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# Positive solutions for fractional differential equations with non-separated type nonlocal multi-point and multi-term integral boundary conditions

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**Abstract.** In this paper, we investigate a class of nonlinear fractional differential equations that contain both the multi-term fractional integral boundary condition and the multi-point boundary condition. By the Krasnoselskii fixed point theorem we obtain the existence of at least one positive solution. Then, we obtain the existence of at least three positive solutions by the Legget-Williams fixed point theorem. Two examples are given to illustrate our main results.

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**Keywords:** Fractional differential equations, Riemann-Liouville fractional derivative, multi-term fractional integral boundary condition, fixed point theorems.

## 1. Introduction

Differential equations of fractional order are one of the fast growing area of research in the field of mathematics and have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find numerous applications of fractional order differential equations in viscoelasticity, electro-chemistry, control theory, movement through porous media, electromagnetics, and signal processing of wireless communication system, etc (see [6, 7, 9, 18, 22, 23, 26, 29, 30]). Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multi-point boundary condition (see [1, 12, 13, 14, 21, 25, 28, 31]), integral boundary condition (see [3, 4, 5, 8, 15, 24, 32, 33]), and many other boundary conditions (see [2, 11, 16, 20, 35]).

In this paper, we are dedicated to considering fractional differential equations that contain both the multi-term fractional integral boundary condition and the multi-point boundary condition:

$$\begin{cases} D^q u(t) + f(t, u(t)) = 0, & 1 < q \le 2,\ 0 < t < 1, \\ u(0) = 0, & u(1) = \sum_{i=1}^{m} \alpha_i \left(I^{p_i} u\right)(\eta) + \sum_{i=1}^{m} \beta_i u(\xi_i), \end{cases} \tag{1.1}$$

where $D^q$ is the standard Riemann-Liouville fractional derivative of order $q$, $I^{p_i}$ is the Riemann-Liouville fractional integral of order $p_i > 0$, $i = 1, 2, ..., m$, $0 < \xi_1 < \xi_2 < ... < \xi_m < 1$ , $0 < \eta < 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $\alpha_i$, $\beta_i \ge 0$ with $i = 1, 2, ..., m$, are real constants such that

$$\Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} + \sum_{i=1}^{m} \beta_i \xi_i^{q-1} < 1.$$

Zhou and Jiang [36] considered the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u'(0) - \beta u(\xi) = 0,\ u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0 \end{cases}$$

where $\alpha$ is a real number with $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 \le \gamma_i \le 1$, $i = 1, 2, ..., m-3$, $0 \le \xi < \eta_1 < \eta_2 < ... < \eta_{m-3} \le 1$, $D_{0+}^{\alpha}$ is the Caputo's derivative. The authors used the fixed point index theory and Krein-Rutman theorem to obtain the existence results.

Ji et al. [17] investigated the existence and multiplicity results of positive solutions for the following boundary value problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f\left(t, u(t), D_{0+}^{\mu} u(t)\right) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) + D_{0+}^{\beta} u(1) = k u(\xi) + l D_{0+}^{\beta} u(\eta), \end{cases}$$

where $D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, $0 \le \beta \le 1$, $\xi,\ \eta \in (0,1)$, $0 \le \mu < 1$, $1 \le \alpha - \beta$, $1 \le \alpha - \mu$, $1 - l\eta^{\alpha-\beta-1}$, and $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous. They used the Leggett-Williams fixed point theorem to obtain the existence and multiplicity results of positive solutions.

Wang et al. [34] considered the following boundary value problem

$$\begin{cases} D^{\sigma} u(t) + f(t, u(t)) = 0, & t \in [0,1], \\ u^{(i)}(0) = 0, & i = 0, 1, 2, ..., n-2, \\ u(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} u(s)\,ds + \sum_{i=1}^{m-2} \gamma_i u(\eta_i), \end{cases}$$

where $D^{\sigma}$ represents the standard Riemann-Liouville fractional derivative of order $\sigma$ satisfying $n - 1 < \sigma \le n$ with $n \ge 3$. The authors used Krasnoselkii's fixed point theorem, Schauder type fixed point theorem, Banach's contraction mapping principle and nonlinear alternative for single-valued maps to obtain the existence results.

Inspired by the above works, in this paper, we establish the existence and multiplicity of positive solutions of the boundary value problem (1.1). Our paper is organized as follows. After this section, some definitions and lemmas will be established in Section 2. In Section 3, we give our main results in Theorems 3.1 and 3.2. Finally, in Section 4, as applications, some examples are presented to illustrate our main results

## 2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus, which can be found in [18, 27, 30]. We also state two fixed-point theorems due to Guo–Krasnosel'skii and Leggett–Williams.

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right side is pointwise defined on $(0, +\infty)$ where $\Gamma(.)$ is the Gamma function.

**Definition 2.2.** The Riemann-Liouville fractional derivative order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = \lceil \alpha \rceil + 1$, $\lceil \alpha \rceil$ denotes the integer part of number $\alpha$, provided that the right side is pointwise defined on $(0, \infty)$.

**Lemma 2.3.** *(i) If $u \in L^p(0,1)$, $1 \le p \le +\infty$, $\beta > \alpha > 0$, then*

$$I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t).$$

*(ii) If $\alpha > 0$ and $\gamma \in (-1, +\infty)$, then*

$$I_{0+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

**Lemma 2.4.** *Let $\alpha > 0$ and for any $y \in L^1(0,1)$. Then, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0 < t < 1$ is given by*

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

*where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = \lceil \alpha \rceil + 1$.*

**Definition 2.5.** Let $E$ be a real Banach space. A nonempty convex closed set $K \subset E$ is said to be a cone provided that

(i) $au \in K$ for all $u \in K$ and all $a \ge 0$, and

(ii) $u, -u \in K$ implies $u = 0$.

**Definition 2.6.** The map $\alpha$ is defined as a nonnegative continuous concave functional on a cone $K$ of a real Banach space $E$ provided that $\alpha : K \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \le t \le 1$.

**Lemma 2.7.** *Let* $\Delta = 1 - \Gamma(q)\sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} > 0$, $\alpha_i$, $\beta_i \geq 0$, $p_i > 0$, $i = 1, 2, ...m$, *and* $h \in C[0,1]$. *The unique solution* $u \in AC[0,1]$ *of the boundary value problem*

$$D^q u(t) + h(t) = 0, \quad t \in (0,1), \; q \in (1,2] \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i (I^{p_i} u)(\eta) + \sum_{i=1}^{m} \beta_i u(\xi_i) \tag{2.2}$$

*is given by*

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{2.3}$$

*where* $G(t,s)$ *is the Green's function given by*

$$G(t,s) = g(t,s) + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} g_i(\eta,s) + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i g(\xi_i,s) \tag{2.4}$$

*where*

$$g(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}, & 0 \leq s \leq t \leq 1, \\ t^{q-1}(1-s)^{q-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.5}$$

*and*

$$g_i(\eta,s) = \begin{cases} \eta^{p_i+q-1}(1-s)^{q-1} - (\eta-s)^{p_i+q-1}, & 0 \leq s \leq \eta < 1, \\ \eta^{p_i+q-1}(1-s)^{q-1}, & 0 < \eta \leq s \leq 1, \end{cases} \tag{2.6}$$

*Proof.* By Lemma 2.4, the general solution for the above equation (2.1) is

$$u(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + c_1 t^{q-1} + c_2 t^{q-2},$$

where $c_1, c_2 \in \mathbb{R}$. The first condition of (2.2) implies that $c_2 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + c_1 t^{q-1}. \tag{2.7}$$

Taking the Riemann-Liouville fractional integral of order $p_i > 0$ for (2.7) and using Lemma 2.3, we get that

$$\begin{aligned}
(I^{p_i} u)(t) &= \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \left( c_1 s^{q-1} - \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} dr \right) h(s) ds \\
&= c_1 \int_0^t \frac{(t-s)^{p_i-1} s^{q-1}}{\Gamma(p_i)} ds - \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} h(r) dsdr \\
&= c_1 \frac{t^{p_i+q-1}\Gamma(q)}{\Gamma(p_i+q)} - \frac{1}{\Gamma(p_i+q)} \int_0^t (t-s)^{p_i+q-1} h(s) ds.
\end{aligned}$$

The second condition of (2.2) yields

$$c_1 - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\, ds = c_1 \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1} \Gamma(q)}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^\eta (\eta - s)^{p_i+q-1} h(s)\, ds + c_1 \sum_{i=1}^{m} \beta_i \xi_i^{q-1} - \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i - s)^{q-1} h(s)\, ds.$$

Then, we have that

$$c_1 = \frac{1}{\Delta} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\, ds - \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^\eta (\eta - s)^{p_i+q-1} h(s)\, ds - \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i - s)^{q-1} h(s)\, ds \right\}.$$

Hence, the solution is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds + \frac{t^{q-1}}{\Delta \Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\, ds \\ &\quad - \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^n (\eta - s)^{p_i+q-1} h(s)\, ds \\ &\quad - \frac{t^{q-1}}{\Delta \Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i - s)^{q-1} h(s)\, ds \\ &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds + \frac{t^{q-1}}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\, ds \\ &\quad + \frac{t^{q-1}}{\Delta} \left\{ \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} + \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \right\} \int_0^1 (1-s)^{q-1} h(s)\, ds \\ &\quad - \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^n (\eta - s)^{p_i+q-1} h(s)\, ds \\ &\quad - \frac{t^{q-1}}{\Delta \Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i - s)^{q-1} h(s)\, ds \\ &= \int_0^1 g(t,s) h(s)\, ds + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^1 g_i(\eta, s) h(s)\, ds \\ &\quad + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i \int_0^1 g(\xi_i, s) h(s)\, ds \\ &= \int_0^1 G(t,s) h(s)\, ds. \end{aligned}$$

□

**Lemma 2.8.** *The Green's function $G(t,s)$ has the following properties:*
$(P_1)$ $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$.
$(P_2)$ $G(t,s)\geq 0$ *for all* $0\leq s,t\leq 1$.
$(P_3)$ $G(t,s)\leq \max\limits_{0\leq t\leq 1} G(t,s)\leq g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)+\sum\limits_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}g_i(\eta,s).$

$(P_4)$ $\int_0^1 \max\limits_{0\leq t\leq 1} G(t,s)\,ds\leq\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)}+\sum\limits_{i=1}^{m}\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right).$

$(P_5)$ $\min\limits_{\eta\leq t\leq 1} G(t,s)\geq\sum\limits_{i=1}^{m}\frac{\alpha_i\eta^{q-1}}{\Delta\Gamma(p_i+q)}g_i(\eta,s)+(q-1)\sum\limits_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}}{\Delta}sg(s,s)$

*for* $s\in[0,1]$.

*Proof.* It is easy to check that $(P_1)$ holds. To prove $(P_2)$, we will show that $g(t,s)\geq 0$ and $g_i(\eta,s)\geq 0$, $i=1,2,...,m$, for all $0\leq s,t\leq 1$. For $t\leq s$, it is clear that $G(t,s)\geq 0$, we only need to prove the case $s\leq t$.
Then

$$\begin{aligned} g(t,s) &= \frac{1}{\Gamma(q)}\left[t^{q-1}(1-s)^{q-1}-(t-s)^{q-1}\right] \\ &= \frac{1}{\Gamma(q)}\left[(t-ts)^{q-1}-(t-s)^{q-1}\right] \\ &\geq \frac{1}{\Gamma(q)}\left[(t-s)^{q-1}-(t-s)^{q-1}\right]=0. \end{aligned}$$

For $0\leq s\leq\eta<1$, we have

$$\begin{aligned} g_i(\eta,s) &= \eta^{p_i+q-1}(1-s)^{q-1}-(\eta-s)^{p_i+q-1} \\ &= \eta^{p_i}(\eta-\eta s)^{q-1}-(\eta-s)^{p_i+q-1} \\ &\geq \eta^{p_i}(\eta-s)^{q-1}-(\eta-s)^{p_i+q-1} \\ &= (\eta-s)^{q-1}\left(\eta^{p_i}-(\eta-s)^{p_i}\right) \\ &\geq 0. \end{aligned}$$

When $0<\eta\leq s\leq 1$, $g_i(\eta,s)=\eta^{p_i+q-1}(1-s)^{q-1}\geq 0$. Therefore, $g_i(\eta,s)\geq 0$, $i=1,2,...,m$ for all $0\leq s\leq 1$.
Now, we prove $(P_3)$. For a given $s\in[0,1]$, when $0\leq s\leq t\leq 1$

$$\Gamma(q)\,g(t,s)=t^{q-1}(1-s)^{q-1}-(t-s)^{q-1}$$

and thus

$$\begin{aligned} \Gamma(q)\frac{\partial}{\partial t}g(t,s) &= (q-1)\,t^{q-2}(1-s)^{q-1}-(q-1)(t-s)^{q-2} \\ &= (q-1)(t-ts)^{q-2}(1-s)-(q-1)(t-s)^{q-2} \\ &\leq (q-1)(t-s)^{q-2}(1-s)-(q-1)(t-s)^{q-2} \\ &= -s(q-1)(t-s)^{q-2}. \end{aligned}$$

Hence, $g(t,s)$ is decreasing with respect to $t$. Then we have $g(t,s) \leq g(s,s)$ for $0 \leq s \leq t \leq 1$. For $0 \leq t \leq s \leq 1$

$$\Gamma(q)\frac{\partial}{\partial t} g(t,s) = (q-1)\, t^{q-2}(1-s)^{q-1} \geq 0,$$

which means that $g(t,s)$ is increasing with respect to $t$. Thus $g(t,s) \leq g(s,s)$ for $0 \leq t \leq s \leq 1$. Therefore $g(t,s) \leq g(s,s)$ for $0 \leq s,t \leq 1$.
From the above analysis, we have for $0 \leq s \leq 1$ that

$$\begin{aligned} G(t,s) \leq \max_{0\leq t\leq 1} G(t,s) &= \max_{0\leq t\leq 1} \Bigg( g(t,s) \\ &\quad + \frac{t^{q-1}}{\Delta}\sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} g_i(\eta,s) + \frac{t^{q-1}}{\Delta}\sum_{i=1}^{m} \beta_i g(\xi_i,s)\Bigg) \\ &\leq g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right) + \sum_{i=1}^{m} \frac{\alpha_i}{\Delta\Gamma(p_i+q)} g_i(\eta,s). \end{aligned}$$

To prove $(P_4)$, by direct integration, we have

$$\begin{aligned} \int_0^1 \max_{0\leq t\leq 1} G(t,s)\,ds &\leq \int_0^1 \left[ g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right) + \sum_{i=1}^{m} \frac{\alpha_i}{\Delta\Gamma(p_i+q)} g_i(\eta,s)\right] ds \\ &= \left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\int_0^1 \frac{s^{q-1}(1-s)^{q-1}}{\Gamma(q)} ds \\ &\quad + \sum_{i=1}^{m} \frac{\alpha_i}{\Delta\Gamma(p_i+q)} \left(\int_\eta^1 \eta^{p_i+q-1}(1-s)^{q-1}\right) ds \\ &\quad + \sum_{i=1}^{m} \frac{\alpha_i}{\Delta\Gamma(p_i+q)} \left(\int_0^\eta \left[\eta^{p_i+q-1}(1-s)^{q-1} - (\eta-s)^{p_i+q-1}\right] ds\right) \\ &= \left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right). \end{aligned}$$

Now, we shall prove $(P_5)$.
Firstly, let $k_1(\xi_i,s) = \frac{g(\xi_i,s)}{g(s,s)}$ for $0 < s < \xi_i < 1$, $i = 1,2,...,m$, then we get

$$k_1(\xi_i,s) = \frac{(\xi_i(1-s))^{q-1} - (\xi_i - s)^{q-1}}{s^{q-1}(1-s)^{q-1}} = \frac{(q-1)\int_{\xi_i-s}^{\xi_i(1-s)} x^{q-2}dx}{s^{q-1}(1-s)^{q-1}}.$$

Since the function $x \longmapsto x^{q-2}$ is continuous and decreasing on $[\xi_i - s,\ \xi_i(1-s)]$, we have

$$\begin{aligned} k_1(\xi_i,s) &\geq \frac{(q-1)(\xi_i(1-s))^{q-2}\left[\xi_i(1-s) - (\xi_i - s)\right]}{s^{q-1}(1-s)^{q-1}} \\ &= \frac{(q-1)\,\xi_i^{q-2}(1-s)^{q-2}\, s(1-\xi_i)}{s^{q-1}(1-s)^{q-1}} \\ &\geq (q-1)\,\xi_1^{q-1}(1-\xi_i)\, s. \end{aligned}$$

Let

$$k_2(\xi_i, s) = \frac{g(\xi_i, s)}{g(s, s)}$$

for $0 < \xi_i \leq s < 1$, $i = 1, 2, ..., m$, then we get

$$k_2(\xi_i, s) = \frac{\xi_i^{q-1}}{s^{q-1}} \geq \frac{\xi_i^{q-1}}{s^{q-2}} = \xi_i^{q-1} s^{2-q} \geq (q-1)\, \xi_i^{q-1}(1-\xi_i)\, s.$$

Therefore, we have

$$g(\xi_i, s) \geq (q-1)\, sg(s, s)\left(\xi_i^{q-1} - \xi_i^{q}\right) \quad for \quad 0 < s, \xi_i < 1 \tag{2.8}$$

Furthermore, the inequality in (2.8) is satisfied for $s \in \{0, 1\}$. Hence

$$g(\xi_i, s) \geq (q-1)\, sg(s, s)\left(\xi_i^{q-1} - \xi_i^{q}\right) \quad for \quad 0 \leq s, \xi_i \leq 1. \tag{2.9}$$

Secondly, from $g(t, s) \geq 0$, $g_i(\eta, s) \geq 0$, $i = 1, 2, ..., m$ and from (2.9), we have

$$\begin{aligned}
\min_{\eta \leq t \leq 1} G(t, s) &= \min_{\eta \leq t \leq 1}\left(g(t, s) + \frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(p_i + q)} g_i(\eta, s)\right. \\
&\quad \left. + \frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\beta_i g(\xi_i, s)\right) \\
&\geq \min_{\eta \leq t \leq 1} g(t, s) + \min_{\eta \leq t \leq 1}\frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(p_i + q)} g_i(\eta, s) \\
&\quad + \min_{\eta \leq t \leq 1}\frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\beta_i g(\xi_i, s) \\
&\geq \sum_{i=1}^{m}\frac{\alpha_i \eta^{q-1}}{\Delta\Gamma(p_i + q)} g_i(\eta, s) + (q-1)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1} - \xi_i^{q}\right)\eta^{q-1}}{\Delta} sg(s, s)
\end{aligned}$$

for $0 \leq s \leq 1$. This completes the proof. □

Let $E = C([0, 1], \mathbb{R})$ be the Banach space of all continuous functions defined on $[0, 1]$ that are mapped into $\mathbb{R}$ with the norm defined as $\|u\| = \sup_{t\in[0,1]} |u(t)|$. If $u \in E$ satisfies the problem (1.1) and $u(t) \geq 0$ for any $t \in [0, 1]$, then $u$ is called a nonnegative solution of the problem (1.1). If $u$ is a nonnegative solution of the problem (1.1) with $\|u\| > 0$, then $u$ is called a positive solution of the problem (1.1). Define the cone $\mathcal{K} \in E$ by

$$\mathcal{K} = \{u \in E : \ u(t) \geq 0\},$$

and the operator $A : K \to E$ by

$$Au(t) := \int_0^1 G(t, s) f(s, u(s))\, ds. \tag{2.10}$$

In view of Lemma 2.7, the nonnegative solutions of problem (1.1) are given by the operator equation $u(t) = Au(t)$

**Lemma 2.9.** *Suppose that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. The operator $A : \mathcal{K} \to \mathcal{K}$ is completely continuous.*

*Proof.* Since $G(t,s) \geq 0$ for $s,t \in [0,1]$, we have $Au(t) \geq 0$ for all $u \in \mathcal{K}$. Therefore, $A : \mathcal{K} \to \mathcal{K}$.
For a constant $R > 0$, we define $\Omega = \{u \in \mathcal{K} : \|u\| < R\}$.
Let

$$L = \max_{0 \leq t \leq 1, 0 \leq u \leq R} |f(t,u)|. \tag{2.11}$$

Then, for $u \in \Omega$, from Lemma 2.8, we have

$$\begin{aligned}
|Au(t)| &= \left| \int_0^1 G(t,s) f(s,u(s))\, ds \right| \\
&\leq L \int_0^1 G(t,s)\, ds \\
&\leq L \int_0^1 \left( g(s,s) \left( 1 + \frac{\sum_{i=1}^m \beta_i}{\Delta} \right) + \sum_{i=1}^m \frac{\alpha_i}{\Delta \Gamma(p_i+q)} g_i(\eta,s) \right) ds \\
&\leq \left( 1 + \frac{\sum_{i=1}^m \beta_i}{\Delta} \right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left( \frac{p_i + q(1-\eta)}{q(p_i+q)} \right).
\end{aligned}$$

Hence, $\|Au\| \leq M$, and so $A(\Omega)$ is uniformly bounded. Now, we shall show that $A(\Omega)$ is equicontinuous. For $u \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$|Au(t_2) - Au(t_1)| \leq L \int_0^1 |G(t_2,s) - G(t_1,s)|\, ds,$$

where $L$ is defined by (2.11). Since $G(t,s)$ is continuous on $[0,1] \times [0,1]$, therefore $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. Hence, for any $\epsilon > 0$, there exists a positive constant

$$\delta = \frac{1}{2} \left[ \frac{\epsilon \Gamma(q)}{L} \left( \frac{1}{\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left( \frac{p_i + q(1-n)}{q(p_i+q)} \right) + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta}} \right) \right]$$

whenever $|t_2 - t_1| < \delta$, we have the following two cases.

**Case 1.** $\delta \leq t_1 < t_2 < 1$.

Therefore,

$$
\begin{aligned}
\left|Au\left(t_{2}\right)-Au\left(t_{1}\right)\right| &\leq L\int_{0}^{1}\left|G\left(t_{2},s\right)-G\left(t_{1},s\right)\right|ds \\
&= L\left[\int_{0}^{t_{1}}\left|G\left(t_{2},s\right)-G\left(t_{1},s\right)\right|ds+\int_{t_{1}}^{t_{2}}\left|G\left(t_{2},s\right)-G\left(t_{1},s\right)\right|ds\right. \\
&\quad \left.+\int_{t_{2}}^{1}\left|G\left(t_{2},s\right)-G\left(t_{1},s\right)\right|ds\right] \\
&\leq \frac{\left(t_{2}^{q-1}-t_{1}^{q-1}\right)L}{\Gamma\left(q\right)}\int_{0}^{1}\left(1-s\right)^{q-1}ds \\
&+\frac{\left(t_{2}^{q-1}-t_{1}^{q-1}\right)L}{\Delta}\int_{0}^{1}\sum_{i=1}^{m}\frac{\alpha_{i}}{\Gamma\left(p_{i}+q\right)}g_{i}\left(\eta,s\right)ds \\
&\quad +\frac{\left(t_{2}^{q-1}-t_{1}^{q-1}\right)L}{\Delta}\sum_{i=1}^{m}\beta_{i}\int_{0}^{1}g\left(s,,s\right)ds \\
&= \frac{\left(t_{2}^{q-1}-t_{1}^{q-1}\right)L}{\Gamma\left(q\right)}\left[\frac{1}{q}+\sum_{i=1}^{m}\frac{\alpha_{i}\eta^{p_{i}+q-1}}{\Delta\Gamma\left(p_{i}+q\right)}\left(\frac{p_{i}+q\left(1-n\right)}{q\left(p_{i}+q\right)}\right)\right. \\
&\qquad\qquad \left.+\frac{\Gamma\left(q\right)}{\Gamma\left(2q\right)}\frac{\sum_{i=1}^{m}\beta_{i}}{\Delta}\right] \\
&\leq \frac{\left(q-1\right)\delta^{q-1}L}{\Gamma\left(q\right)}\left[\frac{1}{q}+\sum_{i=1}^{m}\frac{\alpha_{i}\eta^{p_{i}+q-1}}{\Delta\Gamma\left(p_{i}+q\right)}\left(\frac{p_{i}+q\left(1-n\right)}{q\left(p_{i}+q\right)}\right)\right. \\
&\qquad\qquad \left.+\frac{\Gamma\left(q\right)}{\Gamma\left(2q\right)}\frac{\sum_{i=1}^{m}\beta_{i}}{\Delta}\right] \\
&< \epsilon.
\end{aligned}
$$

**Case 2.** $0\leq t_{1}<1,\ t_{2}<2\delta.$
Hence

$$
\begin{aligned}
&\left|Au\left(t_{2}\right)-Au\left(t_{1}\right)\right| \leq L\int_{0}^{1}\left|G\left(t_{2},s\right)-G\left(t_{1},s\right)\right|ds \\
&< \frac{\left(t_{2}^{q-1}-t_{1}^{q-1}\right)L}{\Gamma\left(q\right)}\left[\frac{1}{q}+\sum_{i=1}^{m}\frac{\alpha_{i}\eta^{p_{i}+q-1}}{\Delta\Gamma\left(p_{i}+q\right)}\left(\frac{p_{i}+q\left(1-n\right)}{q\left(p_{i}+q\right)}\right)+\frac{\Gamma\left(q\right)}{\Gamma\left(2q\right)}\frac{\sum_{i=1}^{m}\beta_{i}}{\Delta}\right] \\
&\leq \frac{t_{2}^{q-1}L}{\Gamma\left(q\right)}\left[\frac{1}{q}+\sum_{i=1}^{m}\frac{\alpha_{i}\eta^{p_{i}+q-1}}{\Delta\Gamma\left(p_{i}+q\right)}\left(\frac{p_{i}+q\left(1-n\right)}{q\left(p_{i}+q\right)}\right)+\frac{\Gamma\left(q\right)}{\Gamma\left(2q\right)}\frac{\sum_{i=1}^{m}\beta_{i}}{\Delta}\right] \\
&< \frac{\left(2\delta\right)^{q-1}L}{\Gamma\left(q\right)}\left[\frac{1}{q}+\sum_{i=1}^{m}\frac{\alpha_{i}\eta^{p_{i}+q-1}}{\Delta\Gamma\left(p_{i}+q\right)}\left(\frac{p_{i}+q\left(1-n\right)}{q\left(p_{i}+q\right)}\right)+\frac{\Gamma\left(q\right)}{\Gamma\left(2q\right)}\frac{\sum_{i=1}^{m}\beta_{i}}{\Delta}\right] \\
&= \epsilon.
\end{aligned}
$$

Thus, $A(\Omega)$ is equicontinuous. In view of the Arzela-Ascoli theorem, we have that $\overline{A(\Omega)}$ is compact, which means $A : \mathcal{K} \to \mathcal{K}$ is a completely continuous operator. This completes the proof. □

**Theorem 2.10.** [10] *Let $E$ be a Banach space, and let $\mathcal{K} \in E$ be a cone. Assume that $\Omega_1$, $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that:*

*(i) $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$; or*

*(ii) $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$. Then $T$ has a fixed point $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

**Theorem 2.11.** [19] *Let $\mathcal{K}$ be a cone in the real Banach space $E$ and $c > 0$ be a constant. Assume that there exists a concave nonnegative continuous functional $\theta$ on $\mathcal{K}$ with $\theta(u) \leq \|u\|$ for all $u \in \overline{\mathcal{K}}_c$. Let $A : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ be a completely continuous operator. Suppose that there exist constants $0 < a < b < d \leq c$ such that the following conditions hold:*
*(i) $\{u \in \mathcal{K}(\theta, b, d) : \theta(u) > b\} \neq \emptyset$ and $\theta(Au) > b$ for $u \in \mathcal{K}(\theta, b, d)$;*
*(ii) $\|Au\| < a$ for $\|u\| \leq a$;*
*(iii) $\theta(Au) > b$ for $u \in \mathcal{K}(\theta, b, c)$ with $\|Au\| > d$.*
*Then $A$ has at least three fixed points $u_1$, $u_2$ and $u_3$ in $\overline{\mathcal{K}}_c$ such that*
*$\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.*

**Remark 2.12.** If there holds $d = c$, then condition $(i)$ implies condition $(iii)$ of Theorem 2.11.

## 3. Main results

In this section, in order to establish some results of existence and multiplicity of positive solutions for BVP (1.1), we will impose growth conditions on $f$ which allow us to apply Theorems 2.10 and 2.11.

For convenience, we denote

$$\Lambda_1 = \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right) + (q-1)\sum_{i=1}^m \frac{\beta_i\left(\xi_i^{q-1}-\xi_i^q\right)\eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)}$$
$$\Lambda_2 = \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right)$$
$$\Lambda_3 = \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+2(q-1)}(1-\eta)^q}{\Delta\Gamma(p_i+q)q} + (q-1)\sum_{i=1}^m \frac{\beta_i\left(\xi_i^{q-1}-\xi_i^q\right)\eta^{q-1}(1-\eta)^{2q}\Gamma(q+1)}{\Delta\Gamma(2q+1)}$$

**Theorem 3.1.** *Let $f : [0,1] \times [0,\infty) \to [0,\infty)$ be a continuous function. Assume that there exist constants $r_2 > r_1 > 0$, $M_1 \in (\Lambda_1^{-1}, \infty)$ and $M_2 \in (0, \Lambda_2^{-1})$ such that:*
*$(H_1)$ $f(t,u) \geq M_1 r_1$, for $(t,u) \in [0,1] \times [0, r_1]$;*
*$(H_2)$ $f(t,u) \leq M_2 r_2$, for $(t,u) \in [0,1] \times [0, r_2]$.*
*Then boundary value problem (1.1) has at least one positive solution $u$ such that $r_1 \leq \|u\| \leq r_2$.*

*Proof.* From Lemma 2.9, the operator $A : \mathcal{K} \to \mathcal{K}$ is completely continuous.
We divide the rest of the proof into two steps.

**Step 1.** Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then for any $u \in \mathcal{K} \cap \Omega_1$, we have $0 \le u(t) \le r_1$ for all $t \in [0,1]$. From $(H_1)$, it follows for $t \in [\eta, 1]$ that

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s) f(s,u(s))\,ds \\
&\ge \int_0^1 \min_{\eta \le t \le 1} G(t,s) f(s,u(s))\,ds \\
&\ge M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{q-1}}{\Delta \Gamma(p_i+q)} \int_0^1 g_i(\eta,s)\,ds \right. \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \int_0^1 s g(s,s)\,ds \right\} \\
&= M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{q-1}}{\Delta \Gamma(p_i+q)} \left( \int_\eta^1 \eta^{p_i+q-1} (1-s)^{q-1}\,ds \right. \right. \\
&\quad \left. + \int_0^\eta \left[ \eta^{p_i+q-1} (1-s)^{q-1} - (\eta - s)^{p_i+q-1} \right] ds \right) \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \right\} \\
&= M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta \Gamma(p_i+q)} \left( \frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \right. \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \right\} \\
&\ge r_1 = \|u\|,
\end{aligned}$$

which means that

$$\|Au\| \ge \|u\| \quad for\ u \in \mathcal{K} \cap \partial\Omega_1. \tag{3.1}$$

**Step 2.** Let $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for any $u \in \mathcal{K} \cap \partial\Omega_2$, we have $0 \le u(t) \le r_2$ for all $t \in [0,1]$. It follows from $(H_2)$ that for $t \in [0,1]$,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s) f(s,u(s))\,ds \\
&\le M_2 r_2 \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left( \frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \right\} \\
&\le r_2 = \|u\|,
\end{aligned}$$

which means that

$$\|Au\| \le \|u\| \quad for\ any\ u \in \mathcal{K} \cap \partial\Omega_2. \tag{3.2}$$

By $(i)$ of Theorem 2.10, we get that $A$ has a fixed point $u$ in $\mathcal{K}$ with $r_1 \le \|u\| \le r_2$, which is also a positive solution of boundary value problem (1.1). □

**Theorem 3.2.** *Let $f : [0,1] \times [0,\infty) \to [0,\infty)$ be a continuous function. Suppose that there exist constants $0 < a < b < c$ such that the following assumptions hold:*
*$(H_3)$ $f(t,u) < \Lambda_2^{-1} a$ for $(t,u) \in [0,1] \times [0,a]$;*
*$(H_4)$ $f(t,u) > \Lambda_3^{-1} b$ for $(t,u) \in [\eta,1] \times [b,c]$;*
*$(H_5)$ $f(t,u) \le \Lambda_2^{-1} c$ for $(t,u) \in [0,1] \times [0,c]$.*
*Then boundary value problem (1.1) has at least one nonnegative solution $u_1$ and two positive solutions $u_2$, $u_3$ in $\overline{\mathcal{K}}_c$ with*

$$\|u_1\| < a, \quad b < \min_{\eta \le t \le 1} u_2(t) \text{ and } \quad a < \|u_3\| \quad \text{with} \quad \min_{\eta \le t \le 1} u_3(t) < b.$$

*Proof.* We show that all the conditions of Theorem 2.11 are satisfied.
If $u \in \overline{\mathcal{K}}_c$, then $\|u\| \le c$. Condition $(H_5)$ implies $f(t,u(t)) \le \Lambda_2^{-1} c$ for $t \in [0,1]$. Consequently,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\le \Lambda_2^{-1} c \int_0^1 \left[ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) g(s,s) + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) g_i(\eta,s) \right] ds \\
&= \Lambda_2^{-1} c \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \right\} \\
&= c,
\end{aligned}$$

which implies $\|Au\| \le c$. Hence, $A : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ is completely continuous.
If $u \in \overline{\mathcal{K}}_a$, then $(H_3)$ yields

$$\begin{aligned}
(Au)(t) &< \Lambda_2^{-1} \int_0^1 \left[ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) g(s,s) + \sum_{i=1}^m \frac{\alpha_i}{\Delta \Gamma(p_i+q)} g_i(\eta,s) \right] ds \\
&= \Lambda_2^{-1} a \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \right\} \\
&= a.
\end{aligned}$$

Thus $\|Au\| < a$. Therefore, condition $(ii)$ of Theorem 2.11 holds.
Define a concave nonnegative continuous functional $\theta$ on $\mathcal{K}$ by

$$\theta(u) = \min_{\eta \le t \le 1} |u(t)|.$$

To check condition $(i)$ of Theorem 2.11, we choose $u(t) = \frac{b+c}{2}$ for $t \in [0,1]$. It is easy to see that $u(t) \in \mathcal{K}(\theta,b,c)$ and $\theta(u) = \theta\left(\frac{b+c}{2}\right) > b$, which means that $\{\mathcal{K}(\theta,b,c) : \theta(u) > b\} \ne \emptyset$. Hence, if $u \in \mathcal{K}(\theta,b,c)$, then $b \le u(t) \le c$ for $t \in [\eta,1]$.

From assumption $(H_4)$, we have

$$\begin{aligned}
\theta\left(Au\right) &= \min_{\eta\leq t\leq 1}\left|\left(Au\right)\left(t\right)\right| \\
&\geq \int_{\eta}^{1}\min_{\eta\leq t\leq 1} G\left(t,s\right) f\left(s,u\left(s\right)\right)ds \\
&> \Lambda_3^{-1} b\left\{\sum_{i=1}^{m}\frac{\alpha_i\eta^{q-1}}{\Delta\Gamma\left(p_i+q\right)}\int_{\eta}^{1} g_i\left(\eta,s\right)ds \right.\\
&\quad \left. +\left(q-1\right)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}}{\Delta}\int_{\eta}^{1} sg\left(s,s\right)ds\right\} \\
&= \Lambda_3^{-1} b\left\{\sum_{i=1}^{m}\frac{\alpha_i\eta^{p_i+2(q-1)}\left(1-\eta\right)^q}{\Delta\Gamma\left(p_i+q\right)q}\right.\\
&\quad \left. +\left(q-1\right)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q}-\xi_i^{q}\right)\eta^{q-1}\left(1-\eta\right)^{2q}\Gamma\left(q+1\right)}{\Delta\Gamma\left(2q+1\right)}\right\} \\
&= b.
\end{aligned}$$

Thus $\theta\left(Au\right) > b$ for all $u\in\mathcal{K}\left(\theta,b,c\right)$. This shows that condition $(i)$ of Theorem 2.11 is also satisfied.

By Theorem 2.11 and Remark 2.12, boundary value problem (1.1) has at least one nonnegative solution $u_1$ and two positive solutions $u_2$, $u_3$, which satisfy

$$\|u_1\| < a, \qquad b < \min_{\eta\leq t\leq 1}\left|u_2\left(t\right)\right| \quad a < \|u_3\| \quad \textit{with} \min_{\eta\leq t\leq 1}\left|u\left(t\right)\right| < b.$$

The proof is complete. □

## 4. Examples

### 4.1. Example

Consider the fractional differential equations with boundary value as follows:

$$\begin{cases} D^{\frac{3}{2}}u\left(t\right)+f\left(t,u\left(t\right)\right)=0, \quad 0<t<1, \\ u\left(0\right)=0 \\ u\left(1\right)=2\left(I^{\frac{3}{2}}u\right)\left(\frac{1}{4}\right)+\frac{1}{2}\left(I^{\frac{\pi}{4}}u\right)\left(\frac{1}{4}\right)+\frac{4}{5}\left(I^{\frac{2}{3}}u\right)\left(\frac{1}{4}\right)+\frac{3}{15}u\left(\frac{1}{3}\right)+\frac{3}{20}u\left(\frac{1}{4}\right)+\frac{1}{4}u\left(\frac{1}{5}\right), \end{cases} \tag{4.1}$$

where

$$f\left(t,u\right)\begin{cases} u(1-u^2)+4\left(1+\frac{3}{4}t\right), \; 0\leq t\leq 1;\; 0\leq u\leq 1 \\ 4\left(1+\frac{3}{4}t\right)e^{1-u}+\sin^2\left(\pi\left(1-u\right)\right), \quad 0\leq t\leq 1;\; 1\leq u\leq 21. \end{cases}$$

Set $m=3$, $\eta=\frac{1}{4}$, $q=\frac{3}{2}$, $\alpha_1=2$, $\alpha_2=\frac{1}{2}$, $\alpha_3=\frac{4}{5}$,$p_1=\frac{3}{2}$, $p_2=\frac{\pi}{4}$, $p_3=\frac{2}{3}$, $\beta_1=\frac{1}{4}$, $\beta_2=\frac{3}{20}$, $\beta_3=\frac{3}{5}$, $\xi_1=\frac{1}{5}$, $\xi_2=\frac{1}{4}$ and $\xi_3=\frac{1}{3}$

Consequently, we can get

$$\Delta = 1 - \Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \approx 0.265299.$$

Then, by direct calculations, we can obtain that

$$\begin{aligned}\Lambda_1 &= \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta\Gamma(p_i+q)} \left( \frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \\ &\quad + (q-1) \sum_{i=1}^{m} \frac{\beta_i \left( \xi_i^{q-1} - \xi_i^{q} \right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \\ &\approx 0.45478 \\ \Lambda_2 &= \left( 1 + \frac{\sum_{i=1}^{m} \beta_i}{\Delta} \right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left( \frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \\ &\approx 2.63219.\end{aligned}$$

Choose $r_1 = 1$, $r_2 = 21$, $M_1 = 3$ and $M_2 = 0.35$, $f(t,u)$ satisfies

$$f(t,u) \geq 4 \geq 3 = M_1 r_1, \qquad \forall (t,u) \in [0,1] \times [0,1]$$

and

$$f(t,u) \leq 7 \leq 7.35 = M_2 r_2 \qquad \forall (t,u) \in [0,1] \times [0,21]$$

Thus, $(H_1)$ and $(H_2)$ hold. By Theorem 3.1, we have that boundary value problem (4.1) has at least one positive solution $u$ such that $1 < \|u\| < 21$.

**4.2. Example**

Consider the following boundary value problem:

$$\begin{cases} D^{\frac{3}{2}} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0 \\ u(1) = \frac{1}{8} \left( I^{\frac{1}{2}} u \right) \left( \frac{1}{8} \right) + \frac{1}{3} \left( I^{\frac{3}{2}} u \right) \left( \frac{1}{8} \right) + \frac{1}{4} \left( I^{\frac{5}{2}} u \right) \left( \frac{1}{8} \right) + \frac{1}{3} u \left( \frac{1}{2} \right) + \frac{1}{5} u \left( \frac{1}{8} \right) + \frac{1}{7} u \left( \frac{1}{6} \right), \end{cases} \tag{4.2}$$

where

$$f(t,u) \begin{cases} u \left( \frac{3}{4} - u \right) + \frac{3}{16} \left( t^2 + 2 \right), & , 0 \leq t \leq 1,\ 0 \leq u \leq \frac{3}{4}, \\ \frac{1}{4} \left( t^2 + 2 \right) \cos^2 \left( \frac{2\pi}{9} u \right) + 120 \left( \frac{3}{4} - u \right)^2, & 0 \leq t \leq 1,\ \frac{3}{4} \leq u \leq \frac{3}{2}, \\ \frac{1}{16} \left( t^2 + 1082 \right) - 10 \sin^2 \left( u - \frac{3}{2} \right) \pi, & , 0 \leq t \leq 1,\ \frac{3}{2} \leq u \leq \infty. \end{cases}$$

Set $m = 3$, $\eta = \frac{1}{8}$, $q = \frac{3}{2}$, $\alpha_1 = \frac{1}{8}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{2}$, $p_3 = \frac{5}{2}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{5}$, $\beta_3 = \frac{1}{7}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{1}{4}$ and $\xi_3 = \frac{1}{6}$.
Consequently, we can get

$$\Delta = 1 - \Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \approx 0.589749.$$

Then, by direct calculations, we can obtain that

$$\Lambda_2 = \left(1 + \frac{\sum_{i=1}^{m} \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \approx 0,97003,$$

$$\Lambda_3 = \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+2(q-1)} (1-\eta)^q}{\Delta\Gamma(p_i+q)\, q} + (q-1) \sum_{i=1}^{m} \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1} (1-\eta)^{2q} \Gamma(q+1)}{\Delta\Gamma(2q+1)} \approx 0.02390086.$$

Choose $a = \frac{3}{4}$, $b = \frac{3}{2}$ and $c = 66$, then $f(t,u)$ satisfies

$$f(t,u) \le \frac{45}{64} < 0.773175 \approx \Lambda_2^{-1} a, \quad \forall (t,u) \in [0,1] \times \left[0, \frac{3}{4}\right],$$

$$f(t,u) \ge 67.62 > 62.73 \approx \Lambda_3^{-1} b, \quad \forall (t,u) \in \left[\frac{1}{8}, 1\right] \times \left[\frac{3}{2}, 66\right]$$

and

$$f(t,u) \le 67.6875 < 68.0391 \approx \Lambda_2^{-1} c, \quad \forall (t,u) \in [0,1] \times [0,66].$$

Thus, $(H_3)$, $(H_4)$ and $(H_5)$ hold. By Theorem 3.2, we have that boundary value problem (4.2) has at least one nonnegative solution $u_1$ and two positive solutions $u_2$, $u_3$ such that $\|u_1\| < \frac{3}{4}$, $\frac{3}{2} < \min_{\frac{1}{8} \le t \le 1} u_2(t)$ and $a < \|u_3\|$ with $\min_{\frac{1}{8} \le t \le 1} u_3(t) < \frac{3}{2}$.

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