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On (p, q)-Opial type inequalities for (p, q)-calculus

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Abstract. In this paper, we establish some (p, q)-Opial type inequalities and generalization of (p, q)-Opial type inequalities.

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1. Introduction

(p, q)-Calculus is more general from q-calculus. There have been many studies on (p, q)-calculus. Recently, Tunç and Göv [27, 28, 29] studied the concept of (p, q)-derivatives and (p, q)-integrals over the intervals of $[a, b] \subset \mathbb{R}$ and settled a number of (p, q) analogues of some well-known results like Hölder inequality, Minkowski inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev and other integral inequalities using classical convexity. The most recently, Alp et al. in [3], proved q-Hermite-Hadamard inequality, some new q-Hermite-Hadamard inequalities, and generalized q-Hermite-Hadamard inequality, also they studied some integral inequalities which provide quantum estimates for the left part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and quasi-convex functions. See [10], [12], [13], [14], [15] for q and (p, q)-analysis.

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about one century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. This class of inequalities includes the Wirtinger, Lyapunov, Landau-Kolmogorov, and Hardy types to which an abundance of literature, including several monographs, have been devoted. Of these inequalities, the earliest one which appeared in print is believed to be a Wirtinger type inequality by L. Sheeffer in 1885 (actually before the result by Wirtinger), which found its motivation in the calculus of variations. Improvements, generalizations, extensions, discretizations, and new applications of these inequalities are constantly being found, making their study an extremely prolific field. These inequalities and their manifold manifestations occupy a central position in mathematical analysis and its applications [1].

In the year 1960, Opial [17], [18] established the following interesting integral inequalities:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(t) > 0$ in $(0,h)$. Then, the following inequalities holds:*

i) If $x(0) = x(h) = 0$, then

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h |x'(t)|^2\,dt. \tag{1.1}$$

ii) If $x(0) = 0$, then

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{2}\int_0^h |x'(t)|^2\,dt. \tag{1.2}$$

In (1.1), the constant $h/4$ is the best possible.

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [5], [7], [8], [11], [20], [22], [23], [24], [30], [4], [9], [16].

In this paper we obtain (p,q)-Opial type inequalities on (p,q)-quantum integral. If $p,q \to 1^-$ are taken, all the results we have obtained provide valid results for classical analysis.

2. Preliminaries and definitions of (p,q)-calculus

Throughout this paper, let $[a,b] \subset \mathbb{R}$ is an interval, $0 < q < p \leq 1$ are constants. The following definitions and theorems for (p,q)- derivative and (p,q)- integral are given in [27, 28].

Definition 2.1. [27, 28]For a continuous function $f : [a,b] \to \mathbb{R}$ then (p,q)- derivative of f at $t \in [a,b]$ is characterized by the expression

$$ {}_aD_{p,q}f(t) = \frac{f(pt + (1-p)a) - f(qt + (1-q)a)}{(p-q)(t-a)},\ t \neq a. \tag{2.1}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, thus we have

$$ {}_aD_{p,q}f(a) = \lim_{t\to a} {}_aD_{p,q}f(t) .$$

The function f is said to be (p,q)- differentiable on $[a,b]$ if ${}_aD_{p,q}f(t)$ exists for all $t \in [a,b]$. If $a = 0$ in (2.1), then ${}_0D_{p,q}f(t) = D_{p,q}f(t)$, where $D_{p,q}f(t)$ is familiar

(p,q)- derivative of f at $t \in [a,b]$ defined by the expression (see [6, 13, 21])

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \ t \neq 0. \tag{2.2}$$

Note also that if $p = 1$ in (2.2), then $D_q f(x)$ is familiar q- derivative of f at $x \in [a,b]$ defined by the expression (see [14])

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \ t \neq 0. \tag{2.3}$$

Corollary 2.2. [21]*For f, g are two functions the rule of multiplicative derivative $D_{p,q}f(t)$ is*

$$D_{p,q}f(t)g(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t).$$

We will use the following proposition throughout our work:

Proposition 2.3.

$$D_{p,q}x^n(t) = \sum_{i=0}^{n-1} x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t) \tag{2.4}$$

Proof. By using rule of multiplicative derivative $D_{p,q}f(t)$ we have

$$\begin{aligned}
D_{p,q}x^n(t) &= D_{p,q}\left[x^{n-1}(t)x(t)\right] \\
&= x^{n-1}(pt)D_{p,q}x(t) + x(qt)D_{p,q}x^{n-1}(t) \\
&= x^{n-1}(pt)D_{p,q}x(t) + x(qt)\left[x^{n-2}(pt)D_{p,q}x(t) + x(qt)D_{p,q}x^{n-2}(t)\right] \\
&= \left[x^{n-1}(pt) + x^{n-2}(pt)\right]D_{p,q}x(t) + x^2(qt)D_{p,q}x^{n-2}(t) \\
&= \left[x^{n-1}(pt) + x^{n-2}(pt) + x^{n-3}(pt)\right]D_{p,q}x(t) + x^3(qt)D_{p,q}x^{n-3}(t) \\
&\ldots \\
&= \sum_{i=0}^{n-1} x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t)
\end{aligned}$$

□

Definition 2.4. [27, 28]. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The definite (p,q) integral on $[a,b]$ is delineated as

$$\int_a^t f(x) \ {}_a d_{p,q}x = (p-q)(t-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \tag{2.5}$$

for $t \in [a, pb + (1-p)a]$. If $c \in (a,t)$, then the (p,q)- definite integral on $[c,t]$ is expressed as

$$\int_c^t f(x) \ {}_a d_{p,q}x = \int_a^t f(x) \ {}_a d_{p,q}x - \int_a^c f(x) \ {}_a d_{p,q}x \ . \tag{2.6}$$

If $p=1$ in (2.5), then one can get the classical q- definite integral on $[a,b]$ defined by (see [25, Definition 2.2])

$$\int_{a}^{t} f\left(x\right)\ \ {}_{a}d_{q}x\ \ =\left(1-q\right)\left(t-a\right)\sum_{n=0}^{\infty}q^{n}f\left(q^{n}t+\left(1-q^{n}\right)a\right). \tag{2.7}$$

If $a=0$ in (2.5), then one can get the classical (p,q)- definite integral defined by (see [21, Definition 4.])

$$\int_{0}^{t} f\left(x\right)\ \ {}_{0}d_{p,q}x\ \ =\int_{0}^{t} f\left(x\right)\ \ d_{p,q}x\ \ =\left(p-q\right)t\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}t\right). \tag{2.8}$$

Note also that if $p=1$ in (2.8), then one can get the classical q- definite integral defined by (see [25, Definition 2.2])

$$\int_{0}^{t} f\left(x\right)\ \ {}_{0}d_{q}x\ \ =\int_{0}^{t} f\left(x\right)\ \ d_{q}x\ \ =\left(1-q\right)t\sum_{n=0}^{\infty}q^{n}f\left(q^{n}t\right). \tag{2.9}$$

3. Main results

First we will prove the (p,q)-Opial inequalities below and some results

Theorem 3.1 ((p,q)-Opial Inequality). *Let $x(t)\in C^{(1)}[0,h]$ be such that*

$$x(0)=x(h)=0,$$

and $x(t)>0$ in $(0,h)$. Then, the following inequality holds:

$$\int_{0}^{h}\left|x(pt)+x(qt)\right|\left|D_{p,q}x\left(t\right)\right|d_{p,q}t\leq\frac{h}{p+q}\int_{0}^{h}\left|D_{p,q}x\left(t\right)\right|^{2}d_{p,q}t. \tag{3.1}$$

Proof. Let choosing $y(t)$ and $z(t)$ functions as

$$\begin{aligned} y(t) &= \int_{0}^{t}\left|D_{p,q}x\left(s\right)\right|d_{p,q}s \qquad (3.2)\\ z(t) &= \int_{t}^{h}\left|D_{p,q}x\left(s\right)\right|d_{p,q}s, \end{aligned}$$

such that

$$\left|D_{p,q}x\left(t\right)\right|=D_{p,q}y\left(t\right)=-D_{p,q}z\left(t\right) \tag{3.3}$$

and for $t \in [0,h]$, it follows that

$$\begin{aligned} |x(t)| &= \left| \int_0^t D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^t |D_{p,q}x(s)|\, d_{p,q}s = y(t) \qquad (3.4) \\ |x(t)| &= \left| \int_t^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_t^h |D_{p,q}x(s)|\, d_{p,q}s = z(t). \end{aligned}$$

$$\begin{aligned} |x(qt)| &= \left| \int_0^{qt} D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^{qt} |D_{p,q}x(s)|\, d_{p,q}s = y(qt) \qquad (3.5) \\ |x(qt)| &= \left| \int_{qt}^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_{qt}^h |D_{p,q}x(s)|\, d_{p,q}s = z(qt). \end{aligned}$$

and

$$\begin{aligned} |x(pt)| &= \left| \int_0^{pt} D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^{pt} |D_{p,q}x(s)|\, d_{p,q}s = y(pt) \qquad (3.6) \\ |x(pt)| &= \left| \int_{pt}^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_{pt}^h |D_{p,q}x(s)|\, d_{p,q}s = z(pt). \end{aligned}$$

Now let calculating the following (p,q)-integral by using partial (p,q)-integration method

$$\int_0^{\frac{h}{p+q}} y(pt) D_{p,q}y(t)\, d_{p,q}t = y^2\left(\frac{h}{p+q}\right) - \int_0^{\frac{h}{p+q}} y(qt) D_{p,q}y(t)\, d_{p,q}t$$

and then

$$\int_0^{\frac{h}{p+q}} \{y(pt) + y(qt)\} D_{p,q}y(t)\, d_{p,q}t = y^2\left(\frac{h}{p+q}\right). \qquad (3.7)$$

By using (3.3), (3.4), (3.5), (3.6) and (3.7) we have the following inequality

$$\begin{aligned} \int_0^{\frac{h}{p+q}} |x(pt) + x(qt)|\, |D_{p,q}x(t)|\, d_{p,q}t &\leq \int_0^{\frac{h}{p+q}} \{|x(pt)| + |x(qt)|\}\, |D_{p,q}x(t)|\, d_{p,q}t \qquad (3.8) \\ &\leq \int_0^{\frac{h}{p+q}} \{y(pt) + y(qt)\} D_{p,q}y(t)\, d_{p,q}t \\ &= y^2\left(\frac{h}{p+q}\right). \end{aligned}$$

Similarly we can write that

$$\begin{aligned}\int_{\frac{h}{p+q}}^{h} |x(pt)+x(qt)|\,|D_{p,q}x(t)|\,d_{p,q}t &\le \int_{\frac{h}{p+q}}^{h} \{|x(pt)|+|x(qt)|\}\,|D_{p,q}x(t)|\,d_{p,q}t \qquad (3.9)\\ &\le -\int_{\frac{h}{p+q}}^{h} \{z(pt)+z(qt)\}\,D_{p,q}z(t)\,d_{p,q}t\\ &= z^2\left(\frac{h}{p+q}\right).\end{aligned}$$

Adding (3.8) and (3.9), we find that

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q}x(t)|\,d_{p,q}t \le y^2\left(\frac{h}{p+q}\right)+z^2\left(\frac{h}{p+q}\right).$$

Finally using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}y^2\left(\frac{h}{p+q}\right) &= \left[\int_0^{\frac{h}{p+q}} |D_{p,q}x(t)|\,d_{p,q}t\right]^2 \qquad (3.10)\\ &= \left[\left(\int_0^{\frac{h}{p+q}} 1^2 d_{p,q}t\right)^{1/2}\left(\int_0^{\frac{h}{p+q}} |D_{p,q}x(t)|^2\,d_{p,q}t\right)^{1/2}\right]^2\\ &= \frac{h}{p+q}\int_0^{\frac{h}{p+q}} |D_{p,q}x(t)|^2\,d_{p,q}t.\end{aligned}$$

Similarly we have

$$z^2\left(\frac{h}{p+q}\right)=\left[\int_{\frac{h}{p+q}}^{h} |D_{p,q}x(t)|\,d_{p,q}t\right]^2=\frac{h}{p+q}\int_{\frac{h}{p+q}}^{h} |D_{p,q}x(t)|^2\,d_{p,q}t. \qquad (3.11)$$

Therefore, from (3.10) and (3.11) we obtain that

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q}x(t)|\,d_{p,q}t \le \frac{h}{p+q}\int_0^h |D_{p,q}x(t)|^2\,d_{p,q}t$$

and the proof is completed. □

Remark 3.2. In Theorem 3.1 if we take $p \to 1^-$, we recapture the following q-Opial inequality in [2]:

$$\int_0^h |x(t)+x(qt)|\,|D_q x\,(t)|\,d_q t \le \frac{h}{1+q}\int_0^h |D_q x\,(t)|^2\,d_q t.$$

Remark 3.3. In Theorem 3.1 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the (1.1) inequality.

Theorem 3.4. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0)=0$ and $x(t)>0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q} x\,(t)|\,d_{p,q} t \le h\int_0^h |D_{p,q} x\,(t)|^2\,d_{p,q} t. \tag{3.12}$$

Proof. Let choosing $y(t)$ functions as (3.2) such that

$$\begin{aligned} |x\,(t)| &\le y(t) \\ |D_{p,q} x\,(t)| &= D_{p,q} y\,(t) \end{aligned} \tag{3.13}$$

and then

$$\int_0^h y(pt) D_{p,q} y\,(t)\,d_{p,q} t = y^2\,(h) - \int_0^h y(qt) D_{p,q} y\,(t)\,d_{p,q} t,$$

i.e

$$\int_0^h \{y(pt)+y(qt)\}\,D_{p,q} y\,(t)\,d_{p,q} t = y^2\,(h)\,. \tag{3.14}$$

Now by using Cauchy-Schwarz inequality for $y^2\,(h)$, we have

$$y^2\,(h) = \left[\int_0^h |D_{p,q} x\,(s)|\,d_{p,q} s\right]^2 \le h\int_0^h |D_{p,q} x\,(s)|^2\,d_{p,q} s.$$

Finally by using (3.13), then we have

$$\begin{aligned} \int_0^h |x(pt)+x(qt)|\,|D_{p,q} x\,(t)|\,d_{p,q} t &\le \int_0^h \{y(pt)+y(qt)\}\,D_{p,q} y\,(t)\,d_{p,q} t \\ &\le h\int_0^h |D_{p,q} x\,(t)|^2\,d_{p,q} t \end{aligned}$$

and the proof is completed. □

Remark 3.5. In Theorem 3.4 if we take $p \to 1^-$, we recapture the following q-Opial inequality in [2]:

$$\int_0^h |x(t) + x(qt)| \, |D_q x(t)| \, d_q t \leq h \int_0^h |D_q x(t)|^2 \, d_q t.$$

Remark 3.6. In Theorem 3.4 if we take $q \to 1^-$, we recapture the (1.2) inequality.

Theorem 3.7. *Let $k(t)$ be a nonnegative and continuous function on $[0, h]$ and $x(t) \in C^{(1)}[0, h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0, h)$. Then, the following inequality holds:*

$$\int_0^h k(t) \, |x(pt) + x(qt)| \, |D_{p,q} x(t)| \, d_{p,q} t \leq \left(h \int_0^h k^2(t) \, d_{p,q} t \right)^{\frac{1}{2}} \int_0^h |D_{p,q} x(t)|^2 \, d_{p,q} t$$

Proof. In proof of Theorem 3.1, we obtained that

$$|x(t)| \leq y(t) \quad \text{and} \quad |x(t)| \leq z(t)$$

Thus we get

$$\begin{aligned} |x(pt)| &\leq \frac{y(pt) + z(pt)}{2} \\ &= \frac{\int_0^{pt} D_{p,q} x(s) \, d_{p,q} s + \int_{pt}^h D_{p,q} x(s) \, d_{p,q} s}{2} \\ &= \frac{1}{2} \int_0^h |D_{p,q} x(s)| \, d_{p,q} s. \end{aligned} \tag{3.15}$$

$$\begin{aligned} |x(qt)| &\leq \frac{y(qt) + z(qt)}{2} \\ &= \frac{\int_0^{qt} D_{p,q} x(s) \, d_{p,q} s + \int_{qt}^h D_{p,q} x(s) \, d_{p,q} s}{2} \\ &= \frac{1}{2} \int_0^h |D_{p,q} x(s)| \, d_{p,q} s. \end{aligned} \tag{3.16}$$

By using the (3.15) and from Cauchy-Schwarz inequality for (p,q)-integral,

$$\begin{aligned}
&\int_0^h k(t)\left|x(pt)\right|^2 d_{p,q}t \qquad (3.17)\\
\leq\ & \frac{1}{4}\int_0^h k(t)\left[\int_0^h \left|D_{p,q}x(s)\right| d_{p,q}s\right]^2 d_{p,q}t\\
\leq\ & \frac{1}{4}\left(\int_0^h k(t)\, d_{p,q}t\right)\left(\int_0^h d_{p,q}s\right)\left(\int_0^h \left|D_{p,q}x(s)\right|^2 d_{p,q}s\right)\\
\leq\ & \frac{h}{4}\left(\int_0^h k(t)\, d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right).
\end{aligned}$$

Similarly from (3.16) we have

$$\int_0^h k(t)\left|x(qt)\right|^2 d_{p,q}t \leq \frac{h}{4}\left(\int_0^h k(t)\, d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right). \qquad (3.18)$$

From Cauchy-Schwarz inequality and (3.17), we have

$$\begin{aligned}
&\int_0^h k(t)\left|x(pt)\, D_{p,q}x(t)\right| d_{p,q}t \qquad (3.19)\\
\leq\ & \left(\int_0^h k^2(t)\left|x(pt)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\\
\leq\ & \left(\frac{h}{4}\left(\int_0^h k^2(t)\, d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\\
\leq\ & \frac{1}{2}\left(h\int_0^h k^2(t)\, d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right).
\end{aligned}$$

Similarly, by using (3.18) we can write

$$\begin{aligned}
&\int_0^h k(t)\left|x(qt)\, D_{p,q}x(t)\right| d_{p,q}t \qquad (3.20)\\
\leq\ & \frac{1}{2}\left(h\int_0^h k^2(t)\, d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)
\end{aligned}$$

Finally by adding (3.19) and (3.20) we have

$$\begin{aligned}
&\int_0^h k(t)\left|x(pt)+x(qt)\right|\left|D_{p,q}x(t)\right|d_{p,q}t \\
\leq\ &\int_0^h k(t)\left\{\left|x(pt)\right|+\left|x(qt)\right|\right\}\left|D_{p,q}x(t)\right|d_{p,q}t \\
\leq\ &\left(h\int_0^h k^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)
\end{aligned}$$

which is complete the proof. □

Remark 3.8. In Theorem 3.7 if we take $p\to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h k(t)\left|x(t)+x(qt)\right|\left|D_q x(t)\right|d_q t\leq\left(h\int_0^h k^2(t)\,d_q t\right)^{\frac{1}{2}}\int_0^h \left|D_q x(t)\right|^2 d_q t$$

Remark 3.9. In Theorem 3.7 if we take $p\to 1^-$ and $q\to 1^-$, we recapture the following inequality

$$\int_0^h k(t)\left|x(t)x'(t)\right|dt\leq\left(\frac{h}{4}\int_0^h k^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^h \left|x'(t)\right|^2 dt\right)$$

which is proved by Trable in [26].

Theorem 3.10. *Let $x(t)\in C^{(1)}[0,h]$ be such that $x(0)=x(h)=0$, and $x(t)>0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h \left|x(s)\right|^{m(R+r)}d_{p,q}s\leq\frac{\left[K(m)\right]^{(R+r)}}{p^{R-1}}\int_0^h \left|D_{p,q}x(s)\right|^{m(R+r)}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)}d_{p,q}s \tag{3.21}$$

where

$$K(m)=\int_0^h\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}d_{p,q}t.$$

Proof. Firstly we can write (p,q)-derivative of $x^n(t)$ from (2.4)

$$D_{p,q}x^n(t)=\sum_{i=0}^{n-1}x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t) \tag{3.22}$$

using (3.22) we have

$$\int_0^t D_{p,q}x^{R+r}(s)d_{p,q}s = x^{R+r}(t) \tag{3.23}$$

on the other hand we can write

$$\int_0^t D_{p,q}x^{R+r}(s)d_{p,q}s = \int_0^t \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\left(s\right)d_{p,q}s. \tag{3.24}$$

From (3.23)-(3.24) we get

$$x^{R+r}(t) = \int_0^t \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\left(s\right)d_{p,q}s. \tag{3.25}$$

Similarly, we can write

$$x^{R+r}(t) = -\int_t^h \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\left(s\right)d_{p,q}s. \tag{3.26}$$

Using the Hölder's inequality for (p,q)-integral with indices m, $\frac{m}{m-1}$ in (3.25) and (3.26), we have

$$\begin{aligned}
&\left|x(t)\right|^{m(R+r)} \qquad (3.27)\\
\leq\ & \left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\left(s\right)\right| d_{p,q}s\right)^m\\
\leq\ & \left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m \left|D_{p,q}x\left(s\right)\right|^m d_{p,q}s\right)\left(\int_0^t d_{p,q}s\right)^{m-1}\\
\leq\ & t^{m-1}\left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m \left|D_{p,q}x\left(s\right)\right|^m d_{p,q}s\right).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left|x(t)\right|^{m(R+r)} \qquad (3.28)\\
\leq\ & (h-t)^{m-1}\left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m \left|D_{p,q}x\left(s\right)\right|^m d_{p,q}s\right).
\end{aligned}$$

Multiplying the (3.27) and (3.28) respectively by t^{1-m} and $(h-t)^{1-m}$ and summing these inequalities, we have

$$\begin{aligned}&\left[t^{1-m}+(h-t)^{1-m}\right]|x(t)|^{m(R+r)} \qquad (3.29)\\ \leq\ &\left(\int_0^h\left|\sum_{i=0}^{R+r-1}x^{R+r-1-i}(ps)x^i(qs)\right|^m|D_{p,q}x(s)|^m d_{p,q}s\right)\end{aligned}$$

and for $t\in[0,h]$ we get

$$\begin{aligned}|x(t)|^{m(R+r)}&\leq\left[t^{1-m}+(h-t)^{1-m}\right]^{-1} \qquad (3.30)\\ &\times\left(\int_0^h\left|\sum_{i=0}^{R+r-1}x^{R+r-1-i}(ps)x^i(qs)\right|^m|D_{p,q}x(s)|^m d_{p,q}s\right)\\ &=\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}\\ &\times\left(\int_0^h|x(ps)|^{m(R+r-1)}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m|D_{p,q}x(s)|^m d_{p,q}s\right)\\ &=\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}\\ &\times\left(\int_0^h|x(ps)|^{mR/r}|D_{p,q}x(s)|^m|x(ps)|^{m(R+r-1)-mR/r}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m d_{p,q}s\right).\end{aligned}$$

Integrating (3.30) on $[0,h]$ and using the Hölder's inequality for (p,q)-integral with indices r, $\frac{r}{r-1}$ we have

$$\begin{aligned}\int_0^h|x(t)|^{m(R+r)}d_{p,q}t&\leq\int_0^h\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}d_{p,q}t \qquad (3.31)\\ &\times\left(\int_0^h|x(ps)|^{mR/r}|D_{p,q}x(s)|^m|x(ps)|^{m(R+r-1)-mR/r}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m d_{p,q}s\right)\\ &\leq K(m)\left(\int_0^h|x(ps)|^{mR}|D_{p,q}x(s)|^{mr}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{mr}d_{p,q}s\right)^{\frac{1}{r}}\\ &\times\left(\int_0^h|x(ps)|^{m(R+r)}d_{p,q}s\right)^{\frac{r-1}{r}}\end{aligned}$$

which by dividing the both sides of (3.31) with $\left(\int_0^h|x(ps)|^{m(R+r)}d_{p,q}s\right)^{\frac{r-1}{r}}$ and taking the rth power on both sides of resulting inequaliy. Finally by using the Hölder's inequality for (p,q)-integral with indices $\frac{R+r}{R}$, $\frac{R+r}{r}$ then, we get

$$\begin{aligned}
&\int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \qquad (3.32)\\
&\leq [K(m)]^r \left(\int_0^h |x(ps)|^{mR} |D_{p,q}x(s)|^{mr} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{mr} d_{p,q}s\right)\\
&\leq [K(m)]^r \left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{R}{R+r}}\\
&\quad\times\left(\int_0^h |D_{p,q}x(s)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s\right)^{\frac{r}{R+r}}
\end{aligned}$$

which by dividing the both sides of (3.32) with $\left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{R}{R+r}}$

$$\begin{aligned}
&\int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} \qquad (3.33)\\
&\leq [K(m)]^r \left(\int_0^h |D_{p,q}x(s)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s\right)^{\frac{r}{R+r}}
\end{aligned}$$

Here since

$$\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s = \frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s$$

from $|x(s)|^{m(R+r)} \geq 0$ and $ph \leq h$ we can say

$$\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s = \frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s \leq \frac{1}{p}\int_0^h |x(s)|^{m(R+r)}\, d_{p,q}s$$

so

$$\begin{aligned}
\left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} &= \left(\frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} \qquad (3.34)\\
&\geq \left(\frac{1}{p}\int_0^h |x(s)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}}.
\end{aligned}$$

From (3.34)

$$\begin{aligned}
&\int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \left(\int_0^h |x\left(ps\right)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} \qquad (3.35)\\
\geq\ & \int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \left(\frac{1}{p}\int_0^h |x\left(s\right)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}}\\
=\ & p^{\frac{R}{R+r}} \left(\int_0^h |x\left(s\right)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{r}{R+r}}.
\end{aligned}$$

From (3.33) and (3.35) we get

$$\begin{aligned}
&p^{\frac{R}{R+r}} \left(\int_0^h |x\left(s\right)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{r}{R+r}} \qquad (3.36)\\
&\leq [K(m)]^r \left(\int_0^h |D_{p,q}x\left(s\right)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s\right)^{\frac{r}{R+r}}.
\end{aligned}$$

Finally by taking the $\frac{R+r}{r}$th power on both sides of (3.36) we have

$$\begin{aligned}
&\int_0^h |x(s)|^{m(R+r)}\, d_{p,q}s\\
&\leq \frac{[K(m)]^{(R+r)}}{p^{R-1}} \int_0^h |D_{p,q}x(s)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s
\end{aligned}$$

and the proof is completed. □

Remark 3.11. In Theorem 3.10 if we take $p \to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h |x\left(s\right)|^{m(R+r)}\, d_q s \leq [K(m)]^{(R+r)} \int_0^h |D_q x\left(s\right)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(s)}\right)^i\right|^{m(R+r)} d_q s$$

which is proved by Pachpatte in [19].

Remark 3.12. In Theorem 3.10 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the following result

$$\int_0^h |x(t)|^{m(R+r)}\, dt \leq \left[(R+r)^m K(m)\right]^{(R+r)} \int_0^h |x'\left(s\right)|^{m(R+r)}\, ds$$

which is proved by Pachpatte in [19].

Theorem 3.13. *Let $x(t)$ be absulately continuous on $[0,h]$, and $x(0)=0$. Further let $\alpha \geq 0$. Then, the following inequality holds:*

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x\left(t\right) \right| d_{p,q}t \leq h^\alpha \int_0^h \left|D_{p,q}x\left(s\right)\right|^{\alpha+1} d_{p,q}s.$$

Proof. By (p,q)-derivative of $x^n\left(t\right)$ from (2.4) we have

$$D_{p,q}y^{\alpha+1}\left(t\right) = \sum_{i=0}^{\alpha} y^{\alpha-i}(pt)y^i(qt)D_{p,q}y\left(t\right). \tag{3.37}$$

and choosing $y\left(t\right)$ as

$$y(t) = \int_0^t \left|D_{p,q}x\left(s\right)\right| d_{p,q}s \tag{3.38}$$

such that

$$\left|x\left(t\right)\right| \leq y(t).$$

From (3.37) we get

$$\begin{aligned} \int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x\left(t\right) \right| d_{p,q}t &\leq \int_0^h \sum_{i=0}^{\alpha} y^{\alpha-i}(pt)y^i(qt)D_{p,q}y\left(t\right) d_{p,q}t \qquad (3.39) \\ &= \int_0^h D_{p,q}y^{\alpha+1}\left(t\right) d_{p,q}t \\ &= y^{\alpha+1}\left(h\right). \end{aligned}$$

By using the Hölder's inequality and (3.39) with (3.38) for (p,q)-integral with indices $\alpha+1$, $\frac{\alpha+1}{\alpha}$, we get

$$\begin{aligned} y^{\alpha+1}\left(h\right) &= \left[\int_0^h \left|D_{p,q}x\left(s\right)\right| d_{p,q}s \right]^{\alpha+1} \\ &\leq \left[\left(\int_0^h d_{p,q}s \right)^{\frac{\alpha}{\alpha+1}} \left(\int_0^h \left|D_{p,q}x\left(s\right)\right|^{\alpha+1} d_{p,q}s \right)^{\frac{1}{\alpha+1}} \right]^{\alpha+1} \\ &= h^\alpha \int_0^h \left|D_{p,q}x\left(s\right)\right|^{\alpha+1} d_{p,q}s \end{aligned}$$

and

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x\left(t\right) \right| d_{p,q}t \leq h^\alpha \int_0^h \left|D_{p,q}x\left(s\right)\right|^{\alpha+1} d_{p,q}s$$

which is completes the proof. ☐

Remark 3.14. In Theorem 3.13 if we take $p \to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(t) x^i(qt) D_q x(t) \right| d_q t \le h^\alpha \int_0^h |D_q x(s)|^{\alpha+1} d_q s.$$

Remark 3.15. In Theorem 3.13 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the following result

$$\int_0^h |x^\alpha(t) x'(t)|\, dt \le \frac{h^\alpha}{\alpha+1} \int_0^h |x'(s)|^{\alpha+1} ds$$

which is proved by Hua in [11].

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