0^1 [h(t)+h(1-t)]\frac{\varphi((b-a)t)}{t}dt\right) \end{aligned}$$

and thus the proof is completed. □

Theorem 2.7. *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$, $a < b$. Suppose that $|f'|$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$ and the function $t \in [0,1] \to L_{w(t)}$ belongs to $L^1[0,1]$, where $w(t) = \frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$. Then, we have the following inequality:*

$$\begin{aligned} & T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b) + {}_{b^-}I_{\varphi}f(a)\right]\right|,\right. \\ & \qquad \left. |f'(a)|, |f'(b)|, t\right) + \int_0^1 L_{w(t)}dt \leq 0. \end{aligned} \tag{2.7}$$

Proof. Since $|f'|$ is F-convex, we have

$$F(|f'(ta+(1-t)b)|, |f'(a)|, |f'(b)|, t) \leq 0, \ \forall t \in [0,1].$$

Using axiom (A3) with $w(t) = \frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$, we get

$$F(w(t)|f'(ta+(1-t)b)|, w(t)|f'(a)|, w(t)|f'(b)|, t) + L_{w(t)} \leq 0, \ \forall t \in [0,1].$$

Integrating over $[0,1]$ and using axiom (A2), we obtain

$$T_{F,w}\left(\int_0^1 w(t)|f'(ta+(1-t)b)|dt, |f'(a)|, |f'(b)|, t\right) + \int_0^1 L_{w(t)}dt \leq 0,$$

$\forall t \in [0,1]$. From Lemma 1.8, we have

$$\begin{aligned} &\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right| \\ \leq &\int_0^1 w(t)\left|f'(ta+(1-t)b)\right|dt. \end{aligned}$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we establish

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|,\right.$$

$$\left.\left|f'(a)\right|,\left|f'(b)\right|,t\right)+\int_0^1 L_{w(t)}dt \leq 0.$$

The proof of Theorem 2.7 is completed. □

Corollary 2.8. *Under assumptions of Theorem 2.7, if we choose*

$$F(e_1,e_2,e_3,e_4)=e_1-e_4e_2-(1-e_4)e_3-\varepsilon,$$

then the function $|f'|$ is ε-convex on $[a,b]$, $\varepsilon \geq 0$ and we have the following new inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f(b)+{}_{b^-}I_{\varphi}f(a)\right]\right|$$

$$\leq \frac{(b-a)}{2\Lambda(1)}\left(\int_0^1 t\left|\Lambda(1-t)-\Lambda(t)\right|dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \tag{2.8}$$

$$+\varepsilon\frac{(b-a)}{2\Lambda(1)}\int_0^1\left|\Lambda(1-t)-\Lambda(t)\right|dt.$$

Proof. From (1.4) with $w(t)=\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}$, we have

$$\begin{aligned} \int_0^1 L_{w(t)}dt &= \varepsilon\int_0^1 \left(1-\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}\right)dt \\ &= \varepsilon\left(1-\int_0^1\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right). \end{aligned}$$

Using (1.3) with $w(t) = |\Lambda(1-t) - \Lambda(t)|$, we get

$$\begin{aligned} T_{F,w}(e_1, e_2, e_3) &= e_1 - \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) e_2 \\ &\quad - \left(\int_0^1 (1-t) \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) e_3 - \varepsilon \\ &= e_1 - \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) (e_2 + e_3) - \varepsilon \end{aligned}$$

for $e_1, e_2, e_3 \in \mathbb{R}$. Then, by Theorem 2.7, we have

$$0 \geq T_{F,w}\left(\frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right|, \right.$$

$$\left. |f'(a)|, |f'(b)|, t \right) + \int_0^1 L_{w(t)} dt$$

$$= \frac{2}{b-a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right|$$

$$- \left(\int_0^1 t \frac{|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) [|f'(a)| + |f'(b)|]$$

$$+\varepsilon \left(1 - \int_0^1 \frac{(|\Lambda(1-t) - \Lambda(t)|}{\Lambda(1)} dt \right) - \varepsilon.$$

This completes the proof. □

Remark 2.9. If we choose $\varepsilon = 0$ in Corollary 2.8, then $|f'|$ is convex and we have the inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{a^+}I_\varphi f(b) + {}_{b^-}I_\varphi f(a) \right] \right|$$

$$\leq \frac{(b-a)}{2\Lambda(1)} \left(\int_0^1 t \left| \Lambda(1-t) - \Lambda(t) \right| dt \right) [|f'(a)| + |f'(b)|], \tag{2.9}$$

which is given by Sarikaya and Ertuğral in [22].

Corollary 2.10. *Under assumption of Theorem* 2.7*, if we choose*

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3,$$

then the function $|f'|$ is h-convex on $[a,b]$ and we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{\Lambda(1)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]\left(\int_0^1 h(t)\left|\Lambda(1-t)-\Lambda(t)\right|dt\right), \tag{2.10}$$

which is given by Ali et al. in [1].

Proof. From (1.8) with $w(t)=|\Lambda(1-t)-\Lambda(t)|$, we have

$$\begin{aligned}T_{F,w}(e_1,e_2,e_3) &= e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_2\\ &\quad-\left(\int_0^1 h(1-t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_3\\ &= e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_2-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)e_3\\ &= e_1-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)(e_2+e_3)\end{aligned}$$

for $e_1,e_2,e_3\in\mathbb{R}$. Then, by Theorem 2.7, we have

$$\begin{aligned}&T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|,|f'(a)|,|f'(b)|,t\right)\\ &=\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right]\right|\\ &-\left(\int_0^1 h(t)\frac{|\Lambda(1-t)-\Lambda(t)|}{\Lambda(1)}dt\right)\left[|f'(a)|+|f'(b)|\right]\leq 0.\end{aligned}$$

This completes the proof. □

Remark 2.11. If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.7, then the inequality (2.7) reduces to the inequality (1.13).

Corollary 2.12. *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.7, then we have the following inequalities for k-Riemann-Liouville fractional integrals*

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+1)}{2(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{a+,\ k}f(b)+I^{\alpha}_{b-,\ k}f(a)\right]\right|,|f'(a)|,|f'(b)|,t\right)$$
$$+\int_0^1 L_{w(t)}dt\leq 0,\ \text{where } w(t)=\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|.$$

3. Conclusion

In the development of this work, using the definition of F–convex functions some new Hermite–Hadamard type inequalities via generalized fractional integrals have been deduced. Also, this class of functions can be applied to obtain several results in convex analysis, related optimization theory, etc. The authors hope that these results will serve as a motivation for future work in this fascinating area.

Acknowledgements. This project is partially supported by the National Natural Science Foundation of China (No. 11971241). Also, the authors would like to thank the honorable referees and editors for valuable comments and suggestions for improved the manuscript.

References

[1] Ali, M.A., Budak, H., Abbas, M., Sarikaya, M.Z., Kashuri, A., *Hermite–Hadamard type inequalities for h–convex functions via generalized fractional integrals*, Submitted, (2019).

[2] Budak, H., Sarikaya, M.Z., *On Ostrowski type inequalities for F-convex function*, AIP Conference Proceedings, 1833, 020057 (2017), doi: 10.1063/1.4981705.

[3] Budak, H., Sarikaya, M.Z., Yildiz, M.K., *Hermite-Hadamard type inequalities for F-convex function involving fractional integrals*, Filomat, **32**(2018), no. 16, 5509-5518.

[4] Budak, H., Tunç, T., Sarikaya, M.Z., *On Hermite-Hadamard type inequalities for F-convex functions*, Miskolc Math. Notes, **20**(2019), no. 1, 169-191.

[5] Defnetti, B., *Sulla strati cazioni convesse*, Ann. Math. Pura. Appl., **30**(1949), 173-183.

[6] Dragomir, S.S., Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**(1998), no. 5, 91-95.

[7] Dragomir, S.S., Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[8] Gorenflo, R., Mainardi, F., *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer Verlag, Wien, 1997, 223-276.

[9] Hudzik, H., Maligranda, L., *Some remarks on s-convex functions*, Aequationes Math., **48**(1994), 100-111.

[10] Hyers, D.H., Ulam, S.M., *Approximately convex functions*, Proc. Amer. Math. Soc., **3**(1952), 821-828.

[11] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, 2006.

[12] Kirmaci, U.S., *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., **147**(2004), 91-95.

[13] Mangasarian, O.L., *Pseudo-convex functions*, SIAM Journal on Control., **3**(1965), 281-290.

[14] Miller, S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993.

[15] Mohammed, P.O., Sarikaya, M.Z., *Hermite-Hadamard type inequalities for F-convex function involving fractional integrals*, J. Inequal. Appl., **2018**(2018), Art. no. 359.

[16] Pearce, C.E.M., Pecarič, J., *Inequalities for differentiable mappings with application to special means and quadrature formula*, Appl. Math. Lett., **13**(2000), 51-55.

[17] Pečarić, J.E., Proschan, F., Tong, Y.L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[18] Podlubni, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] Polyak, B.T., *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl., **7**(1966), 72-75.

[20] Samet, B., *On an implicit convexity concept and some integral inequalities*, J. Inequal. Appl., **2016**(2016), Art. no. 308.

[21] Sarikaya, M.Z., Aktan, N., *On the generalization some integral inequalities and their applications,* Math. Comput. Modelling, **54**(2011), no. 9-10, 2175-2182.

[22] Sarikaya, M.Z., Ertuğral, F., *On the generalized Hermite-Hadamard inequalities*, Annals of the University of Craiova – Mathematics and Computer Science Series (in press).

[23] Sarikaya, M.Z., Saglam, A., Yildirim, H., *On some Hadamard-type inequalities for* $h-$*convex functions*, J. Math. Inequal., **2**(2008), no. 3, 335-341.

[24] Sarikaya, M.Z., Saglam, A., Yildirim, H., *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex*, Int. J. Open Probl. Comput. Sci. Math., IJOPCM, **5**(2012), no. 3, 1-14.

[25] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, **57**(2013), 2403-2407.

[26] Sarikaya, M.Z., Tunç, T., Budak, H., *Simpson's type inequality for* F*-convex function*, Facta Univ., Ser. Math. Inf., **32**(2018), no. 5, 747-753.

[27] Set, E., Mumcu, I., *Hermite-Hadamard type inequalities for* F*-convex functions via Katugampola fractional integral*, (submitted), (2018).

[28] Varosanec, S., *On h-convexity*, J. Math. Anal. Appl., **326**(2007), no. 1, 303-311.

Hüseyin Budak
Department of Mathematics,
Faculty of Science and Arts,
Düzce University, Düzce-Turkey
e-mail: hsyn.budak@gmail.com

Muhammad Aamir Ali
Jiangsu Key Laboratory for NSLSCS,
School of Mathematical Sciences,
Nanjing Normal University, 210023, China
e-mail: mahr.muhammad.aamir@gmail.com

Artion Kashuri
Department of Mathematics,
Faculty of Technical Science,
University Ismail Qemali, Vlora, Albania
e-mail: artionkashuri@gmail.com