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On the starlikeness of iterative integral operators

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Abstract. The main object of the present paper is to investigate starlikeness of certain integral operators, which are defined here by means of iterative in the open disk $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$ with $R \geq 1$. Also we prove that these result are best possible.

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1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions $f(z)$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ which are in the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots .$$

Also, let S and S^* denote the subclasses of $\mathcal{A}$ consisting of the univalent and starlike functions respectively. Studying the geometric properties of certain integral operators were considered by many authors during the last years. For example, some results of integral operator $F_\alpha(z) = \int_0^z (f(t)/t)^\alpha dt$ were obtained by Merkes and Wright [3]. Other type of integral operator such as $G_\alpha(z) = \int_0^z (f'(t))^\alpha dt$ was studied by the authors in [3] and [6]. Recently, the authors in [5] defined integral operators $L^k f(z)$ and $L_k f(z)$ which are iterative and take normalized analytic functions into the class S when restricted to $\mathbb{D}$. In this note, we define two new iterative integral operators $F^n(\gamma)(f(z)), F_n(\gamma)(f(z))$ and investigate the starlikeness of them in $\mathbb{D}$.

2. Integral Operators $F^n(\gamma)(f(z))$ and $F_n(\gamma)(f(z))$

Suppose that $\mathcal{A}_R$ denote the class of normalized analytic functions $f(z)$ in $\mathbb{D}_R$ with radius of convergence R and $R \geq 1$. We recall the generalized Bernardi integral operator $F(\gamma) : \mathcal{A} \to \mathcal{A}$, with $\gamma > -1$ as following (see [4])

$$F(\gamma)(f(z)) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt; \;\; (z \in \mathbb{D}, f \in \mathcal{A}). \tag{2.1}$$

Note that all powers in (2.1) are principal ones.

We now introduce the following two operators defined on $\mathcal{A}_R$ with $R \geq 1$.

Definition 2.1. *For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}_R$, let*

$$F^1(\gamma)(f(z)) = F(\gamma)(f(z)) = z + \sum_{k=2}^{\infty} \frac{1+\gamma}{\gamma+k} a_k z^k$$

$$F^2(\gamma)(f(z)) = F(\gamma)(F(\gamma)(f(z))) = z + \sum_{k=2}^{\infty} \left(\frac{1+\gamma}{\gamma+k}\right)^2 a_k z^k.$$

In general, for $n \in \mathbb{N}$ we define

$$F^n(\gamma)(f(z)) = F(\gamma)(F^{n-1}(\gamma)(f(z))) = z + \sum_{k=2}^{\infty} \left(\frac{1+\gamma}{\gamma+k}\right)^n a_k z^k. \tag{2.2}$$

Definition 2.2. *For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}_R$ we define*

$$F_1(\gamma)(f(z)) = F(\gamma)(f(z)) = z + \sum_{k=2}^{\infty} \frac{1+\gamma}{\gamma+k} a_k z^k,$$

$$\begin{aligned} F_2(\gamma)(f(z)) &= \frac{(1+\gamma)(2+\gamma)}{z^{\gamma+1}} \int_0^z \int_0^{t_2} t_1^{\gamma-1} f(t_1)\,dt_1 dt_2 \\ &= z + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)}{(\gamma+k)(\gamma+k+1)} a_k z^k \end{aligned} \tag{2.3}$$

and in general we have

$$\begin{aligned} F_n(\gamma)(f(z)) &= \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{z^{\gamma+n-1}} \int_0^z \int_0^{t_n} \int_0^{t_{n-1}} ... \int_0^{t_2} t_1^{\gamma-1} f(t_1)\,dt_1 dt_2 ... dt_n \\ &= z + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} a_k z^k. \end{aligned} \tag{2.4}$$

The aim of this note is to show that for $f \in \mathcal{A}_R$ with $R > 1$ there exists a positive integer N such that for $n \geq N$, $F^n(\gamma)(f(z))$ and $F_n(\gamma)(f(z))$ are starlike. Also, we show that these results are sharp.

To prove our main results, we need each of the following lemmas.

Lemma 2.3. ([2]) If $f \in \mathcal{A}$ satisfies

$$\Re(f'(z) + \alpha z f''(z)) > \frac{\frac{-1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-t}{1+t}\,dt}{1 - \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1-t}{1+t}\,dt};\ (z \in \mathbb{D})$$

for $\alpha \geq \frac{1}{3}$, then $f \in S^*$. The result is sharp.

Theorem 2.4. (Bieberbach's Theorem [1]) If $f \in S$, then $|a_n| \leq n$. The equality holds if and only if f is a rotation of the Koebe function.

3. Main Results

Theorem 3.1. Suppose that $f \in \mathcal{A}_R$, where $R > 1$. There exists a positive integer N such that for every $n \geq N$, $F^n(\gamma)(f(z))$ when restricted to $\mathbb{D}$ is starlike.

Proof. Let $n \in \mathbb{N}, \alpha \geq \frac{1}{3}$ and $f \in \mathcal{A}_R$. From (2.2) we obtain

$$(F^n(\gamma)(f(z)))' = 1 + \sum_{k=2}^{\infty} k \left(\frac{1+\gamma}{\gamma+k}\right)^n a_k z^{k-1}$$

and

$$\alpha z(F^n(\gamma)(f(z)))'' = \sum_{k=2}^{\infty} \frac{\alpha k(k-1)(1+\gamma)^n}{(\gamma+k)^n} a_k z^{k-1}.$$

So we obtain

$$Re\{(F^n(\gamma)(f(z)))' + \alpha z(F^n(\gamma)(f(z)))''\} := 1 + G(z)$$

where

$$G(z) = \sum_{k=2}^{\infty} \frac{k(1+\alpha(k-1))(1+\gamma)^n}{(\gamma+k)^n} Re(a_k z^{k-1}).$$

From last equality we observe that

$$|G(z)| \leq \sum_{k=2}^{\infty} \frac{k(1+\alpha(k-1))(1+\gamma)^n}{(\gamma+k)^n} |a_k|; \ (|z| < 1).$$

Since the radius of convergence of f (i.e. R) is greater than one, so there exists an $\epsilon > 0$ such that $C := \frac{1}{R} + \epsilon < 1$. In view of $R = \frac{1}{\limsup |a_k|^{\frac{1}{k}}}$ and the property of limit superior, there exists $N_1 \in \mathbb{N}, N_1 \geq 3$ such that for every $k \geq N_1$ we have $|a_k| < C^k$. Let $C_1 = \max\{|a_2|, |a_3|, ..., |a_{N_1-1}|\}$, then we obtain

$$\begin{aligned} |G(z)| &\leq (1+\gamma)^n \left(\sum_{k=2}^{N_1-1} \frac{k(1+\alpha(k-1))}{(\gamma+k)^n} C_1 + \sum_{k=N_1}^{\infty} \frac{k(1+\alpha(k-1))}{(\gamma+k)^n} C^k \right) \\ &\leq C_1 M_1 \left(\frac{1+\gamma}{2+\gamma}\right)^n + M_2 \left(\frac{1+\gamma}{\gamma+N_1}\right)^n \end{aligned}$$

where

$$M_1 := \sum_{k=2}^{N_1-1} k(1+\alpha(k-1)), M_2 := \sum_{k=N_1}^{\infty} k(1+\alpha(k-1))C^k < \infty.$$

Now from the last inequality we observe that there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|G(z)| < \frac{1}{1-\beta}$, where

$$0 < \beta = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t}\, dt < 1.$$

With this N we see that, $1 + G(z) > \frac{-\beta}{1-\beta}$, and by lemma 2.3 we conclude that $F^n(\gamma)(f(z))$ is starlike in $\mathbb{D}$ whenever $n \geq N$. $\square$

Theorem 3.2. Let $f \in \mathcal{A}_R$ with $R > 1$. There exists $N \in \mathbb{N}$ such that for every $n \geq N, F_n(\gamma)(f(z))$ is starlike in $\mathbb{D}$.

Proof. Let $n \in \mathbb{N}$ and $f \in \mathcal{A}_R$. From (2.4) we have

$$(F_n(\gamma)(f(z)))' = 1 + \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)ka_k}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} z^{k-1}$$

and

$$\alpha z(F_n(\gamma)(f(z)))'' = \sum_{k=2}^{\infty} \frac{\alpha(1+\gamma)(2+\gamma)...(n+\gamma)k(k-1)a_k}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} z^{k-1}.$$

So we obtain

$$\Re\{(F_n(\gamma)(f(z)))' + \alpha z(F_n(\gamma)(f(z)))''\} = 1 + H(z),$$

where

$$H(z) = \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)k(1+\alpha(k-1))}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} Re(a_k z^{k-1}).$$

Now the last equality implies that

$$|H(z)| \leq \sum_{k=2}^{\infty} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)k(1+\alpha(k-1))}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)} |a_k|; \;\; (z \in \mathbb{D}).$$

Since the radius of convergence of f is greater than one, hence there exists an $\epsilon > 0$ such that $B = \frac{1}{R} + \epsilon < 1$. Now using $\limsup |a_k|^{\frac{1}{k}} = \frac{1}{R}$ and the property of limit superior, there exists $N_1 \in \mathbb{N}, N_1 \geq 3$ such that for $k \geq N_1$ we have $|a_k| < B^k$. Let $C_1' = \max\{|a_2|, |a_3|, ..., |a_{N_1-1}|\}$, then we obtain

$$\begin{aligned} |H(z)| &\leq C_1' \sum_{k=2}^{N_1-1} \frac{(1+\gamma)k(1+\alpha(k-1))}{\gamma+n+1} + \sum_{k=N_1}^{\infty} A_n k(1+\alpha(k-1))B^k \\ &= C_1' M_1' \left(\frac{1+\gamma}{n+\gamma+1}\right) + A_n M_2', \end{aligned} \tag{3.1}$$

where

$$M_1' := \sum_{k=2}^{N_1-1} k(1+\alpha(k-1)), M_2' := \sum_{k=N_1}^{\infty} k(1+\alpha(k-1))B^k < \infty$$

and

$$A_n = \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{(\gamma+N_1)(\gamma+N_1+1)...(\gamma+N_1+n-1)}.$$

It is easy to see that $\lim_{n\to\infty} A_n = 0$. Using this fact, the relation (3.1) shows that there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|H(z)| < \frac{1}{1-\beta}$, where

$$0 < \beta = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} \frac{1-t}{1+t} \, dt < 1.$$

With this N we see that, $1 + H(z) > \frac{-\beta}{1-\beta}$ and $F_n(\gamma)(f(z))$ is starlike in $\mathbb{D}$ for $n \geq N$. $\square$

Now we shall see that the radius of convergence of f (*i.e.* $R > 1$) is best possible. To this end, let

$$L(z) = z + \sum_{k=2}^{\infty} a_k z^k, \text{ where } a_k = \begin{cases} (l+\gamma)^l, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Since $\limsup |a_k|^{\frac{1}{k}} = 1$, so the radius of convergence of $L(z)$ is one. In fact, this shows that $L(z) \in \mathcal{A}$. We then show that $F^n(\gamma)(L(z))$ and $F_n(\gamma)(L(z))$ are not starlike in $\mathbb{D}$ for every positive integer n.

Theorem 3.3. $F^n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$ for every $n \in \mathbb{N}$.

Proof. For fixed $n \in \mathbb{N}$, we have

$$F^n(\gamma)(L(z)) = z + \sum_{k=2}^{\infty} b_k z^k, \text{ with } b_k = \begin{cases} \frac{(1+\gamma)^n (l+\gamma)^l}{(k+\gamma)^n}, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\gamma > -1$ and $k = (3+\lfloor\gamma\rfloor)^l$, then we obtain

$$b_k = \frac{(1+\gamma)^n (l+\gamma)^l}{((3+\lfloor\gamma\rfloor)^l + \gamma)^n} > \left(\frac{1+\gamma}{2}\right)^n \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l.$$

Since

$$\lim_{l\to\infty} \left(\frac{1+\gamma}{2}\right)^{\frac{n}{l}} \frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n} = \infty,$$

there is $N \in \mathbb{N}$ such that for $l \geq N$ we have

$$\left(\frac{1+\gamma}{2}\right)^{\frac{n}{l}} \frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n} > 3 + \lfloor\gamma\rfloor,$$

or equivalently

$$\left(\frac{1+\gamma}{2}\right)^n \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l > (3+\lfloor\gamma\rfloor)^l = k.$$

Therefore we conclude that $b_k > k$, and by theorem 2.4 $F^n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$. Since $n \in \mathbb{N}$ is arbitrary, the proof is complete. $\square$

Theorem 3.4. $F_n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$ for every $n \in \mathbb{N}$.

Proof. For a fixed $n \in \mathbb{N}$ we obtain $F_n(\gamma)(L(z)) = z + \sum_{k=2}^{\infty} c_k z^k$, where

$$c_k = \begin{cases} \frac{(1+\gamma)(2+\gamma)...(n+\gamma)(l+\gamma)^l}{(\gamma+k)(\gamma+k+1)...(\gamma+k+n-1)}, & \text{if } k = (3+\lfloor\gamma\rfloor)^l, l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

There is $N_1 \in \mathbb{N}$ such that for $l \geq N_1$ we have

$$0 < \gamma + (3+\lfloor\gamma\rfloor)^l + n - 1 < 2\ (3+\lfloor\gamma\rfloor)^l.$$

Now for $\gamma > -1, k = (3+\lfloor\gamma\rfloor)^l$ and $l \geq N_1$ we obtain

$$\begin{aligned} c_k &\geq \frac{(1+\gamma)(2+\gamma)...(n+\gamma)(l+\gamma)^l}{(\gamma+(3+\lfloor\gamma\rfloor)^l+n-1)^n} \\ &> \frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{2^n} \left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l. \end{aligned} \tag{3.3}$$

As in the proof of theorem 3.3, it is easy to see that there is $N \in \mathbb{N}, N \geq N_1$ such that for $l \geq N$ we have

$$\frac{(1+\gamma)(2+\gamma)...(n+\gamma)}{2^n}\left(\frac{l+\gamma}{(3+\lfloor\gamma\rfloor)^n}\right)^l > (3+\lfloor\gamma\rfloor)^l = k.$$

Hence for $l \geq N$ we have $c_k > k$, and $F_n(\gamma)(L(z))$ is not starlike in $\mathbb{D}$. This completes the proof. □

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